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Computation of Structural Invariants of Generalized State-space Systems*

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A numerically stable method is proposed for computing the zeros and the Kronecker indices of a system given in generalized state-space form.

Key Words—Generalized state-space systems; singular systems; zeros; minimal indices; structural invariants; numerical methods.

Abstract—In this paper, we develop an algorithm for computing the zeros of a generalized state-space model described by the matrix 5-tuple (E, A, B, C, D), where E may be a singular matrix but $\det(A - \lambda E) \neq 0$. The characterization of these zeros is based on the system matrix of the corresponding 5-tuple. Both the characterization and the computational algorithm are extensions of equivalent results for state-space models described by the 4-tuples (A, B, C, D). We also extend these results to the computation of infinite zeros, and left and right minimal indices of the system matrix. Several non-trivial numerical examples are included to illustrate the proposed results.

1. INTRODUCTION

A LINEAR MULTIVARIABLE system can always be represented by the following polynomial set of equations:

$$\begin{aligned} T(\lambda)x(t) &= U(\lambda)u(t), \\ y(t) &= V(\lambda)x(t) + W(\lambda)u(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{F}^n$, $u(t) \in \mathbb{F}^m$ and $y(t) \in \mathbb{F}^p$; $\mathbb{F}$ denotes the appropriate field (of real or complex numbers in the present case) and $T(\lambda)$, $U(\lambda)$, $V(\lambda)$ and $W(\lambda)$ are polynomial matrices in λ and have dimensions $(n \times n)$, $(n \times m)$, $(p \times n)$ and $(p \times m)$, respectively. $T(\lambda)$ is assumed 'regular' (i.e. $\det(T(\lambda)) \neq 0$) (Rosenbrock, 1970). The operator λ could represent the differential operator d/dt (continuous-time) as well as the advance operator $\mathscr{D}$ (discrete-time).

From this representation one can define the 'system matrix' of (1) as:

$$\mathscr{S}(\lambda) = \left[\begin{array}{c|c} -T(\lambda) & U(\lambda) \\ \hline V(\lambda) & W(\lambda) \end{array}\right]. \tag{2}$$

The transfer function matrix of the system in (1) is given by $R(\lambda) = V(\lambda)T(\lambda)^{-1}U(\lambda) + W(\lambda)$. Note, that if $T(\lambda) = (\lambda I - A)$, $U(\lambda) = B$, $V(\lambda) = C$ and $W(\lambda) = D$, we get the standard state-space model

$$\begin{aligned} \lambda x(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{3}$$

for which numerous analysis and design methods exist. In recent years, there has been considerable interest in the study of modified state-space systems where now $T(\lambda) = (\lambda E - A)$, and E is a general matrix that may be singular (see e.g. Dervişoğlu and Desoer (1975), Luenberger (1977), Verghese *et al.* (1979), Campbell (1980), Verghese *et al.* (1981), Van Dooren (1981), Cobb (1984), Lewis (1985a, 1985b, 1986), Bender and Laub (1987), Misra and Patel (1989a, 1989b), Miminis (1993) and the references therein). The equations corresponding to this case are given by

$$\begin{aligned} \lambda Ex(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{4}$$

where, as mentioned earlier, $\det(\lambda E - A) \neq 0$. The systems described by (4) are frequently referred to as generalized state-space (GSS) systems or descriptor systems. For the sake of conciseness, in the sequel we will denote the system (4) by its parameters in the 5-tuple (E, A, B, C, D). Their importance arises from their applications in representing and resolving problems concerning differential equations with perturbed coefficients, singular perturbations

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(Saxena *et al.*, 1984), noncausal systems (Bernhard, 1982), identification (Adams *et al.*, 1984), economic systems (Luenberger, 1977), interconnected systems (Rosenbrock and Pugh, 1974) and modeling of electronic circuits (Chua and P.-M. Lin, 1975).

A system described by (4) is said to be 'non-singular' if E has full rank and 'singular' otherwise. The zeros of a non-singular GSS system are identical to those of the corresponding standard state-space system described by

$$\begin{aligned} \lambda x(t) &= E^{-1}Ax(t) + E^{-1}Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{5}$$

Definition and properties of the zeros of a standard state-space system are well understood (Davison and Wang, 1974a, 1974b; Desoer and Schulman, 1974) and a numerically stable algorithm for their computation was proposed by Emami-Naeini and Van Dooren (1982). However, if E is a singular matrix, then the characterization and computation of the zeros is not so straightforward. The primary difficulty stems from the fact that such systems cannot be transformed to the equivalent standard state-space system (5). It was shown by Rosenbrock (1970) that it is always possible to reduce the system in (4) to an equivalent system given by

$$\begin{aligned} \lambda x(t) &= \tilde{A}x(t) + \tilde{B}u(t), \\ y(t) &= \tilde{C}x(t) + \tilde{D}(\lambda)u(t), \end{aligned} \tag{6}$$

where $\tilde{D}(\lambda)$ is a polynomial matrix and λ represents the differential operator $\mathrm{d}/\mathrm{d}t$. Since the derivation of Rosenbrock (1970) is purely algebraic the same essentially holds for discrete time systems where λ is now the advance operator $\mathscr{D}$. A computational scheme for obtaining (6) from (4) was developed by Misra and Patel (1989a, 1989b), but this representation does not simplify the problem of determining the zeros, since now the system matrix $\mathscr{S}(\lambda)$ in (6) is not necessarily a first order polynomial matrix (or 'pencil').

In this paper, we present a computational technique for finding the zeros of generalized state-space systems, where E may be singular or may be poorly conditioned (with respect to inversion). Note, that in the latter case, it is theoretically possible to obtain the standard state-space model (5). However, due to numerical ill-conditioning (Golub and Van Loan, 1989) of E, the computed zeros may be far from accurate. Characterization of the zeros of generalized state-space systems proposed in this paper is parallel to that of standard state-space systems. Based on the proposed characterization, we develop a numerically reliable algorithm for their computation, which is also a generalization of the corresponding algorithm for the standard state-space systems (Emami-Naeini and Van Dooren, 1982). The proposed characterization and computational schemes are based upon the earlier results reported by the authors in Misra *et al.* (1990) and Varga (1991). In addition to finding the finite transmission and decoupling zeros, the proposed algorithm also computes the order of zeros at infinity, and row and column minimal indices of the system matrix. The layout of this paper is as follows. In Section 2, we review some useful results regarding the definition and the properties of zeros of rational and polynomial matrices and some essential techniques from numerical linear algebra. The state-space characterization of transmission zeros of singular systems is developed in Section 3, where it is shown that they can be interpreted as the transmission zeros of a lower-order non-singular generalized state-space system. Based on this characterization, an $\mathcal{O}(n^3)$ computational scheme is developed in Section 4. Finally, in Section 5, several examples are presented to illustrate various features of the proposed algorithm.

2. BACKGROUND MATERIAL

2.1. *Zeros of rational and polynomial matrices*

It is well known that any $(p \times m)$ rational matrix $R(\lambda)$ can be reduced by means of unimodular transformations to its Smith–McMillan form given by

$$M(\lambda)R(\lambda)N(\lambda) = G(\lambda), \tag{7}$$

where $M(\lambda)$ and $N(\lambda)$ are $(p \times p)$ and $(m \times m)$ polynomial matrices, respectively, with constant non-zero determinants (i.e. unimodular matrices). The matrix $G(\lambda)$ has the following structure:

$$G(\lambda) = \left[\begin{array}{cccc|c} \dfrac{\phi_{11}(\lambda)}{\psi_{11}(\lambda)} & 0 & \cdots & 0 & \\ 0 & \dfrac{\phi_{22}(\lambda)}{\psi_{22}(\lambda)} & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & O \\ 0 & 0 & \cdots & \dfrac{\phi_{l,l}(\lambda)}{\psi_{l,l}(\lambda)} & \\ \hline & & O & & O \end{array}\right], \tag{8}$$

where, $\phi_{ii} \mid \phi_{i+1,i+1}$, $\psi_{i+1,i+1} \mid \psi_{ii}$, $i = 1, \ldots, (l-1)$. The normal rank of $R(\lambda)$ is l, which clearly is the rank of $R(\lambda)$, for almost all values of λ. The 'finite zeros' of $R(\lambda)$ are defined as the zeros of the numerator polynomials of $R(\lambda)$, i.e. the values of λ for which $R(\lambda)$ has rank lower than l (Rosenbrock, 1970). For a polynomial matrix $P(\lambda)$, essentially, the same decomposition applies except that $G(\lambda)$ will then also be a polynomial matrix and, hence, all $\psi_{ii} = 1$. The above form then is called the Smith form of the polynomial matrix $P(\lambda)$.

