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Structural Invariants of Generalized State Space Systems

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Abstract

In this paper, we study the structural invariants of a generalized state space model described by the matrix 5-tuple (E, A, B, C, D), where E may be a singular matrix but $\det(A - \lambda E) \neq 0$. The characterization of these zeros is based on the system matrix of the corresponding 5-tuple and is an extension of equivalent results for state space models described by the 4-tuples (A, B, C, D).

1. Introduction

A linear multivariable system can always be represented by the following polynomial set of equations:

$$\begin{aligned} T(\lambda)x(t) &= U(\lambda)u(t) \\ y(t) &= V(\lambda)x(t) + W(\lambda)u(t) \end{aligned} \tag{1.1}$$

where $x(t) \in \mathbb{F}^n$, $u(t) \in \mathbb{F}^m$ and $y(t) \in \mathbb{F}^p$. $T(\lambda), U(\lambda), V(\lambda)$ and $W(\lambda)$ are polynomial matrices in λ and have dimensions $(n \times n)$, $(n \times m)$, $(p \times n)$ and $(p \times m)$, respectively. $T(\lambda)$ is assumed *regular* (i.e., $\det(T(\lambda)) \neq 0$) [15].

From this representation one can define the *system matrix* of (1.1) as:

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} -T(\lambda) & U(\lambda) \\ \hline V(\lambda) & W(\lambda) \end{array}\right]. \tag{1.2}$$

The transfer function matrix of the system in (1.1) is given by $R(\lambda) = V(\lambda)T(\lambda)^{-1}U(\lambda) + W(\lambda)$. Note that if $T(\lambda) = (\lambda E - A)$, $U(\lambda) = B$, $V(\lambda) = C$ and $W(\lambda) = D$, we get the state space model

$$\begin{aligned} \lambda E x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{1.3}$$

If $E = I$ in (1.3), we get the standard state model for which numerous analysis and design methods exist. In recent years, there has been considerable interest in the study of the case where $E \neq I$ may even be singular (see e.g., [1]–[11], etc. The systems described by (1.3) are referred to as *generalized state space* (GSS) systems or *descriptor systems*. For the sake of conciseness, in the sequel we will denote the system (1.3) by its parameters in the 5-tuple (E, A, B, C, D).

Definition and properties of the zeros of a standard state space system are well understood [12] and a numerically stable algorithm for their computation was proposed by Emami-Naeini and Van Dooren [13]. In this paper, we extend the results to generalized state space systems and also illustrate how the order of zeros at infinity and row and column minimal indices of the system matrix can be found.

2. Background Material

2.1. Zeros of Rational and Polynomial Matrices

It is well known that any $(p \times m)$ rational matrix $R(\lambda)$ can be reduced by means of unimodular transformations to its Smith-McMillan form given by

$$M(\lambda)R(\lambda)N(\lambda) = G(\lambda), \tag{2.1}$$

where $M(\lambda)$ and $N(\lambda)$ are $(p \times p)$ and $(m \times m)$ polynomial matrices respectively, with constant non-zero determinants (i.e., *unimodular matrices*). The matrix $G(\lambda)$ is $\mathrm{diag}(\phi_{ii}(\lambda)/\psi_{ii}(\lambda))$, $i = 1, \ldots, \ell$. Further, $\phi_{ii} | \phi_{i+1,i+1}$, $\psi_{i+1,i+1} | \psi_{ii}$, $i = 1, \ldots, (\ell - 1)$. The *normal rank* of $R(\lambda)$ is ℓ, which clearly is the rank of $R(\lambda)$ for almost all values of λ. The *finite zeros* of $R(\lambda)$ are defined as the zeros of the numerator polynomials of $R(\lambda)$, i.e., the values of λ for which $R(\lambda)$ has rank *lower* than ℓ [15]. For a polynomial matrix $P(\lambda)$ essentially the same decomposition applies except that $G(\lambda)$ will then also be polynomial matrix and hence all $\psi_{ii} = 1$. The above form then is called the Smith form of the polynomial matrix $P(\lambda)$.

