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Maximizing the stability radius: An LMI approach

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Abstract

Given a stabilizable linear system $E\dot{x} = Ax + Bu$ with $sE - A$ regular, we analyze the stability robustness of the closed-loop system $(E+BK)\dot{x} = (A+BF)x+v$, obtained by proportional and derivative (PD) state feedback $u = Fx - K\dot{x} + v$. Our goal is to maximize the *stability radius* of the closed-loop system matrix $s(E + BK) - (A + BF)$ over all stabilizing PD state feedback control laws. This problem turns out to be equivalent to a particular H^∞ control problem for a generalized state-space system and reduces to a system of matrix inequalities. Under certain conditions the problem actually reduces to an LMI system. We also show how to apply these ideas to higher order dynamical systems.

1 Introduction

Subsequently the following notations will be adopted. By $\mathbb{C}$ and $\mathbb{C}^{n\times m}$ we denote the complex field and the set of $n \times m$ complex matrices, respectively. Further, $\mathbb{C}^-$ will stand for the open left part of the complex plane, *i.e.* $\{s \in \mathbb{C} \,:\, \mathrm{Re}\, s < 0\}$, while $\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| < 1\}$ denotes the open unit disc.

Let us briefly recall some basic facts concerning stability radius theory. Consider a partition of the complex plane $\mathbb{C}$ into two disjoint sets $\mathbb{C}_g$ and $\mathbb{C}_b$, $\mathbb{C} = \mathbb{C}_g \dot{\cup} \mathbb{C}_b$, such that $\mathbb{C}_g$ is open and non-empty. Let also $E,\ A \in \mathbb{C}^{n\times n}$ such that $\Lambda(\lambda E - A) \subset \mathbb{C}_g$, that is, the pencil $\lambda E - A$ is $\mathbb{C}_g$-stable (or, simply, stable). The two regions that are typically considered for $\mathbb{C}_g$ are $\mathbb{C}^-$ and $\mathbb{D}$. The unstructured *complex stability radius* of the pair (E, A) with respect to $\mathbb{C}_g$ and the perturbation $\Delta := \begin{bmatrix} \Delta_E & \Delta_A \end{bmatrix}$ is

$$r_{\mathbb{C}}(E, A, \mathbb{C}_g;\, \Delta) := \inf_{\Delta \in \mathbb{C}^{n\times n}} \{\|\Delta\|_2 \,:\, \exists\, \lambda \in \mathbb{C}_b \text{ s.t. } \det(\lambda(E + \Delta_E) - (A + \Delta_A)) = 0\}, \quad (1)$$

i.e, $r_{\mathbb{C}}$ is the norm of the smallest perturbation Δ causing at least one eigenvalue of $\lambda(E + \Delta_E) - (A + \Delta_A)$ to leave the "good" region $\mathbb{C}_g$ for $\mathbb{C}_b$. Here $\|\Delta\|_2 := \sigma_1(\Delta)$, where σ_1 denotes the largest singular value of Δ. Notice also that E is nonsingular since $(\lambda E - A)$ is stable. We also restrict to perturbations that do not change the infinite eigenvalues of the pencil, *i.e* $E+\Delta_E$ is nonsingular as well.

Remark 1. If $\lambda E - A$ is stable, *i.e* $\Lambda(\lambda E - A) \subset \mathbb{C}^-$, then

$$r_{\mathbb{C}}(E, A, \mathbb{C}^-;\, \Delta) = \left[\sup_{\omega\in\mathbb{R}} \left\| \begin{bmatrix} j\omega I \\ I \end{bmatrix} (j\omega E - A)^{-1} \right\|_2\right]^{-1} = \left\| \begin{bmatrix} sI \\ I \end{bmatrix} (sE - A)^{-1} \right\|_\infty^{-1}. \quad (2)$$

For more details on stability radii of descriptor and higher order systems see [8], [3].