While the definition of finite zeros from the Smith-McMillan form of a rational matrix $R(\lambda)$ is straightforward, this form is not recommended for their computation. It was shown in Rosenbrock (1970) that the polynomial matrix description (1) reduces this problem to one involving only the polynomial matrix $\mathscr{S}(\lambda)$, provided that the quadruple $\{T(\lambda), U(\lambda), V(\lambda), W(\lambda)\}$ has the property that the polynomial matrices

$$[-T(\lambda) \mid U(\lambda)], \quad \left[\begin{array}{c} -T(\lambda) \\ \hline V(\lambda) \end{array}\right], \tag{9}$$

have no finite zeros. This is equivalent to requiring that both matrices in (9) have full rank n where n is the dimension of the square invertible matrix $T(\lambda)$. These conditions are also called minimality conditions of the corresponding polynomial matrix description. Notice, that for a standard state-space model these conditions correspond to the system being controllable and observable. This connection was used in Emami-Naeini and Van Dooren (1982) to compute the zeros of a proper $R(\lambda)$ from a minimal standard state-space realization of $R(\lambda)$ at the points where the rank of $\mathscr{S}(\lambda)$ drops below its normal rank $n + l$. When $\mathscr{S}(\lambda)$ corresponds to a minimal order system, the points ($\lambda \in \mathbb{F}$) for which

$$\operatorname{rank}(\mathscr{S}(\lambda)) = \operatorname{rank}\left[\begin{array}{c|c} A - \lambda I_n & B \\ \hline C & D \end{array}\right] < n + l, \tag{10}$$

are indeed the McMillan zeros of the transfer function matrix $R(\lambda)$. Based on this definition of zeros, Emami-Naeini and Van Dooren (1982) developed a numerically backward stable algorithm for their computation.

In order to define 'infinite zeros' of rational and polynomial matrices one merely needs to perform the change of variables $\lambda = 1/\mu$, which maps the point $\lambda = \infty$ to $\mu = 0$, and then use the new Smith–McMillan decomposition of the transformed rational matrix $R(1/\mu)$ to extract its zeros at $\mu = 0$. Notice, that the new unimodular matrices $M(\lambda)$ and $N(\lambda)$ in (7) and polynomials ϕ and ψ in (8) will be different. The relation with the standard state-space system matrix now fails to hold, even if $R(\lambda)$ is proper, and one needs to use, instead, the concept of generalized state-space systems and generalized eigenvalues.

2.2. *Generalized eigenvalue problems*

In this section we review some basic facts about first order polynomial matrices or 'matrix pencils'.

2.2.1. *Singular pencils and the Kronecker canonical form.* Given an arbitrary pencil $(F - \lambda G)$, there exist invertible transformations S and T yielding a block diagonal decomposition

$$S(F - \lambda G)T = \operatorname{diag}\{\lambda I - J_f, I - \lambda J_\infty, L_{\epsilon_1}, \dots, L_{\epsilon_s}, L_{\eta_1}^{\mathrm{T}}, \dots, L_{\eta_t}^{\mathrm{T}}\}, \tag{11}$$

where J_f and J_∞ are in Jordan form (with J_∞ nilpotent) and describe the finite and infinite eigenvalues, respectively. The matrix L_k is the bidiagonal matrix of dimension $(k \times (k+1))$:

$$L_k = \begin{bmatrix} -\lambda & 1 & & & \\ & -\lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & -\lambda & 1 \end{bmatrix}. \tag{12}$$

The index sets $\{\epsilon_i, i = 1, \dots, s\}$ and $\{\eta_j, j = 1, \dots, t\}$ are the left and right minimal indices of $(F - \lambda G)$ (Wilkinson, 1978). The relationship of this canonical form to the Smith form of the first-order polynomial matrix $(F - \lambda G)$ is as follows (see e.g. Verghese *et al.* (1979) for details).

- To each Jordan block of size k at the eigenvalue α there corresponds an elementary divisor $(\lambda - \alpha)^k$ of a polynomial ϕ in the Smith form of $(F - \lambda G)$.
- To each Jordan block of size k at the eigenvalue ∞ there corresponds an elementary divisor $(\mu)^{k-1}$ of a polynomial ϕ in the Smith–McMillan form of $(F - 1/\mu G)$. (Notice the difference of one for the structure at ∞).
- To each minimal index $k = \epsilon$, respectively $k = \eta$, there is a polynomial vector of appropriate length whose only non-zero elements are $[1, \lambda, \lambda^2, \dots, \lambda^k]^{\mathrm{T}}$ in the right, respectively left null space of $(F - \lambda G)$.

The above connections indicate that the problem of finding the eigenstructure of a first-order polynomial matrix reduces to the computation of the Kronecker structure of this pencil.

2.2.2. *Generalized Schur decomposition.* For an arbitrary pencil $(F - \lambda G)$, there exist unitary transformations Q and Z yielding the block triangular decomposition

$$Q^*(F - \lambda G)Z = \left[\begin{array}{c|c|c} F_r - \lambda G_r & * & * \\ \hline O & F_f - \lambda G_f & * \\ \hline O & O & F_c - \lambda G_c \end{array}\right], \quad (13)$$

where $F_r - \lambda G_r$ has full row rank for all finite λ, $F_c - \lambda G_c$ has full column rank for all finite λ, and G_f is invertible. This decomposition is proven in Van Dooren (1979), where an algorithm is also given to find such a decomposition. Moreover, it is easy to check that the eigenvalues of $G_f^{-1}F_f$ are the only finite points where the rank of $(F - \lambda G)$ drops below its normal value, and hence are the finite zeros of $(F - \lambda G)$.

2.2.3. *QZ algorithm and finite zeros.* Given a square matrix pencil $(F_f - \lambda G_f)$, with $\det(F_f - \lambda G_f) \neq 0$, then there exist updating unitary matrices Q_f^* and Z_f of appropriate dimension, such that $Q_f^* F_f Z_f$ and $Q_f^* G_f Z_f$ are both upper triangular matrices (Wilkinson, 1978). Let f_{ii} and g_{ii} represent the ith elements along the diagonals of the upper triangular matrices $Q_f^* F_f Z_f$ and $Q_f^* G_f Z_f$, respectively. Then the ratios f_{ii}/g_{ii} represent the finite eigenvalues (or finite zeros) of $(F_f - \lambda G_f)$ and also of $(F - \lambda G)$. Note, that the QZ algorithm which performs this triangularization, also works for rank deficient G_f. However, this was not required here because of the preliminary reduction (13).

2.2.4. *Minimal indices and finite zeros.* In the matrix decomposition (13), one may choose Q^* and Z, such that the subpencils $(F_r - \lambda G_r)$ and $(F_c - \lambda G_c)$ have the special forms:

$$F_r - \lambda G_r = \begin{bmatrix} F^r_{1,1} & F^r_{1,2} - \lambda G^r_{1,2} & \cdots & F^r_{1,k-1} - \lambda G^r_{1,k-1} & F^r_{1,k} - \lambda G^r_{1,k} \\ O & F^r_{2,2} & \cdots & F^r_{2,k-1} - \lambda G^r_{2,k-1} & F^r_{2,k} - \lambda G^r_{2,k} \\ O & O & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & F^r_{k-1,k-1} & F^r_{k-1,k} - \lambda G^r_{k-1,k} \\ O & O & \cdots & O & F^r_{k,k} \end{bmatrix}, \quad (14)$$

where the diagonal matrices $F^r_{i,i}$ have full row rank $\hat{\mu}_i$, and the principal super diagonal matrices $G^r_{i,i+1}$ have full column rank $\hat{\tau}_{i+1}$, and

$$F_c - \lambda G_c = \begin{bmatrix} F^c_{j,j} & F^c_{j,j-1} - \lambda G^c_{j,j-1} & \cdots & F^c_{j,2} - \lambda G^c_{j,2} & F^c_{j,1} - \lambda G^c_{j,1} \\ O & F^c_{j-1,j-1} & \cdots & F^c_{j-1,2} - \lambda G^c_{j-1,2} & F^c_{j-1,1} - \lambda G^c_{j-1,1} \\ O & O & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & F^c_{2,2} & F^c_{2,1} - \lambda G^c_{2,1} \\ O & O & \cdots & O & F^c_{1,1} \end{bmatrix}, \quad (15)$$

where the matrices $F^c_{i,i}$ have full column rank μ_i, and the principal super diagonal matrices $G^c_{i+1,i}$ have full row rank τ_{i+1}. It is clear that these rank conditions guarantee the full rank properties of $(F_r - \lambda G_r)$ and $(F_c - \lambda G_c)$ for all finite λ, but in addition to this, it was shown in Van Dooren (1979) that the minimal indices and infinite zero structure of $(G - \lambda F)$ can be derived from this as well. We show later how they relate to the index sets $\{\hat{\mu}_i\}$, $\{\hat{\tau}_i\}$, $\{\mu_i\}$ and $\{\tau_i\}$.