It was shown by Rosenbrock [15] that the polynomial matrix description (1.1) reduces this problem to one involving only the polynomial matrix $\mathcal{S}(\lambda)$, *provided* that the quadruple $\{T(\lambda), U(\lambda), V(\lambda), W(\lambda)\}$ has the property that the polynomial matrices

$$\left[\begin{array}{c|c} -T(\lambda) & U(\lambda) \end{array}\right], \quad \left[\begin{array}{c} -T(\lambda) \\ \hline V(\lambda) \end{array}\right] \tag{2.2}$$

have *no finite zeros*. This is equivalent to requiring that both matrices in (2.2) have full rank n where n is the dimension of the square invertible matrix $T(\lambda)$. These conditions are also called *minimality conditions* of the corresponding polynomial matrix description. Notice that for a standard state space model these conditions correspond to the system being controllable and observable. This connection was used by Emami-Naeini and Van Dooren [13] to compute the zeros of a proper $R(\lambda)$ from a *minimal* standard state space realization of $R(\lambda)$ as the points where the rank of $\mathcal{S}(\lambda)$ drops below its normal rank $n + \ell$. When $\mathcal{S}(\lambda)$ corresponds to a minimal order system, the points $(\lambda \in \mathbb{F})$ for which

$$\mathrm{rank}\,(\mathcal{S}(\lambda)) = \mathrm{rank} \left[\begin{array}{c|c} A - \lambda I_n & B \\ \hline C & D \end{array}\right] < n + \ell, \tag{2.3}$$

are indeed the McMillan zeros of the transfer function matrix $R(\lambda)$.

In order to define *infinite zeros* of rational and polynomial matrices one merely needs to perform the change of variables $\lambda = 1/\mu$, which maps the point $\lambda = \infty$ to $\mu = 0$ and then use the new Smith-McMillan decomposition of the transformed rational matrix $R(1/\mu)$ to extract its zeros at $\mu = 0$. Notice that the new unimodular matrices M and N in (2.1) and polynomials ϕ and ψ will be different. The relation with the standard state space system matrix now fails to hold, even if $R(\lambda)$ is proper, and one needs to use instead the concept of generalized state space systems and generalized eigenvalues.

2.2. Generalized Eigenvalue Problems

In this section we review some basic facts about first order polynomial matrices or *matrix pencils.*

Singular pencils and the Kronecker canonical form

Given an arbitrary pencil $(F - \lambda G)$, there exist *invertible* transformations S and T yielding a block diagonal decomposition

$$S(F-\lambda G)T = \mathrm{diag}\{\lambda I - J_f, I - \lambda J_\infty, L_{\epsilon_1}, \ldots, L_{\epsilon_s}, L_{\eta_1}^T, \ldots, L_{\eta_t}^T\}$$

where J_f and J_∞ are in Jordan form (with J_∞ nilpotent) and describe the finite and infinite eigenvalues, respectively. The matrix L_k is a bidiagonal matrix of dimension $(k \times (k+1))$, with elements $l_{i,i} = -\lambda$ and $l_{i,i+1} = 1$. Further, the index sets $\{\epsilon_i, i = 1, \ldots, s\}$ and $\{\eta_j, j = 1, \ldots, t\}$ are the left and right minimal indices of $(F - \lambda G)$ [19]. The relationship of this canonical form to the Smith form of the first order polynomial matrix $(F - \lambda G)$ may be found in [10]. The connections in [10] indicate that the problem of finding the eigenstructure of a first order polynomial matrix reduces to the computation of the Kronecker structure of the corresponding pencil.

Generalized Schur decomposition

For an arbitrary pencil $(F-\lambda G)$, there exist *unitary* transformations Q and Z yielding the block triangular decomposition

$$Q^*(F-\lambda G)Z = \left[\begin{array}{c|c|c} F_r - \lambda G_r & * & * \\ \hline O & F_f - \lambda G_f & * \\ \hline O & O & F_c - \lambda G_c \end{array}\right], \tag{2.4}$$

where $F_r - \lambda G_r$ has full row rank for all finite λ, $F_c - \lambda G_c$ has full column rank for all finite λ, and G_f is invertible. This decomposition is proven in [17] where an algorithm is also given to find such a decomposition. Moreover the eigenvalues of $G_f^{-1}F_f$ are the only *finite* points where the rank of $(F - \lambda G)$ drops below its normal value, and hence are the finite zeros of $(F - \lambda G)$.