2 Problem formulation

Consider the generalized continuous-time system

$$E\dot{x} = Ax + Bu, \quad (3)$$

with $A, E \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, such that the pencil $sE - A$ is regular and (E, A, B) is stabilizable, *i.e.* $\mathrm{rank}[A - \lambda E \quad B] = n \ \forall\, \lambda \in \mathbb{C} \setminus \mathbb{C}^-$, λ finite, and $\mathrm{rank}[E \quad B] = n$. Equivalently, there exist $F_0, K_0 \in \mathbb{C}^{m\times n}$ such that the pencil $s(E + BK_0) - (A + BF_0)$ has all its eigenvalues in $\mathbb{C}^-$. Moreover, under these conditions, let $\alpha, \beta \in \mathbb{C}$, not both zero, be such that α/β is not an eigenvalue of $sE - A$ and $\alpha/\beta \notin \mathbb{C}^-$. Then there exist $F \in \mathbb{C}^{m\times n}$ such that $\Lambda(s(E + \beta BF) - (A + \alpha BF)) \subset \mathbb{C}^-$. More details on the generalized eigenvalue assignment problems can be found in [6]. Consider a proportional and derivative (PD) state feedback control law

$$u = Fx - K\dot{x} + v.$$

Then the system (3) becomes

$$E_K\dot{x} = A_F x + Bv, \quad (4)$$

where

$$E_K := E + BK, \quad A_F := A + BF$$

and $\Lambda(sE_K - A_F) \subset \mathbb{C}^-$.

Our goal is to *maximize* the complex *stability radius* of the pair (E_K, A_F) over all PD stabilizing feedback matrices (F, K), subject to the perturbation

$$\Delta := \begin{bmatrix} \Delta_E & \Delta_A & \Delta_B \end{bmatrix} \in \mathbb{C}^{n\times(2n+m)}.$$

In other words, solve

$$\sup_{F,K\in\mathbb{C}^{m\times n}} r_{\mathbb{C}}(E_K, A_F, \mathbb{C}^-; \Delta). \tag{5}$$

The *complex* stability radius of (E_K, A_F) with respect to Δ is defined as

$$\begin{aligned} r_{\mathbb{C}}(E_K, A_F, \mathbb{C}^-; \Delta) &= \inf_{\Delta}\{\|\Delta\|_2 : \exists\,\lambda \in \mathbb{C}\setminus\mathbb{C}^- \\ \text{s.t. } &\det(\lambda(E_\Delta + B_\Delta K) - (A_\Delta + B_\Delta F)) = 0\}, \end{aligned} \tag{6}$$

where $E_\Delta := E + \Delta_E$, $A_\Delta := A + \Delta_A$ and $B_\Delta := B + \Delta_B$.

Next we derive a closed formula for $r_{\mathbb{C}}(E_K, A_F, \mathbb{C}^-; \Delta)$.

Proposition 2. *The complex stability radius of (E_K, A_F) with respect to $\Delta \in \mathbb{C}^{n\times(2n+m)}$ is given by*

$$r_{\mathbb{C}}(E_K, A_F, \mathbb{C}^-; \Delta) = \|(C_F - sG_K)(sE_K - A_F)^{-1}\|_\infty^{-1} \tag{7}$$

where

$$C_F := \begin{bmatrix} I \\ F \end{bmatrix}, \quad G_K := \begin{bmatrix} I \\ K \end{bmatrix}. \tag{8}$$