2.2.5. *Singular value decomposition and row/column compression.* Given an arbitrary $(n \times m)$ complex matrix A, there always exists unitary $(n \times n)$ and $(m \times m)$ matrices U and V such that

$$A = U\Sigma V^*, \quad \Sigma = \begin{bmatrix} \Sigma_r & O \\ O & O \end{bmatrix}, \quad (16)$$

where $\Sigma_r = \operatorname{diag}\{\sigma_1, \ldots, \sigma_r\}$, r is the rank of the matrix A and σ_i are its singular values in descending order of their magnitude. An arbitrary $(n \times m)$ matrix A can also be transformed by means of unitary transformations to

$$T_1^* A = \begin{bmatrix} A_r \\ O \end{bmatrix} \quad AT_2 = [A_c \quad O] \quad (17)$$

where A_r has r independent rows and A_c has r independent columns. Clearly, r is the rank of the matrix A. We will refer to $[A_r^{\mathrm{T}} \ O]^{\mathrm{T}}$ as a row compressed matrix and to $[A_c \ O]$ as a column compressed matrix. It is easily seen that the required unitary matrices T_1 and T_2 can be obtained from the singular value decomposition of the matrix A.

Note, that it is also possible to find unitary transformations, such that the row compression yields $[O^{\mathrm{T}} \ A_r^{\mathrm{T}}]^{\mathrm{T}}$ and the column compression yields $[O \ A_c]$. Moreover, such decompositions can also be obtained via rank revealing QR factorizations (Chan, 1987). Of course, the non-zero matrices A_r and A_c are far from unique in the row and column compressed representations. It is also clear that in order to obtain decompositions of the type (14) and (15), these row and column compressions will have to be used extensively.

3. CHARACTERIZATION OF SYSTEM ZEROS

In this section, we define different types of zeros of a singular system described by (E, A, B, C, D) where the $n \times n$ pencil $(A - \lambda E)$ is non-singular (i.e. $\det(A - \lambda E) \neq 0$). In the sequel, we will always refer to both finite and infinite zeros as defined in Section 2 via the Smith–McMillan and Kronecker canonical forms.

Definition 3.1. The zeros of the pencil $[A - \lambda E \;\; B]$ are called the 'input decoupling zeros' of the system (E, A, B, C, D).

Definition 3.2. The zeros of the pencil

$$\begin{bmatrix} A - \lambda E \\ C \end{bmatrix}$$

are called the 'output decoupling zeros' of the system (E, A, B, C, D).

Definition 3.3. If the system (E, A, B, C, D) has no input and no output decoupling zeros, then the zeros of the system matrix

$$\mathscr{S}(\lambda) = \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right]$$

are called the 'transmission zeros' of the system (E, A, B, C, D).

We will be making extensive use of transformations of the kind

$$\left[\begin{array}{c|c} U & O \\ \hline O & W \end{array}\right]\left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c|c} V & O \\ \hline O & Z \end{array}\right] = \left[\begin{array}{c|c} U(A - \lambda E)V & UBZ \\ \hline WCV & WDZ \end{array}\right], \quad (18)$$

which we will refer to as 'generalized state-space transformations'. The matrices U, V, W and Z are invertible, and their dimensions are determined by the number of states (for U and V), the outputs and inputs, respectively, of the given system. It is obvious that the transfer function matrix $R(\lambda) = C(\lambda E - A)^{-1}B + D$ of such a generalized state-space system changes as $WR(\lambda)Z = WC(\lambda E - A)^{-1}BZ + WDZ$, since the other transformation matrices cancel out.

Theorem 3.1. The different zeros of the generalized state-space system (4) are invariant under generalized state-space transformations.

Proof. The proof is trivial since zeros are determined in terms of the Kronecker form of the pencils and these are unchanged under invertible left and right transformations. □

One special case of such transformations results from choosing U and V such that the matrix UEV is both row and column compressed:

$$UEV = \begin{bmatrix} E_{11} & O \\ O & O \end{bmatrix}, \quad (19)$$

where E_{11} is thus invertible. This can be achieved with the singular value decomposition or some other alternative suggested at the end of the previous section. Transforming and partitioning the matrices A, B and C conformably yields a new generalized state-space system with special properties.

Definition 3.4. Let the system (E, A, B, C, D) have a singular E matrix and let U and V yield a row and column compression of E. Then from

$$\left[\begin{array}{c|c} U & O \\ \hline O & I \end{array}\right]\left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c|c} V & O \\ \hline O & I \end{array}\right] =: \left[\begin{array}{c:c|c} A_{11} - \lambda E_{11} & A_{12} & B_1 \\ \hdashline A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right], \quad (20)$$

we define a 'compressed generalized state-space system' with E_{11} of full rank, given by

$$\begin{aligned} \lambda E_{11}\hat{x}(t) &= A_{11}\hat{x}(t) + [A_{12} \;\; B_1]\hat{u}(t), \\ \hat{y}(t) &= \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix}\hat{x}(t) + \begin{bmatrix} A_{22} & B_2 \\ C_2 & D \end{bmatrix}\hat{u}(t). \end{aligned} \quad (21)$$

This now leads to the following theorem.

Theorem 3.2. If the system (E, A, B, C, D) has no input or output decoupling zeros then neither does the compressed system (21). Moreover, their transmission zeros are then equal.

Proof. By transforming the system to a compressed form as in (20), we only performed a generalized state-space transformation. Clearly

$$\left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right], \quad \left[\begin{array}{c:c|c} A_{11} - \lambda E_{11} & A_{12} & B_1 \\ \hdashline A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

and

$$\left[\begin{array}{c|cc} A_{11} - \lambda E_{11} & A_{12} & B_1 \\ \hline A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{array}\right] \quad (22)$$

have the same Kronecker canonical form and, hence, the same zeros. The latter are then the transmission zeros of (21) if the compressed system has no input or output decoupling zeros under the condition that the original system has none. This is shown as follows. Since E_{11} is invertible, the pencils

$$[A_{11} - \lambda E_{11} \quad A_{12} \quad B_1]$$

and

$$\begin{bmatrix} A_{11} - \lambda E_{11} \\ A_{21} \\ C_1 \end{bmatrix} \tag{23}$$

can only have finite zeros. But, if there are finite points in which these pencils drop rank then this must also be the case for

$$[A - \lambda E \quad B] \quad \text{and} \quad \begin{bmatrix} A - \lambda E \\ C \end{bmatrix}, \tag{24}$$

and this contradicts the assumption that the original system had no input or output decoupling zeros. □

Notice that we can derive a compressed system as well for the triples (E, A, B) and (E, A, C), which will in both cases yield 5-tuples if E is singular:

$$\left[\begin{array}{c|cc} A_{11} - \lambda E_{11} & A_{12} & B_1 \\ \hline A_{21} & A_{22} & B_2 \end{array}\right] \tag{25}$$

and

$$\left[\begin{array}{c|c} A_{11} - \lambda E_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ C_1 & C_2 \end{array}\right]. \tag{26}$$

Since the zeros of these pencils are not affected by the compression, we have the following obvious, but useful, result.

Corollary 3.1. The zeros of the pencils

$$[A - \lambda E \quad B] \quad \text{and} \quad \begin{bmatrix} A - \lambda E \\ C \end{bmatrix}$$

(i.e. the input and output decoupling zeros of the system (E, A, B, C, D)) are those of the compressed system matrices (25) and (29), respectively.