QZ algorithm and finite zeros

Given a square matrix pencil $(F_f - \lambda G_f)$, with $\det(F_f - \lambda G_f) \neq 0$, there exist unitary matrices Q_f^* and Z_f of appropriate dimension such that $Q_f^* F_f Z_f$ and $Q_f^* G_f Z_f$ are both upper triangular matrices [18]. Let f_{ii} and g_{ii} represent the i-th elements along the diagonals of the upper triangular matrices $Q_f^* F_f Z_f$ and $Q_f^* G_f Z_f$, respectively, then the ratio's f_{ii}/g_{ii} represent the finite eigenvalues (or *finite zeros*) of $(F_f - \lambda G_f)$ and also of $(F - \lambda G)$. Note that the QZ algorithm which performs this triangularization, also works for rank deficient G_f. However, this was not required here because of the preliminary reduction (2.4).

Minimal indices and infinite zeros

In the matrix decomposition (2.4), one may choose Q^* and Z such that the subpencils $(F_r - \lambda G_r)$ and $(F_c - \lambda G_c)$ have the special forms:

$$F_r - \lambda G_r = \begin{bmatrix} F_{1,1}^r & \cdots & F_{1,k}^r - \lambda G_{1,k}^r \\ & \ddots & \vdots \\ & & F_{k,k}^r \end{bmatrix}, \tag{2.5}$$

where the diagonal matrices $F_{i,i}^r$ have full row rank $\hat{\mu}_i$, and the principal super diagonal matrices $G_{i,i+1}^r$ have full column rank $\hat{\tau}_{i+1}$, and

$$F_c - \lambda G_c = \begin{bmatrix} F_{j,j}^c & \cdots & F_{j,1}^c - \lambda G_{j,1}^c \\ & \ddots & \vdots \\ & & F_{1,1}^c \end{bmatrix}, \tag{2.6}$$

where the matrices $F_{i,i}^c$ have full column rank μ_i, and the principal super diagonal matrices $G_{i+1,i}^c$ have full row rank τ_{i+1}. It is clear that these rank conditions guarantee the full rank properties of $(F_r - \lambda G_r)$ and $(F_c - \lambda G_c)$ for all finite λ, but in addition to this it was shown in Van Dooren (1979) that the minimal indices and infinite zero structure of $(G - \lambda F)$ can be derived from this as well. We show later how they relate to the index sets $\{\hat{\mu}_i\}$, $\{\hat{\tau}_i\}$, $\{\mu_i\}$ and $\{\tau_i\}$.

3. Characterization of System Zeros

In this section, we define different types of zeros of a singular system described by (E, A, B, C, D) where the $n \times n$ pencil $(A - \lambda E)$ is non-singular (i.e. $\det(A - \lambda E) \neq 0$). In the sequel we will always refer to both *finite and infinite* zeros as defined in SECTION 2 via the Smith-McMillan and Kronecker canonical forms.

Definition 3.1. *The zeros of the pencils $[A - \lambda E,\ B]$ and $\begin{bmatrix} A - \lambda E \\ C \end{bmatrix}$ are called the input and output decoupling zeros of the system (E, A, B, C, D) respectively.*

Definition 3.2. *If the system (E, A, B, C, D) has no output and no input decoupling zeros, then the zeros of the system matrix $S(\lambda) = \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right]$ are called the transmission zeros of the system (E, A, B, C, D).*

We will be making extensive use of transformations of the kind,

$$\left[\begin{array}{c|c} U & O \\ \hline O & W \end{array}\right] \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} V & O \\ \hline O & Z \end{array}\right] = \left[\begin{array}{c|c} U(A-\lambda E)V & UBZ \\ \hline WCV & WDZ \end{array}\right] \tag{3.1}$$

which we will refer to as *generalized state space transformations*. The matrices U, V, W and Z are invertible [14].

Theorem 3.1. *The zeros of the generalized state space system (1.3) are invariant under generalized state space transformations.*

One special case of such transformations consists in choosing U and V such that the matrix UEV has the following form

$$UEV = \begin{bmatrix} E_{11} & O \\ O & O \end{bmatrix} \tag{3.2}$$

where E_{11} is invertible. This can be achieved with the singular value decomposition. Transforming and partitioning the matrices A, B and C conformably yields a new generalized state space system with special properties.