Proof: Since $sE_K - A_F$ is stable, it follows that E_K is nonsingular. Then the eigenvalues of $sE_K - A_F$ move continuously with the perturbations Δ_E, Δ_A, Δ_B, and the eigenvalue "leaving" $\mathbb{C}^-$ must actually lie on its boundary $\partial\mathbb{C}^-$, *i.e.* on the $j\omega$-axis. Hence

$$r_{\mathbb{C}}(E_K, A_F, \mathbb{C}^-; \Delta) = r_{\mathbb{C}}(E_K, A_F, \partial\mathbb{C}^-; \Delta)$$

$$= \inf_{\lambda\in\partial\mathbb{C}^-}\Big(\inf_{\Delta}\{\|\Delta\|_2 : \det\big(\lambda(E + \Delta_E + (B + \Delta_B)K) - (A + \Delta_A) - (B + \Delta_B)F\big) = 0\}\Big)$$

$$= \inf_{\lambda\in\partial\mathbb{C}^-}\Big(\inf_{\Delta}\{\|\Delta\|_2 : \det(\lambda E_K - A_F - \begin{bmatrix} \Delta_E & \Delta_A & \Delta_B \end{bmatrix}\begin{bmatrix} -\lambda I \\ I \\ -\lambda K + F \end{bmatrix}) = 0\}\Big)$$

$$\overset{(\lambda=j\omega)}{=} \inf_{\omega\in\mathbb{R}}\Big(\inf_{\Delta}\{\|\Delta\|_2 : \det(I - \Delta\begin{bmatrix} -j\omega I \\ I \\ -j\omega K + F \end{bmatrix} \times (j\omega E_K - A_F)^{-1}) = 0\}\Big). \tag{9}$$

The last equality is due to the invertibility of $j\omega E_K - A_F$, since $\Lambda(sE_K - A_F) \in \mathbb{C}^-$.

For arbitrary $\omega \in \mathbb{R}$ one has

$$\inf_{\Delta}\{\|\Delta\|_2 : \det(I - \Delta\begin{bmatrix} -j\omega I \\ I \\ -j\omega K + F \end{bmatrix} \times (j\omega E_K - A_F)^{-1}) = 0\}$$

$$= \|\begin{bmatrix} -j\omega I \\ I \\ -j\omega K + F \end{bmatrix}(j\omega E_K - A_F)^{-1})\|_2^{-1}$$

$$= \|\begin{bmatrix} (-j\omega + 1)I \\ -j\omega K + F \end{bmatrix}(j\omega E_K - A_F)^{-1})\|_2^{-1},$$

therefore

$$r_{\mathbb{C}}(E_K, A_F, \mathbb{C}^-; \Delta) = \inf_{\omega\in\mathbb{R}}\|\begin{bmatrix} (-j\omega + 1)I \\ -j\omega K + F \end{bmatrix}(j\omega E_K - A_F)^{-1})\|_2^{-1}$$

$$= \Big[\sup_{\omega\in\mathbb{R}}\|\begin{bmatrix} (-j\omega + 1)I \\ -j\omega K + F \end{bmatrix}(j\omega E_K - A_F)^{-1})\|_2\Big]^{-1}$$

$$= \|\begin{bmatrix} (-s + 1)I \\ -sK + F \end{bmatrix}(sE_K - A_F)^{-1})\|_\infty^{-1},$$

and the proof is complete. □

By substituting now $r_{\mathbb{C}}(\cdot)$ from (7) into (5), the problem can be reformulated in the following manner: Given a stabilizable system (E, A, B), solve

$$\inf_{F,K\in\mathbb{C}^{m\times n}}\|\begin{bmatrix} (-s + 1)I \\ -sK + F \end{bmatrix}(sE_K - A_F)^{-1}\|_\infty$$

$$= \inf_{F,K\in\mathbb{C}^{m\times n}}\|(C_F - sG_K)(sE_K - A_F)^{-1}\|_\infty, \tag{10}$$

where C_F and G_K have been defined by (8).

3 Maximizing the stability radius: An LMI approach

Our aim is to reformulate problem (10) as a convex optimization problem, deriving F and G from the solution of some appropriate LMI's. The crucial result used in our development is the LMI version of the Bounded Real Lemma for generalized state-space systems.