Notice, that if E is non-singular the compression has no effect and some submatrices are void, but even then the above corollary still holds. Further, these results can be applied to any non-degenerate pencil $(A - \lambda E)$, whence the compressed form will yield a 5-tuple similar to (21) as shown below:

$$U(A - \lambda E)V =: \left[\begin{array}{c|c} A_{11} - \lambda E_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]. \tag{27}$$

Again, since the zeros of matrix pencils are unaffected by the compressions, we can state the following.

Corollary 3.2. The zeros of the pencil $(A - \lambda E)$ (i.e. the finite poles and the poles at infinity of the system (E, A, B, C, D)) are those of the compressed system matrix given in (27).

4. COMPUTATION OF TRANSMISSION ZEROS AND STRUCTURE AT INFINITY

In this section, we develop a deflation technique for the computation of transmission zeros, orders of infinite zeros and left and right minimal indices of singular systems. The procedure uses unitary transformation matrices to obtain matrix pencils $(A_f - \lambda E_f)$ and $(A_\infty - \lambda E_\infty)$, where the generalized eigenvalues of the former are the transmission zeros of the given singular system and the latter contains information about the orders of zeros at infinity and left and right minimal indices. Note, that for finite transmission zeros, the pencil is given by

$$\mathcal{S}_{rc}(\lambda) = \left[\begin{array}{c|c} A_{rc} - \lambda E_{rc} & B_{rc} \\ \hline C_{rc} & D_{rc} \end{array}\right], \tag{28}$$

where E_{rc} and D_{rc} are square invertible matrices. Once the reduced order system matrix (28) is obtained, the transmission zeros of the system can be computed as the generalized eigenvalues of the pencil $(A_{rc} - B_{rc}D_{rc}^{-1}C_{rc} - \lambda E_{rc})$ using the QZ algorithm (Stewart, 1973; Golub and Van Loan, 1989). It will be shown later that the generalized eigenvalues can be obtained without explicitly forming the inverse of D_{rc}.

In principle, the reduction procedure corresponds to transforming the variables $x(t)$, $u(t)$ and $y(t)$ to $\hat{x}(t) = V^*x(t)$, $\hat{u}(t) = Z^*u(t)$, $\hat{y}(t) = Wy(t)$, premultiplication of the state equation with the matrix U and deflation. The four matrices U, V, W and Z are chosen to be unitary and are constructed recursively, as described in the rest of this section.

Before starting the reduction procedure, we first transform the system to its compressed coordinates. To achieve this, we compute unitary matrices U and V as in (20) and (21). This transformation performs a rank revealing factorization on the descriptor matrix E such that E_{11} now has full rank r and is upper triangular. This can be achieved in $\mathcal{O}(n^3)$ operations using the singular value decomposition or via a rank revealing QR factorization. Next, we partition the matrices UAV, UB and CV conformably to UEV in (19) and redefine the system matrix as

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} \hat{A} - \lambda \hat{E} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right], \tag{29}$$

where

$$\hat{E} := E_{11},\ \hat{A} := A_{11},\ \hat{B} := [A_{12} \quad B_1] \quad \hat{C} := \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix}$$

and
$$\hat{D} := \begin{bmatrix} A_{22} & B_2 \\ C_2 & D \end{bmatrix}.$$

Clearly, $\hat{E}$ is now an invertible upper triangular matrix. As mentioned in Section 1, premultiplication of the state equation with $\hat{E}^{-1}$ leads to an rth order standard state-space system, the transmission zeros of which can be easily determined. However, conversion to a standard state-space system to determine transmission zeros should be avoided for reasons of numerical instability. Instead, the recursive deflation technique described in the rest of this section may be used.

In the recursive scheme, we will use the concepts of row and column compression as defined in Section 2. For notational convenience, it is assumed that the descriptor matrix is already a full rank upper triangular matrix i.e. $(E, A, B, C, D) := (\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{D})$, where the latter is defined as in (29). Further, let $m := n - r + m$, $p := n - r + p$ and $n := r$.

4.1. *Structure at infinity and row minimal indices*

The ith iteration performs the following operations on the system matrix: the rows of D are compressed by computing a unitary matrix $W^{(i)}$, such that $W^{(i)}D$ is row compressed i.e.

$$W^{(i)}D = \begin{bmatrix} D_1 \\ O \end{bmatrix}, \quad D_1 \in \mathbb{F}^{(p-\tau_i)\times m}, \quad O \in \mathbb{F}^{\tau_i \times m}. \tag{30}$$

Next, the matrix $W^{(i)}C$ is partitioned into C_1 and C_2, where the number of rows in C_1 is the same as that in D_1

$$W^{(i)}C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad C_1 \in \mathbb{F}^{(p-\tau_i)\times n}, \quad C_2 \in \mathbb{F}^{\tau_i \times n} \tag{31}$$

and a unitary $V^{(i)}$ is determined such that $C_2 V^{(i)}$ is column compressed:

$$C_2 V^{(i)} = [O \quad C_{22}], \quad O \in \mathbb{F}^{\tau_i \times (n-\mu_i)}, \quad C_{22} \in \mathbb{F}^{\tau_i \times \mu_i}. \tag{32}$$

Note, that the operation in (32) will destroy the diagonal or upper triangular structure of E. It is, therefore, necessary to perform a column compression and at the same time maintain the triangular structure of E. This is done by simultaneously determining $U^{(i)}$, such that $U^{(i)}EV^{(i)}$ is upper triangular. Details for achieving this are given later in this section.

Next, partition the descriptor E matrix as:

$$U^{(i)}EV^{(i)} = \left[\begin{array}{c|c} E_{11} & E_{12} \\ \hline O & E_{22} \end{array}\right],$$

where

$$E_{11} \in \mathbb{F}^{(n-\mu_i)\times(n-\mu_i)} \quad \text{and} \quad E_{22} \in \mathbb{F}^{\mu_i \times \mu_i}, \tag{33}$$

and perform the strict system equivalence transformation on the system and partition it as shown below:

$$\left[\begin{array}{c|c} U^{(i)} & \\ \hline & W^{(i)} \end{array}\right]\left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c|c} V^{(i)} & \\ \hline & I \end{array}\right]$$

$$= \left[\begin{array}{c:c|c} A_{11} - \lambda E_{11} & A_{12} - \lambda E_{12} & B_1 \\ \hdashline A_{21} & A_{22} - \lambda E_{22} & B_2 \\ \hline C_{11} & C_{12} & D_1 \\ \hdashline O & C_{22} & O \end{array}\right], \tag{34}$$

where the submatrices have appropriate dimensions and the submatrix C_{22} has full column rank.

The basic deflation procedure described is very similar, in principle, to the steps in the REDUCE algorithm for computing the transmission zeros of standard state-space systems, as described by Emami-Naeini and Van Dooren (1982). Following their ideas, it is now easily seen that the recursion can be performed on the reduced order subsystem (E, A, B, C, D) defined as:

$$\left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{c|c} A_{11} - \lambda E_{11} & B_1 \\ A_{21} & B_2 \\ C_{11} & D_1 \end{array}\right], \tag{35}$$

where $A,\ E \in \mathbb{F}^{(n-\mu_i)\times(n-\mu_i)}$, $B \in \mathbb{F}^{(n-\mu_i)\times m}$, $C \in \mathbb{F}^{(p-\tau_i+\mu_i)\times(n-\mu_i)}$ and $D \in \mathbb{F}^{(p-\tau_i+\mu_i)\times m}$. Further, for notational convenience, we define $A^c_{i,i} := C_{22}$. Since, $\mu_i \le \tau_i$, then $p - (\tau_i - \mu_i) \le p$ and $n - \mu_i \le n$, i.e. the dimensions of the state as well as the output vectors in (35) are less than or equal to the corresponding dimensions in the system from the previous recursion.

Note, that the new descriptor matrix E is a full rank upper triangular matrix. The reduction can therefore be repeated until a full row rank D matrix (i.e. $\tau_i = 0$) or a zero rank C_2 matrix (i.e. $\mu_i = 0$) is encountered. The state dimension n is decreased to $n := n - \mu_i$ at each step and the number of outputs to $p := p - (\tau_i - \mu_i)$.