Definition 3.3. *Let the system (E, A, B, C, D) have a singular E matrix and let U and V yield a SVD of E. Then from*

$$\left[\begin{array}{c|c} U & O \\ \hline O & I \end{array}\right]\left[\begin{array}{c|c} A-\lambda E & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c|c} V & O \\ \hline O & I \end{array}\right] =: \left[\begin{array}{c:c|c} A_{11}-\lambda E_{11} & A_{12} & B_1 \\ \hdashline A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \quad (3.3)$$

we define a "compressed generalized state space system" with E_{11} of full rank, given by

$$\begin{aligned} \lambda E_{11}\hat{x}(t) &= A_{11}\hat{x}(t) + [\,A_{12} \quad B_1\,]\,\hat{u}(t) \quad (3.4) \\ \hat{y}(t) &= \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix}\hat{x}(t) + \begin{bmatrix} A_{22} & B_2 \\ C_2 & D \end{bmatrix}\hat{u}(t). \end{aligned}$$

This leads to the following theorem [14].

Theorem 3.2. *If the system (E, A, B, C, D) has no input or output decoupling zeros then neither does the compressed system (3.4). Moreover their transmission zeros are then equal.*

Notice that we can derive a compressed system as well for the triples (E, A, B) and (E, A, C), which in both cases will yield 5-tuples if E is singular :

$$\left[\begin{array}{c|cc} A_{11}-\lambda E_{11} & A_{12} & B_1 \\ \hline A_{21} & A_{22} & B_2 \end{array}\right], \quad (3.5)$$

$$\left[\begin{array}{c|c} A_{11}-\lambda E_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ C_1 & C_2 \end{array}\right]. \quad (3.6)$$

Since the zeros of these pencils are not affected by the transformation, we have the following obvious but useful result.

Corollary 3.1. *The zeros of the pencils $[A - \lambda E \quad B]$ and $\left[\begin{smallmatrix} A-\lambda E \\ C \end{smallmatrix}\right]$ (i.e. the input and output decoupling zeros of the system (E, A, B, C, D)) are those of the compressed system matrices (3.5) and (3.6), respectively, given above.*

These results can also be applied to any non-degenerate pencil $(A - \lambda E)$ whence the compressed form will yield a 5-tuple similar to (3.4) as shown below

$$U(A-\lambda E)V =: \left[\begin{array}{c|c} A_{11}-\lambda E_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \quad (3.7)$$

Again, since the zeros of matrix pencils are unaffected by the compressions, we can state the following

Corollary 3.2. *The zeros of the pencil $(A - \lambda E)$ (i.e. the finite poles and the poles at infinity of the system (E, A, B, C, D)) are those of the compressed system matrix given in (3.7).*

4. Transmission Zeros and Structure at ∞

In this section, we develop a deflation technique for the computation of transmission zeros, orders of infinite zeros and left and right minimal indices of singular systems. The procedure uses unitary transformation matrices to obtain matrix pencils $(A_f - \lambda E_f)$ and $(A_\infty - \lambda E_\infty)$, where the generalized eigenvalues of the former are the transmission zeros of the given singular system and the latter contains information about the orders of zeros at infinity and left and right minimal indices. Note that for finite transmission zeros, the pencil is given by

$$\mathcal{S}_{rc}(\lambda) = \left[\begin{array}{c|c} A_{rc}-\lambda E_{rc} & B_{rc} \\ \hline C_{rc} & D_{rc} \end{array}\right] \quad (4.1)$$

where E_{rc} and D_{rc} are square invertible matrices. Once the reduced order system matrix (4.1) is obtained, the transmission zeros of the system can be computed as the *generalized eigenvalues* of the pencil $(A_{rc} - B_{rc}D_{rc}^{-1}C_{rc} - \lambda E_{rc})$ using the QZ algorithm [19]. It will be shown later that the generalized eigenvalues can be obtained without explicitly forming the inverse of D_{rc}.