Theorem 3. *Consider the first order descriptor system $H(s) = D + (C - sG)(sE - A)^{-1}B$ and let $\gamma > 0$ be given. Then the following two assertions are equivalent:*

1. *$sE - A$ is stable and $\|H(s)\|_\infty < \gamma$.*
2. *There exists $Y > 0$ such that*

$$D(Y) := -\begin{bmatrix} 0 \\ I \end{bmatrix} \gamma^2 \begin{bmatrix} 0 & I \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B^* & D^* \end{bmatrix}$$

$$\begin{bmatrix} A \\ C \end{bmatrix} Y \begin{bmatrix} E^* & G^* \end{bmatrix} + \begin{bmatrix} E \\ G \end{bmatrix} Y \begin{bmatrix} A^* & C^* \end{bmatrix} < 0. \quad (11)$$

3. *There exists a symmetric matrix $Y \geq 0$ such that*

$$R_Y := CYG^* + GYC^* + DD^* - \gamma^2 I < 0$$

and the "generalized" Riccati equation

$$AYE^* + EYA^* + BB^* - (EYC^* + AYG^* + BD^*) \times R_Y^{-1} (CYE^* + GYA^* + DB^*) = 0$$

has a stabilizing solution, i.e the matrix pencil

$$sE - A + (sG - C)(EYC^* + AYG^* + BD^*) R_Y^{-1}$$

is stable.

Remark 4. The Bounded Real Lemma shows that $\|H(s)\|_\infty$ is the *global minimum* of the following linear objective minimization problem:

$$\inf_{\gamma,\, Y=Y^*} \gamma \quad \text{subject to } Y > 0,\ D(Y) < 0. \quad (12)$$

According to Remark 4 and by updating formula (11) to (10) one can express $\|(C_F - sG_K)(sE_K - A_F)^{-1}\|_\infty$ in (10) as the global minimum of

$$\inf_{\gamma>0,\, Y=Y^*} \gamma \quad \text{subject to } Y > 0 \text{ and } D(Y, F, K) < 0. \quad (13)$$

Here

$$D(Y,F,K) := -\begin{bmatrix} 0 & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix} \gamma^2 \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_m \end{bmatrix} +$$

$$\begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \end{bmatrix} + \begin{bmatrix} A_F \\ I_n \\ F \end{bmatrix} Y \begin{bmatrix} E_K^* & I_n & K^* \end{bmatrix} + \begin{bmatrix} E_K \\ I_n \\ K \end{bmatrix} Y \begin{bmatrix} A_F^* & I_n & F^* \end{bmatrix}. \quad (14)$$

Recall that $E_K = E + BK$ and $A_F = A + BF$. By pre and post-multiplying $D(Y,F,K)$ by

$$U := \begin{bmatrix} I & 0 & -B \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{and} \quad U^*,$$

then $D(Y,F,K) < 0$ reads as

$$U\, D(Y,F,K)\, U^* = D_2(B,\gamma) + D_1(Y,F,K) + \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix} (FYK^* + KYF^*) \begin{bmatrix} 0 & 0 & I_m \end{bmatrix} < 0, \quad (15)$$

where

$$D_2(B,\gamma) := -\begin{bmatrix} 0 & -B \\ I_n & 0 \\ 0 & I_m \end{bmatrix} \gamma^2 \begin{bmatrix} 0 & I_n & 0 \\ -B^* & 0 & I_m \end{bmatrix} + \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \end{bmatrix}$$

and

$$\begin{aligned} D_1(Y,F,K) \;:=\; & \begin{bmatrix} A \\ I_n \\ F \end{bmatrix} Y \begin{bmatrix} E^* & I_n & K^* \end{bmatrix} \\ & + \begin{bmatrix} E \\ I_n \\ K \end{bmatrix} Y \begin{bmatrix} A^* & I_n & F^* \end{bmatrix} + \\ & - \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix} (FYK^* + KYF^*) \begin{bmatrix} 0 & 0 & I_m \end{bmatrix}. \end{aligned}$$