Once a full row rank matrix $D := D_r$ is found, at step $(j+1)$, the transformed system (up to a column permutation) has the following structure:

$$\left[\begin{array}{c|c} U & \\ \hline & W \end{array}\right]\left[\begin{array}{c|c} B & A - \lambda E \\ \hline D & C \end{array}\right]\left[\begin{array}{c|c} I_p & \\ \hline & V \end{array}\right] = P_r(\lambda), \tag{36}$$

where

$$P_r(\lambda) = \left[\begin{array}{c|ccccc} \begin{array}{c:c} B_r & A_r - \lambda E_r \\ \hdashline D_r & C_r \end{array} & & & X & & \\ \hline & A^c_{j,j} & A^c_{j,j-1} - \lambda E^c_{j,j-1} & \cdots & A^c_{j,2} - \lambda E^c_{j,2} & A^c_{j,1} - \lambda E^c_{j,1} \\ & 0 & A^c_{j-1,j-1} & \cdots & A^c_{j-1,2} - \lambda E^c_{j-1,2} & A^c_{j-1,1} - \lambda E^c_{j-1,1} \\ O & 0 & 0 & \ddots & \vdots & \vdots \\ & \vdots & \vdots & \ddots & A^c_{2,2} & A^c_{2,1} - \lambda E^c_{2,1} \\ & 0 & 0 & \cdots & 0 & A^c_{1,1} \end{array}\right]$$

$$=: \left[\begin{array}{c|c} A_1 - \lambda E_1 & X \\ \hline O & A_2 - \lambda E_2 \end{array}\right], \tag{37}$$

D_r has full row rank ($\tau_{j+1} = 0$), $E^c_{i+1,i}$ has full row rank τ_{i+1} and $A^c_{i,i}$ has full column rank μ_i.

The pencil $(A_1 - \lambda E_1)$ contains the finite zeros (transmission zeros) of the system (E, A, B, C, D) and the information on the right nullspace of the corresponding system matrix. The pencil $(A_2 - \lambda E_2)$ contains the information on the orders of infinite zeros and left nullspace of the system matrix. This result is essentially the same as proven by Svaricek (1985) for standard state-space systems. The only difference resides in the matrix E which is invertible, because of the use of a compressed state-space system. Therefore, the same reasoning as in Svaricek's paper applies here as well and we quote the next result without proof.

Lemma 4.1. (Van Dooren, 1979). From the structure of the pencil $(A_2 - \lambda E_2)$, we can state that:

(i) there are $d_i = \mu_i - \tau_{i+1}$ infinite elementary divisors of degree i, $(i = 1, \ldots, j)$; and

(ii) there are $r_i = \tau_i - \mu_i$ Kronecker row indices of size $(i-1)$, $(i = 1, \ldots, j)$.

One then links this lemma to the following theorem.

Theorem 4.1. The orders of the infinite elementary divisors of $(A_2 - \lambda E_2)$ are equal to the orders of infinite zeros of the system (E, A, B, C, D).

Proof. The proof is a straightforward generalization of the result by Svaricek (1985) for standard state-space systems. □

It should be pointed out that similar results for standard state-space system were also reported by Kouvaritakis and MacFarlane (1976a, 1976b).

4.2. Finite zeros and column minimal indices

After a full row rank D_r matrix is found, the deflation procedure defined by (30)–(35) is repeated on the pertransposed (i.e. transposed over the anti-diagonal) system

$$\left[\begin{array}{c|c} C^p_r & A^p_r - \lambda E^p_r \\ \hline D^p_r & B^p_r \end{array}\right]$$

until an invertible input–output matrix D_{rc} is found. The resulting transformed pencil has the following structure:

$$\left[\begin{array}{c|c} U & \\ \hline & W \end{array}\right]\left[\begin{array}{c|c} C^p_r & A^p_r - \lambda E^p_r \\ \hline D^p_r & B^p_r \end{array}\right]\left[\begin{array}{c|c} I_m & \\ \hline & V \end{array}\right] = P_{rc}(\lambda), \tag{38}$$

where

$$P_{rc}(\lambda) = \left[\begin{array}{c|ccccc} \begin{array}{c:c} B_{rc} & A_{rc} - \lambda E_{rc} \\ \hdashline D_{rc} & C_{rc} \end{array} & & & X & & \\ \hline & A^r_{k,k} & A^r_{k,k-1} - \lambda E^r_{k,k-1} & \cdots & A^r_{k,2} - \lambda E^r_{k,2} & A^r_{k,1} - \lambda E^r_{k,1} \\ & 0 & A^r_{k-1,k-1} & \cdots & A^r_{k-1,2} - \lambda E^r_{k-1,2} & A^r_{k-1,1} - \lambda E^r_{k-1,1} \\ O & 0 & 0 & \ddots & \vdots & \vdots \\ & \vdots & \vdots & \ddots & A^r_{2,2} & A^r_{2,1} - \lambda E^r_{2,1} \\ & 0 & 0 & \cdots & 0 & A^r_{1,1} \end{array}\right]$$

$$=: \left[\begin{array}{c|c} A_1 - \lambda E_1 & X \\ \hline O & A_2 - \lambda E_2 \end{array}\right], \tag{39}$$

where, D_{rc} is a square invertible matrix, $E^r_{i+1,i}$ has full row rank ($=\hat{\tau}_{i+1}$) and $A^r_{i,i}$ has full column rank ($\hat{\mu}_i$).

Lemma 4.2. From the structure of the pencil $(A_2 - \lambda E_2)$ and with $\hat{\tau}_i$, $\hat{\mu}_i$ as defined above, $c_i = \hat{\tau}_i - \hat{\mu}_i$, $i = 1, \ldots, k$ are the Kronecker column indices of size $(i-1)$ for the system.

In the pencil $(A_1 - \lambda E_1)$, both E_{rc} and D_{rc} are square invertible matrices. Hence, as described in (13), it contains only the finite zeros. Further, since the matrix pencils $(A_2 - \lambda E_2)$ had full column rank in both (37) and (39), clearly the zeros of the non-singular pencil $(A_1 - \lambda E_1)$ correspond to the finite transmission zeros of the original singular system. In addition, the following result provides the numerical means to compute the finite transmission zeros of the system.

Theorem 4.2. The finite transmission zeros of the system are the generalized eigenvalues of the finite structure pencil $(A_f - \lambda E_f)$, where A_f and E_f are defined as

$$\left[\begin{array}{c|c} A_f & * \\ \hline O & D_f \end{array}\right] := \left[\begin{array}{c|c} A_{rc} & B_{rc} \\ \hline C_{rc} & D_{rc} \end{array}\right] W,$$
$$\left[\begin{array}{c|c} E_f & * \\ \hline O & O \end{array}\right] := \left[\begin{array}{c|c} E_{rc} & O \\ \hline O & O \end{array}\right] W, \tag{40}$$

and W is unitary.

Proof. This is completely analogous to the last step in Emami-Naeini and Van Dooren (1982). Since W is an orthogonal matrix

$$\begin{aligned} &\operatorname{rank}\left(\left[\begin{array}{c|c} A_{rc} - \lambda E_{rc} & B_{rc} \\ \hline C_{rc} & D_{rc} \end{array}\right]\right) \\ &= \operatorname{rank}\left(\left[\begin{array}{c|c} A_f - \lambda E_f & * \\ \hline O & D_f \end{array}\right]\right) \\ &= \operatorname{rank}(D_f) + \operatorname{rank}(A_f - \lambda E_f). \end{aligned} \tag{41}$$

By the reduction procedure, D_f has full rank. Therefore, $(A_1 - \lambda E_1)$ in (39) is rank deficient only at the generalized eigenvalues of $(A_f - \lambda E_f)$. Notice also, that E_f is invertible as shown by Emami-Naeini and Van Dooren (1982). □

We point out here that when pertransposing the decompositions in (39) and (40) again and embedding them in (37) we obtain exactly the required result (13)–(15).

4.3. *Implementation and computational complexity*

From the implementation point of view, the procedure is very similar to Emami-Naeini and Van Dooren (1982), except that an additional step is needed to keep E upper triangular at each stage of the recursion. Maintaining an upper triangular structure of E is crucial for reducing the dimension of the system in each iteration as described by (36) and (38). This will be discussed in more detail later in the section.