In principle, the reduction procedure corresponds to transforming the variables $x(t)$, $u(t)$ and $y(t)$ to $\hat{x}(t) = V^*x(t)$, $\hat{u}(t) = Z^*u(t)$, $\hat{y}(t) = Wy(t)$, premultiplication of the state equation with the matrix U and deflation. The four matrices U, V, W and Z are chosen to be *unitary* and are constructed recursively as described in the rest of this section.

Before starting the reduction procedure, we first transform the system to its compressed coordinates. To achieve this, we compute unitary matrices U and V as in (3.3)-(3.4). This transformation performs a rank revealing factorization on the descriptor matrix E such that now E_{11} has full rank r and is upper triangular. Next, we partition the matrices UAV, UB and CV conformably to UEV in (3.2) and redefine the system matrix as

$$\hat{\mathcal{S}}(\lambda) = \left[\begin{array}{c|c} \hat{A}-\lambda\hat{E} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] \quad (4.2)$$

where, $\hat{E} := E_{11}$, $\hat{A} := A_{11}$, $\hat{B} := [A_{12} \quad B_1]$, $\hat{C} := \left[\begin{smallmatrix} A_{21} \\ C_1 \end{smallmatrix}\right]$ and $\hat{D} := \left[\begin{smallmatrix} A_{22} & B_2 \\ C_2 & D \end{smallmatrix}\right]$. Clearly, $\hat{E}$ is now an *invertible upper triangular* matrix and by premultiplication of state equation with $\hat{E}^{-1}$ leads to an r-th order standard state space system, whose transmission zeros can be easily determined. However, conversion to a standard state space system to determine transmission zeros should be avoided for reasons of numerical stability. Instead, the recursive deflation technique described in the rest of this section may be used.

For notational convenience, it is assumed that the descriptor matrix is already a full rank upper triangular matrix i.e., $(E, A, B, C, D) := (\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{D})$, where the latter is defined as in (4.2). Further, let $m := n - r + m$, $p := n - r + p$ and $n := r$.

4.1. Structure at ∞ and Row Minimal Indices

The i-th iteration performs the following operations on the system matrix: The rows of D are compressed by computing a unitary matrix $W^{(i)}$ such that $W^{(i)}D$ is row compressed i.e.,

$$W^{(i)}D = \begin{bmatrix} D_1 \\ O \end{bmatrix}, \quad D_1 \in \mathbb{F}^{(p-\tau_i)\times m}, \quad O \in \mathbb{F}^{\tau_i\times m}. \quad (4.3)$$

Next, the matrix $W^{(i)}C$ is partitioned into C_1 and C_2, where the number of rows in C_1 is the same as that in D_1,

$$W^{(i)}C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad C_1 \in \mathbb{F}^{(p-\tau_i)\times n}, \quad C_2 \in \mathbb{F}^{\tau_i\times n} \quad (4.4)$$

and a unitary $V^{(i)}$ is determined such that $C_2V^{(i)}$ is column compressed:

$$C_2V^{(i)} = [\,O \quad C_{22}\,], \quad C_{22} \in \mathbb{F}^{\tau_i\times\mu_i}. \quad (4.5)$$

Note that the operation in (4.5) will destroy the diagonal or upper triangular structure of E, therefore it is necessary to perform a column compression and at the same time maintain the triangular structure of E. This is done by simultaneously determining $U^{(i)}$ such that $U^{(i)}EV^{(i)}$ is upper triangular. Details for achieving this can be found in in [14].

Next, partition the descriptor E matrix as:

$$U^{(i)}EV^{(i)} = \left[\begin{array}{c|c} E_{11} & E_{12} \\ \hline O & E_{22} \end{array}\right], \qquad E_{11} \in \mathbb{F}^{(n-\mu_i)\times(n-\mu_i)}, \tag{4.6}$$

where $E_{22} \in \mathbb{F}^{\mu_i \times \mu_i}$ and perform the strict system equivalence transformation on the system and partition it as shown below

$$\left[\begin{array}{c|c} U^{(i)} & \\ \hline & W^{(i)} \end{array}\right] \left[\begin{array}{c|c} A-\lambda E & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} V^{(i)} & \\ \hline & I \end{array}\right] =$$

$$\left[\begin{array}{c:c|c} A_{11}-\lambda E_{11} & A_{12}-\lambda E_{12} & B_1 \\ \hdashline A_{21} & A_{22}-\lambda E_{22} & B_2 \\ \hline C_{11} & C_{12} & D_1 \\ \hdashline O & C_{22} & O \end{array}\right] \tag{4.7}$$

where the submatrices have appropriate dimensions and the submatrix C_{22} has full column rank.