Assume that $M := FYF^* + KYK^* < \alpha I_m$ for given $\alpha > 0$. We relax the matrix inequality (15) to

$$D_2(B,\gamma) + D_1(Y,F,K) + \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix} (F+K)Y(F^* + K^*) \begin{bmatrix} 0 & 0 & I_m \end{bmatrix} < 0. \quad (16)$$

Since $Y > 0$, if (Y,F,K) is feasible for (16) then it is also feasible for (15). The above relaxation together with an appropriate change of variables allows us to rewrite the matrix inequality (16) as a LMI.
Replace F and K in (15) by introducing two new variables, $P := FY \in \mathbb{C}^{m\times n}$ and $Q := KY \in \mathbb{C}^{m\times n}$, respectively. One also has $FYK^* = PY^{-1}Q^*$, $KYF^* = QY^{-1}P^*$. With the above considerations in mind, the inequalities (15) and (16) become

$$D_2(B,\gamma) + D_1(Y, PY^{-1}, Q^{-1}) + \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix} (PY^{-1}Q^* + QY^{-1}P^*) \begin{bmatrix} 0 & 0 & I_m \end{bmatrix} < 0. \quad (17)$$

and

$$D_2(B,\gamma) + D_1(Y, PY^{-1}, QY^{-1})$$
$$+\begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix} (P+Q)Y^{-1}(P^*+Q^*) \begin{bmatrix} 0 & 0 & I_m \end{bmatrix} < 0, \tag{18}$$

respectively. Furthermore,

$$M = FYF^* + KYK^* < \alpha I_m \quad \text{is equivalent to}$$

$$\begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} Y^{-1} & 0 \\ 0 & Y^{-1} \end{bmatrix} \begin{bmatrix} P^* \\ Q^* \end{bmatrix} - \alpha I < 0$$

$$\iff \left[\begin{array}{cc|c} -Y & 0 & P \\ 0 & -Y & Q \\ \hline P^* & Q^* & -\alpha I \end{array}\right] < 0. \tag{19}$$

Let

$$D_3(Y,P,Q,\gamma) := D_2(B,\gamma) + D_1(Y, PY^{-1}, QY^{-1}).$$

Then, since $Y > 0$, one also has that (18) rewritten as

$$D_3(Y,P,Q) + \begin{bmatrix} 0 \\ 0 \\ P+Q \end{bmatrix} Y^{-1} \begin{bmatrix} 0 & 0 & P^*+Q^* \end{bmatrix} < 0 \tag{20}$$

is equivalent to

$$\left[\begin{array}{c|c} -Y & 0 \quad 0 \quad P^*+Q^* \\ \hline \begin{array}{c} 0 \\ 0 \\ P+Q \end{array} & D_3(Y,P,Q,\gamma) \end{array}\right] < 0. \tag{21}$$

Problem (10) can be finally reduced to the following (relaxed) optimization problem:

Given a stabilizable triple (E, A, B) and $\alpha > 0$ solve

$$\inf_{\gamma>0,Y,P,Q} \gamma \quad \text{subject to } Y > 0 \text{ and to (19), (21).} \tag{22}$$

If γ_0, $Y > 0$, P and Q are a solution to (22), then

$$F := PY^{-1}, \quad K := QY^{-1}$$

are a solution to the problem (10).

Note. Rather than relaxing the problem as in (16), one can also consider PD feedback control laws where $K = \beta F$, such that $\operatorname{Re}\beta > 0$ (or β real and positive) and $1/\beta$ is not a generalized eigenvalue of $sE - A$. Let $\delta > 0$, $\delta^2 = 2\operatorname{Re}\beta$. Replace now Q in the matrix inequality (17) by βP and obtain directly a linear matrix inequality in Y and P, which is similar to (21)

$$\left[\begin{array}{c|c} -Y & 0 \quad 0 \quad \delta P^* \\ \hline \begin{array}{c} 0 \\ 0 \\ \delta P \end{array} & D_3(Y,P,\beta P,\gamma) \end{array}\right] < 0. \tag{23}$$

These ideas can be applied to more general systems, like higher order dynamical systems. For sake of simplicity, we concentrate on systems of second order, but a similar methodology can be employed for higher order systems.