Based on the results presented in previous sections, we next outline a formal algorithm for the computation of finite zeros, row and column minimal indices and the orders of zeros at infinity. For simplicity of presentation, we assume that the system (E, A, B, C, D) is already a compressed generalized state-space system as defined by (21). The main operations are performed by 'Algorithm *S*(ingular)-*REDUCE*' given below:

Algorithm S-REDUCE.
input (E, A, B, C, D, n, m, p),
output $(E_r, A_r, B_r, C_r, D_r, n_r, m_r, p_r, r, d)$.
step *i*.
comment. compress rows of D with the unitary output transformation matrix $W^{(i)}$ and transform C:

$$\left[\begin{array}{c|c} C_1 & D_1 \\ C_2 & O \end{array}\right] := W^{(i)}[C \mid D],$$

$$D_1 \in \mathbb{F}^{(p-\tau_i)\times n}, \quad C_2 \in \mathbb{F}^{\tau_i \times n}$$

if $\tau_i = 0$, go to exit 1, end;
comment. using *TRIANGULARIZE*, compress columns of C_2 with $V^{(i)}$ and maintain E upper triangular with $U^{(i)}$; transform the system and partition as:

$$\left[\begin{array}{c|c} U^{(i)} & \\ \hline & W^{(i)} \end{array}\right] \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} V^{(i)} & \\ \hline & I \end{array}\right]$$
$$= \left[\begin{array}{c:c|c} A_{11} - \lambda E_{11} & A_{12} - \lambda E_{12} & B_1 \\ \hdashline A_{21} & A_{22} - \lambda E_{22} & B_2 \\ \hline C_{11} & C_{12} & D_1 \\ \hdashline O & C_{22} & O \end{array}\right];$$

set $\left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{c|c} A_{11} - \lambda E_{11} & B_1 \\ \hline A_{21} & B_2 \\ C_{11} & D_1 \end{array}\right]$;

if $\mu_i = 0$, then begin $p := p - \tau_i$, go to exit 2, end;
comment. update

$n := n - \mu_i$, $p := p - (\tau_i - \mu_i)$,
$i := i + 1$ go to step i;

exit 1.
$(E_r, A_r, B_r, C_r, D_r) := (E, A, B, C, D)$;
$n_r := n$; $m_r := m$; $p_r := p$ and
$r_k := \tau_k - \mu_k$, $d_k := \mu_k - \tau_{k+1}$,
$k = 1, \ldots, (i-1)$

exit 2.

$(E_r, A_r, B_r, C_r, D_r) := (E, A, B, C, D);$

$n_r := n; \quad m_r := m; \quad p_r := p$ and

$r_k := \tau_k - \mu_k, \quad d_k := \mu_k - \tau_{k+1},$

$k = 1, \ldots, (i-1), \quad r_i := \tau_i.$

Using algorithms *S-REDUCE*, above, and *TRIANGULARIZE*, given later in this section, we outline the algorithm *S*(ingular)-*ZEROS* for the computation of various structural invariants. Let ρ denote the rank defect of the original E. Then:

Algorithm S-ZEROS.

comment. reduce (E, A, B, C, D) to a system $(E_r, A_r, B_r, C_r, D_r)$ with the same structural invariants, with D_r of full row rank and calculate the orders of zeros at infinity and the row minimal indices;

call *S-REDUCE* (E, A, B, C, D, n, m, p), result $(E_r, A_r, B_r, C_r, D_r, n_r, m_r, p_r, r, d)$

rank $:= p_r - \rho$; if $n_r + p_r = 0$, then begin $n_f = 0$, go to exit, end;

comment. reduce $(E_r, A_r, B_r, C_r, D_r)$ to $(E_{rc}, A_{rc}, B_{rc}, C_{rc}, D_{rc})$ with the same structural invariants and with D_{rc} invertible,

call *S-REDUCE*$(E_r^P, A_r^P, C_r^P, B_r^P, D_r^P, n_r, p_r, m_r)$, result $(E_{rc}, A_{rc}, B_{rc}, C_{rc}, D_{rc}, n_{rc}, m_{rc}, p_{rc}, c)$;

if $n_{rc} = 0$, then begin $n_f = 0$, go to exit, end;

comment. compress columns of $[C_{rc} \; D_{rc}]$ to $[O \; D_f]$ and transform the system

$$\left[\begin{array}{c|c} A_f & * \\ \hline O & D_f \end{array}\right] := \left[\begin{array}{c|c} A_{rc} & B_{rc} \\ \hline D_{rc} & D_{rc} \end{array}\right] W;$$

$$\left[\begin{array}{c|c} E_f & * \\ \hline O & O \end{array}\right] := \left[\begin{array}{c|c} E_{rc} & O \\ \hline O & O \end{array}\right] W;$$

comment. compute finite transmission zeros using the QZ algorithm

$\{\lambda_1, \ldots, \lambda_{nf}\} := QZ(A_f, E_f);$

Exit. stop.

Next, we discuss the computational complexity of the algorithm. The operations for obtaining row compressed D are applied to the input and output matrices only. Hence, they will have an overall complexity $\mathcal{O}((m+n)(p+n)n)$ for the whole procedure. Although this compression has to be performed more than once, the total amount of computation needed is still proportional to μn^2, where μ is small compared to the state dimension. If we succeed in keeping the matrix E upper triangular using some method with complexity μn^2 as well, then the overall algorithm will remain cubic in complexity. At first sight, keeping E upper triangular does not seem simple because $EV^{(i)}$ is a full matrix, and triangularization of $EV^{(i)}$ may require $\mathcal{O}(n^3)$ operations. This would make the overall complexity of the algorithm $\mathcal{O}(n^4)$.

However, if $V^{(i)}$ is constructed using Givens rotations between the adjacent columns only, then the triangular form of E can be restored by applying one Givens rotation on the rows of E for each rotation of the columns of E. This is demonstrated by means of an illustrative example below, with $n = 6$ and $\tau = 1$. Let

$$\left[\frac{E}{c}\right] = \left[\begin{array}{cccccc} \times & \times & \times & \times & \times & \times \\ \otimes_1 & \times & \times & \times & \times & \times \\ & \otimes_2 & \times & \times & \times & \times \\ & & \otimes_3 & \times & \times & \times \\ & & & \otimes_4 & \times & \times \\ & & & & \otimes_5 & \times \\ \hline \times_1 & \times_2 & \times_3 & \times_4 & \times_5 & \times_6 \end{array}\right], \quad (42)$$

where the elements $\otimes_i$, $i = 1, \ldots, 5$ are zero to start with.

For the convenience of notation, denote $V^{(i)}$ by V and $U^{(i)}$ by U. The elements $\times_1, \ldots, \times_5$ have to be transformed to zero by postmultiplication with V, while UEV has to be maintained upper triangular. To achieve this, we select V as a product of Givens rotations $\mathcal{G}_{i,i+1}$ over appropriate angles θ_i and between columns i and $(i+1)$

$$V = \mathcal{G}_{12}(\theta_1)\mathcal{G}_{23}(\theta_2) \cdots \mathcal{G}_{56}(\theta_5)$$

and U as a 'reversed' product of Givens rotations $\mathcal{G}_{i,i+1}$ over some angles ϕ_i:

$$U = \mathcal{G}_{56}(\phi_5)\mathcal{G}_{45}(\phi_4) \cdots \mathcal{G}_{12}(\phi_1).$$

Clearly θ_1 can be chosen to annihilate $\times_1$ in (42), but it will introduce a non-zero element in position $\otimes_1$ of the equation. This newly introduced non-zero element is eliminated by the rotation $\mathcal{G}_{12}(\phi_1)$ of U, such that in $C\mathcal{G}_{12}(\theta_1)$, $\times_1 = 0$ and in $\mathcal{G}_{12}(\phi_1)E\mathcal{G}_{12}(\theta_1)$, $\otimes_1 = 0$ as well. By induction, each $\mathcal{G}_{i,i+1}(\theta_i)$ annihilates an element $\times_i$ in CV and each $\mathcal{G}_{i,i+1}(\phi_i)$ preserves $\otimes_i = 0$ in UEV. It is easy to see that using this approach, the triangularization step $U(EV)$ has a complexity $\mathcal{O}(\mu n^2)$. Therefore, the complexity of the overall algorithm is $\mathcal{O}(n^3)$.