The recursion can be performed on the reduced order subsystem (E, A, B, C, D) defined as:

$$\left[\begin{array}{c|c} A-\lambda E & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{c|c} A_{11}-\lambda E_{11} & B_1 \\ A_{21} & B_2 \\ C_{11} & D_1 \end{array}\right] \tag{4.8}$$

where $A,\ E \in \mathbb{F}^{(n-\mu_i)\times(n-\mu_i)}$, $B \in \mathbb{F}^{(n-\mu_i)\times m}$, $C \in \mathbb{F}^{(p-\tau_i+\mu_i)\times(n-\mu_i)}$ and $D \in \mathbb{F}^{(p-\tau_i+\mu_i)\times m}$. Further, for notational convenience, we define $A^c_{i,i} = C_{22}$. Since, $\mu_i \le \tau_i$, therefore, $p - (\tau_i - \mu_i) \le p$ and $n - \mu_i \le n$, i.e., the dimension of state as well as output vectors in (4.8) are less than or equal to the corresponding dimensions in the system from previous recursion.

Note that the new descriptor matrix E is a full rank upper triangular matrix. The reduction can therefore be repeated until a full row rank D matrix (i.e., $\tau_i = 0$) or a zero rank C_2 matrix (i.e., $\mu_i = 0$) is encountered. As long as this is not the case, the state dimension n is decreased to $n := n - \mu_i$ at each step and the number of outputs to $p := p - (\tau_i - \mu_i)$.

Once a full row rank matrix $D := D_r$ is found, at step $(j+1)$, the transformed system (up to a column permutation) has the following structure:

$$\left[\begin{array}{c|c} U & \\ \hline & W \end{array}\right] \left[\begin{array}{c|c} B & A-\lambda E \\ \hline D & C \end{array}\right] \left[\begin{array}{c|c} I_p & \\ \hline & V \end{array}\right] = P_r(\lambda) \tag{4.9}$$

where

$$\begin{aligned} P_r(\lambda) &= \left[\begin{array}{c:c|c} B_r & A_r - \lambda E_r & X \\ \hdashline D_r & C_r & \\ \hline & & A^c - \lambda E^c \end{array}\right] \\ &= \left[\begin{array}{c|c} A_1 - \lambda E_1 & X \\ \hline O & A_2 - \lambda E_2 \end{array}\right], \end{aligned} \tag{4.10}$$

and $[A^c - \lambda E^c]$ has the following structure

$$\begin{bmatrix} A^c_{j,j} & A^c_{j,j-1} - \lambda E^c_{j,j-1} & \cdots & A^c_{j,1} - \lambda E^c_{j,1} \\ & A^c_{j-1,j-1} & \cdots & A^c_{j-1,1} - \lambda E^c_{j-1,1} \\ & & \ddots & \vdots \\ & & & A^c_{1,1} \end{bmatrix}.$$

D_r has full row rank ($\tau_{j+1} = 0$), $E^c_{i+1,i}$ has full row rank τ_{i+1} and $A^c_{i,i}$ has full column rank μ_i.

The pencil $(A_1 - \lambda E_1)$ contains the finite transmission zeros and the information on the right nullspace of the corresponding system matrix. The pencil $(A_2 - \lambda E_2)$ contains the information on the orders of infinite zeros and left nullspace of the system matrix. This result is essentially the same as proven by Svaricek [16] for standard state space systems. The only difference resides in the matrix E which is invertible, because of the use of a compressed state space system. Therefore the same reasoning as in Svaricek's paper applies here as well and we quote the next result from [17] without proof.

Lemma 4.1. *From the structure of the pencil $(A_2 - \lambda E_2)$, we can state that*

1. *There are $d_i = \mu_i - \tau_{i+1}$ infinite elementary divisors of degree i, $(i = 1, \ldots, j)$*
2. *There are $r_i = \tau_i - \mu_i$ Kronecker row indices of size $(i-1)$, $(i = 1, \ldots, j)$*

Which leads to the following theorem [14] and [16].