The following segment shows how the above update can be accomplished efficiently on the entire system (E, A, B, C, D). Double subscript notation $a_{*,k}(a_{k,*})$ is used to denote the kth column (row) of A. For the ith iteration,

Algorithm TRIANGULARIZE.
step i.
  $\mu_i := 0$,
  for $k = p, -1, p - \tau_i - 1$,
    for $j = 1, n - 1 - \mu_i$,
      comment. compute (2×2) unitary $\mathcal{G}_{j,j+1}(\theta_j)$ such that $[c_{k,j}, c_{k,j+1}]$ is column compressed
        $[c_{k,j}, c_{k,j+1}]\mathcal{G}_{j,j+1}(\theta_j) =: [0, c_{k,j+1}]$,
        if $\theta_j = 0$, exit, else;
      comment. perform strict system equivalence transformation

$$\text{set } [e_{*,j}, e_{*,j+1}] := [e_{*,j}, e_{*,j+1}]\mathcal{G}_{j,j+1}(\theta_j),$$
$$[a_{*,j}, a_{*,j+1}] := [a_{*,j}, a_{*,j+1}]\mathcal{G}_{j,j+1}(\theta_j),$$
$$[c_{*,j}, c_{*,j+1}] := [c_{*,j}, c_{*,j+1}]\mathcal{G}_{j,j+1}(\theta_j);$$

      comment. compute (2×2) unitary $\mathcal{G}_{j,j+1}(\phi_j)$ such that $[e_{j,j}, e_{j+1,j}]^{\mathrm{T}}$ is row compressed

$$\mathcal{G}_{j,j+1}(\phi_j)\begin{bmatrix} e_{j,j} \\ e_{j+1,j} \end{bmatrix} =: \begin{bmatrix} e_{j,j} \\ 0 \end{bmatrix};$$

      comment. perform strict system equivalence transformation

$$\text{set } \begin{bmatrix} e_{j,*} \\ e_{j+1,*} \end{bmatrix} := \mathcal{G}_{j,j+1}(\phi_j)\begin{bmatrix} e_{j,*} \\ e_{j+1,*} \end{bmatrix},$$
$$\begin{bmatrix} a_{j,*} \\ a_{j+1,*} \end{bmatrix} := \mathcal{G}_{j,j+1}(\phi_j)\begin{bmatrix} a_{j,*} \\ a_{j+1,*} \end{bmatrix},$$
$$\begin{bmatrix} b_{j,*} \\ b_{j+1,*} \end{bmatrix} := \mathcal{G}_{j,j+1}(\phi_j)\begin{bmatrix} b_{j,*} \\ b_{j+1,*} \end{bmatrix};$$

    end;
  if $\theta_{n-1-\mu_i} \neq 0$, $\mu_i := \mu_i + 1$
  end;

where τ_i and μ_i are as defined by equations (30) and (32), respectively.

We conclude this section by pointing out that this algorithm is in fact an efficient implementation of the general algorithm described in Van Dooren (1979). The efficiency is obtained by a careful ordering of Givens rotations and hence, we are still using orthogonal transformations at all stages of the algorithm. As a consequence of this, the error analysis in Van Dooren (1979) still holds here and we can conclude that the computed zeros are in fact the exact zeros of a slightly perturbed system matrix $\mathcal{S}(\lambda)$. In other words, the present algorithm is backward stable. Notice, that the same result also applies to the algorithm described in Emami-Naeini and Van Dooren (1982).

5. NUMERICAL EXAMPLES AND DISCUSSION

In this section, we present several examples to illustrate the proposed technique. The numerical computations reported in this section were performed in double precision, on an IMB PC compatible (386/387) machine using MATLAB.

Example 5.1. The first example is a scalar system. We can easily compare the transmission zeros for the system computed using the proposed technique with the roots of the numerator polynomial of the transfer function computed using the technique described in Misra (1989). For this example, we selected a 5th order, one input, one output system with rank $(E) = 4$. Various parameters of the system for this example are given below.

$$E = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 \end{bmatrix},$$

$$b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix},$$

$$c = [1 \quad 2 \quad 2 \quad 1 \quad 2], \quad \text{and } d = 1.$$

The transfer function of the above descriptor system is given by

$$R(\lambda) = \frac{0.25\lambda^4 + 3.5\lambda^3 - \lambda^2 + 2.75\lambda + 1.5}{\lambda^4 - 1.75\lambda^3 + 1.5\lambda^2 - 1.5\lambda - 2.5}.$$

Table 1 compares the roots of the numerator polynomial with the transmission zeros of the non-singular lower-order generalized state-space subsystem.

The above example is for the sake of illustration only and not to demonstrate any numerical properties of the proposed algorithm. In fact, for this system, the matrix E in the compressed representation was very well conditioned. The results will be accurate, even if the transmission zeros were obtained by transforming it to standard state-space form and computing zeros by applying the algorithm of Emami-Naeini and Van Dooren (1982).

TABLE 1. FINITE TRANSMISSION ZEROS IN EXAMPLE 5.1

Roots of numerator polynomial	Finite zeros (proposed method)
$-1.433064593655173e^{+01}$	$-1.433064593655172e^{+01}$
$+3.674820146082842e^{-01}$	$+3.674820146082841e^{-01}$
$\pm j9.489394451132227e^{-01}$	$\pm j9.489394451132229e^{-01}$
$-4.043180926648485e^{-01}$	$-4.043180926648483e^{-01}$

The next few examples illustrate the computation of various structural invariants.

Example 5.2. This example is a polynomial matrix $D(\lambda) = D_0 + D_1\lambda + D_2\lambda^2$ from Van Dooren and Dewilde (1983), where

$$D_0 = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 4 & 2 \\ 0 & -1 & -2 \end{bmatrix}$$

and

$$D_2 = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 1 & 4 & 2 \end{bmatrix}.$$

An irreducible 4th order state-space realization of $D(\lambda)$ is the system (E, A, B, C, D) with

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 4 & 2 \\ 0 & -1 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The compressed form (21) of the above system can be immediately seen to be the 2nd-order subsystem $(\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{D})$ where

$$\hat{E} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\hat{D} = \begin{bmatrix} 0 & 0 & 1 & 4 & 2 \\ -1 & 0 & 0 & -1 & -2 \\ -1 & -1 & 1 & 2 & -2 \\ -1 & 0 & 0 & -1 & -2 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

By applying the algorithm *S-ZEROS* to this compressed system we obtained the following results.

- The polynomial matrix $D(\lambda)$ has a finite zero at $\lambda_1 = 1$ (computed without rounding errors).
- The normal rank of $D(\lambda)$ is 2.
- $D(\lambda)$ has no zeros at infinity.
- The computed row and column minimal indices are $r_1 = 0$, $r_2 = 1$, $c_1 = 1$.

Example 5.3. This example illustrates the usage of algorithm *S-ZEROS* to compute the poles of a GSS system. In particular, we determined the structure at infinity of the pencil $(A - \lambda E)$, where A and E are the matrices from the previous example. The compressed system $(\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{D})$ for the pair (E, A) is (see (21) and (27)):

$$\hat{E} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\hat{D} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

By applying the algorithm *S-ZEROS* to this compressed system we obtained the following results.

- The pencil $(A - \lambda E)$ has no finite zeros.
- The pencil $(A - \lambda E)$ has no infinite elementary divisors of degree 1 but has an infinite elementary divisor of degree 2.
- The pencil $(A - \lambda E)$ does not have any row or column minimal indices.