Theorem 4.1. *The orders of the infinite elementary divisors of $(A_2 - \lambda E_2)$ are equal to the orders of infinite zeros of the system (E, A, B, C, D).*

4.2. Finite Zeros and Column Minimal Indices

After a full row rank D_r matrix is found, the deflation procedure defined by (4.3)- (4.8) is repeated on the pertransposed (i.e. transposed over the anti-diagonal) system $\left[\begin{array}{c|c} C^P_r & A^P_r - \lambda E^P_r \\ \hline D^P_r & B^P_r \end{array}\right]$ until an invertible input output matrix D_{rc} is found. The resulting transformed pencil becomes

$$\left[\begin{array}{c|c} U & \\ \hline & W \end{array}\right] \left[\begin{array}{c|c} C^P_r & A^P_r - \lambda E^P_r \\ \hline D^P_r & B^P_r \end{array}\right] \left[\begin{array}{c|c} I_m & \\ \hline & V \end{array}\right] = P_{rc}(\lambda) \tag{4.11}$$

where

$$\begin{aligned} P_{rc}(\lambda) &= \left[\begin{array}{c:c|c} B_{rc} & A_{rc} - \lambda E_{rc} & X \\ \hdashline D_{rc} & C_{rc} & \\ \hline & O & A^r - \lambda E^r \end{array}\right] \\ &= \left[\begin{array}{c|c} A_1 - \lambda E_1 & X \\ \hline O & A_2 - \lambda E_2 \end{array}\right], \end{aligned} \tag{4.12}$$

and $A_2 - \lambda E_2$ is given by

$$\begin{bmatrix} A^r_{k,k} & A^r_{k,k-1} - \lambda E^r_{k,k-1} & \cdots & A^r_{k,1} - \lambda E^r_{k,1} \\ & A^r_{k-1,k-1} & \cdots & A^r_{k-1,1} - \lambda E^r_{k-1,1} \\ & & \ddots & \vdots \\ & & & A^r_{1,1} \end{bmatrix}$$

D_{rc} is a square invertible matrix, $E^r_{i+1,i}$ has full row rank $(= \hat{\tau}_{i+1})$ and $A^r_{i,i}$ has full column rank $(\hat{\mu}_i)$.

Lemma 4.2. *From the structure of the pencil $(A_2 - \lambda E_2)$ and with $\hat{\tau}_i$, $\hat{\mu}_i$ as defined above, $c_i = \hat{\tau}_i - \hat{\mu}_i$, $i = 1, \ldots, k$ are the Kronecker column indices of size $(i-1)$ for the system.*

The following result provides a numerical way to compute the finite transmission zeros of the system [14].

Theorem 4.2. *The finite transmission zeros of the system are the generalized eigenvalues of the finite structure pencil* $(A_f - \lambda E_f)$ *where* A_f *and* E_f *are defined as:*

$$\left[\begin{array}{c|c} A_f & * \\ \hline O & D_f \end{array}\right] := \left[\begin{array}{c|c} A_{rc} & B_{rc} \\ \hline C_{rc} & D_{rc} \end{array}\right] W$$
$$\left[\begin{array}{c|c} E_f & * \\ \hline O & O \end{array}\right] := \left[\begin{array}{c|c} E_{rc} & O \\ \hline O & O \end{array}\right] W. \qquad (4.13)$$

and W *is unitary.*

5. Concluding Remarks

In this paper, we presented a state space characterization of the transmission zeros of *singular* linear multivariable systems that is analogous to that of standard systems. It was shown that from the given singular system, using unitary coordinate transformations, we can obtain a non-singular subsystem whose transmission zeros are identical to the transmission zeros of the original singular system.

It should, perhaps, be emphasized that an algorithm such as proposed in this paper can be viewed as an (almost) universal analysis tool for linear time-invariant systems. Properties such as stability, controllability, observability, stabilizability or detectability, row and column minimal indices of the corresponding system matrix, *etc.* can be easily obtained by computing zeros of appropriate system matrices (for $p = 0$ and/or $m = 0$).

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