Example 5.4. This example illustrates the usage of algorithm *S-ZEROS* for the analysis of controllability of a GSS. We consider the following 9th order observable but uncontrollable realization of the polynomial matrix from Example 5.2.

$$E = \begin{bmatrix} I & O & O \\ O & I & O \\ O & O & O \end{bmatrix}, \quad A = \begin{bmatrix} O & I & O \\ O & O & I \\ I & O & O \end{bmatrix},$$

$$B = \begin{bmatrix} D_1 \\ O \\ D_2 \end{bmatrix}, \quad C = [O \quad -I \quad O]$$

and

$$D = D_0,$$

where each submatrix has order three. In order to analyze the controllability of this system, we compute the zeros of the 6th order compressed system corresponding to the triple (E, A, B) (see

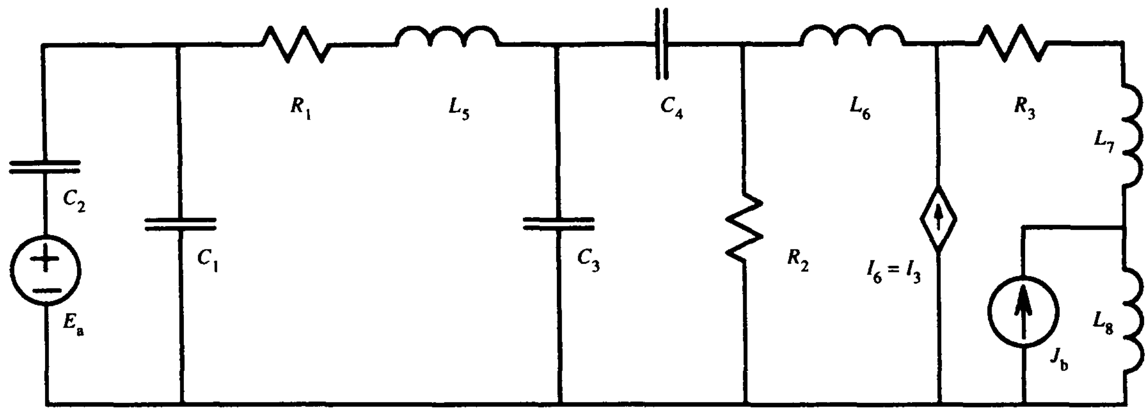

FIG. 1. Circuit for Example 5.5.

(25)):

$$\hat{E} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} O & I \\ O & O \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} O & D_1 \\ I & O \end{bmatrix}, \quad \hat{C} = [I \quad O]$$

and

$$\hat{D} = [O \quad D_2].$$

By applying the algorithm *S-ZEROS* to this compressed system we obtained the following results.

- The pencil $[A - \lambda E\, B]$ has two infinite elementary divisors of degree 2 ($d_1 = 0$, $d_2 = 2$), which correspond to the uncontrollable infinite eigenvalues.
- The pencil $[A - \lambda E\, B]$ has the column minimal indices $c_1 = 1$, $c_2 = 2$ and no row minimal indices.

The next example illustrates the numerical properties of the proposed algorithms. The example is an electrical circuit, hence the data describing its parameters are realistic.

Example 5.5. The final example is an RLC electrical circuit (Fig. 1.) with an independent loop containing capacitors and voltage sources only and an independent cutset with inductors and current sources only. E_a and J_b represent the inputs, the voltages across C_1, C_3 and C_4 are the outputs of the circuit and I_3 is the current through C_3. The matrices representing the state description of this circuit for $C_1 = C_2 = 1000\mu$F, $C_3 = C_4 = 5000\mu$F, $L_5 = L_6 = 100$ mH, $L_7 = L_8 = 200$ mH, $R_1 = 1000\,\Omega$, $R_2 = 500\,\Omega$ and $R_3 = 250\,\Omega$ are

$$E = \begin{bmatrix} -1.00 & -1.00 & 5.0000e^{-03} & -5.0000e^{-03} & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & -5.0000e^{-03} & -5.0000e^{-03} & 0.00 & 0.00 & 0.00 & 0.00 \\ -1.0000e^{-03} & 0.00 & 0.00 & -2.5000e^{-01} & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & -5.00 & 0.00 & 1.0000e^{-01} & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 2.5000e^{-01} & -2.5000e^{-01} & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & -7.5000e^{+02} & 0.00 & 1.0000e^{-01} & -2.0000e^{-01} & -2.0000e^{-01} \end{bmatrix},$$

$$A = \begin{bmatrix} 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 5.0000e^{+02} & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -1.00 & 1.00 \\ 1.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 1.00 \\ -1.00 & 0.00 \\ 1.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \end{bmatrix}.$$

Applying the *S-ZEROS* algorithm, to the pencil $(A - \lambda E)$ it was found that:

- the above system has five finite poles at $-4.997798310302830e^{+03}$, $-4.001159494478821e^{+00}$, $-2.005301982357856e^{-01}$, $-1.819955490315653e^{-18}$ and $1.426452528199640e^{-17}$. Note, that the last two poles are at the origin;
- there are two non-dynamic modes at infinity and an infinite elementary divisor of order 1, which corresponds to a true dynamical impulsive mode.

Next, from the analysis of the system pencil it was determined that:

- the system has two finite zeros at the origin;
- the system has one infinite elementary divisor of order 1, i.e. $d_1 = 1$;
- the row minimal indices are $r_1 = r_2 = 1$ and the column minimal index is $c_1 = 1$;
- the normal rank of the transfer function matrix is 1.

To show the numerical performance, its finite decoupling zeros were computed by (a) finding a full rank E_{11}, premultiplying the state equation by its inverse ($\mathrm{cond}(E_{11}) = 1.8944e^{+08}$) and computing zeros of the resulting standard state-space system and (b) applying the proposed algorithm. It is easy to see that the system has two input decoupling zeros at the origin. The results obtained using the two approaches are listed in Table 2. Note, that there is significant improvement in the numerical values of the input decoupling zeros (known to be at the origin) when using the proposed method compared to using the inverse of (full rank) the descriptor matrix and the algorithm for standard systems. To further verify our results, we computed the singular value decomposition of the pencil

$$\left[\begin{array}{c|c} A - \lambda_i E & B \\ \hline C & D \end{array}\right],$$

where λ_i are the decoupling zeros in the second column of Table 2. For the two cases, the last three singular values were found to be at

$1.290987778971753e^{-16}$, $3.382764823262971e^{-17}$
$1.237027516183035e^{-17}$
and
$6.426132695085887e^{-17}$, $1.140918261013493e^{-17}$,
$7.959844515347648e^{-18}$.

Clearly, the singular values verify the observations deduced from the system pencil information.

Summarizing the above observations, we can state that:

(i) The poles at the origin are uncontrollable because $\mathrm{rank}([A\ B]) = 6$. Thus, the computed zeros are in fact input-decoupling zeros. At least one of these poles is also unobservable because $\mathrm{rank}([A^T\ C^T]) = 7$. Clearly, one of the computed zeros is an input–output decoupling zero. Note, that this information can also be obtained directly from the analysis of pencils $[A - \lambda E\ B]$ or $[A^T - \lambda E^T\ C^T]$.
(ii) The zero at infinity coincides with the pole

TABLE 2. FINITE INPUT DECOUPLING ZEROS IN EXAMPLE 5.5

Zeros using E_{11}^{-1} (compressed)	Zeros using proposed method
$1.031363086958463e^{-11}$	$-7.755365867415895e^{-18}$
$-8.213043648840842e^{-14}$	$1.305271529760255e^{-17}$

at infinity. This zero is an output-decoupling zero because $\text{rank}([E\ B]) = 8$ and $\text{rank}([E^T\ C^T]) = 6$.

(iii) By taking into account the above facts, it follows that the circuit can be modeled by a state-space model of order $8 -$ (number of finite input-decoupling zeros) $-$ (number of infinite output-decoupling zeros) $= 5$. In fact, by extracting a non-zero direct feedthrough matrix D, the order can be further reduced to 4.

6. CONCLUDING REMARKS

In this paper, we presented a state-space characterization of the transmission zeros of singular linear multivariable systems that is analogous to that of standard systems. Based on the results reported in this paper, we developed an efficient technique for their computation. It was shown that from the given singular system, using unitary coordinate transformations, we can obtain a non-singular subsystem whose transmission zeros are identical to the transmission zeros of the original singular system. The proposed characterization and the computational procedure based on it were illustrated by means of some examples.

It should, perhaps, be emphasized that an algorithm such as proposed in this paper can be viewed as an (almost) universal analysis tool for linear time-invariant systems. Properties such as stability, controllability, observability, stabilizability or detectability, as well as the row and column minimal indices of the corresponding system matrix, can be easily obtained by computing zeros of appropriate system matrices (for $p = 0$ and/or $m = 0$). It is also a valid alternative to computing the Kronecker structure of an arbitrary singular pencil (Beelen *et al.*, 1986; Beelen and Van Dooren, 1988). An implementation of the proposed computational method is available in the descriptor systems subroutines library DESCRIPT (Varga, 1992). For additional information regarding these subroutines, please contact Andras Varga.

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