4 Robust stabilization of second order dynamical systems

Consider the following second order system

$$A_2\ddot{x} + A_1\dot{x} + A_0x = Bu, \tag{24}$$

where $A_i \in \mathbb{C}^{n\times n}$, $i = 0,1,2$, $B \in \mathbb{C}^{n\times m}$, and the state-feedback control law

$$u = -F_2\ddot{x} - F_1\dot{x} - F_0x + v.$$

Then, the closed-loop system is given by

$$(A_2 + BF_2)\ddot{x} + (A_1 + BF_1)\dot{x} + (A_0 + BF_0)x = Bv$$

and its associated characteristic equation is

$$\det P_F(s) := \det\big(P(s) + BF(s)\big) = 0, \tag{25}$$

where

$$P(s) = A_2s^2 + A_1s + A_0, \quad F(s) = F_2s^2 + F_1s + F_0.$$

Assuming that $P_F(s)$ is *stable* (*i.e* all the zeros of (25) lie in $\mathbb{C}^-$), one can now define the stability radius of P_F with respect to $\Delta = \begin{bmatrix} \Delta_{A_2} & \Delta_{A_1} & \Delta_{A_0} & \Delta_B \end{bmatrix}$, as follows:

$$r_{\mathbb{C}}(P_F, \mathbb{C}^-; \Delta) := \inf_{\Delta\in\mathbb{C}^{n\times(3n+m)}} \{\|\Delta\|_2 : \exists\, \lambda \in \overline{\mathbb{C}^+} \text{ s.t. } \det(P_F(\lambda) + \Delta P_F(\lambda)) = 0\}. \tag{26}$$

Here

$$\begin{aligned} \Delta P_F(s) &= \Delta P(s) + \Delta_B F(s) \\ &= \Delta_{A_2}s^2 + \Delta_{A_1}s + \Delta_{A_0} + \Delta_B F(s) \end{aligned}$$

It can be shown that (see [3])

$$r_{\mathbb{C}}(P_F, \mathbb{C}^-; \Delta) = \left\| \begin{bmatrix} d(s)I_n \\ F(s) \end{bmatrix} P_F^{-1}(s) \right\|_\infty, \tag{27}$$

where $d(s) = d_2s^2 + d_1s + d_0$. Moreover, for second order systems, one gets $d_2 = d_0 = 1$, $d_1 = \sqrt{3}$.

We may now formulate the same kind of synthesis problem as that given by (10):
Given a *stabilizable* system (24), maximize the complex stability radius (27) over all F_0, F_1, F_2. Equivalently, solve

$$\inf_{F_0,F_1,F_2\in\mathbb{C}^{m\times n}} \left\| \begin{bmatrix} d(s)I_n \\ F(s) \end{bmatrix} P_F^{-1}(s) \right\|_\infty. \tag{28}$$

It is easy to see that (27) can be immediately extended to linear dynamical systems of order k; accordingly, problem (28) can be also reformulated in an appropriate manner.

Some tedious manipulation yields

$$\begin{bmatrix} d(s)I_n \\ F(s) \end{bmatrix} P_F^{-1}(s) = (\widehat{C} - s\widehat{G})(s\widehat{E} - \widehat{A})^{-1}\widehat{B}, \quad (29)$$

where

$$\widehat{A} = \begin{bmatrix} 0 & I_n \\ -(A_0 + BF_0) & -(A_1 + BF_1) \end{bmatrix},$$

$$\widehat{E} = \begin{bmatrix} I_n & 0 \\ 0 & A_2 + BF_2 \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} 0 \\ I_n \end{bmatrix},$$

$$\widehat{G} = \begin{bmatrix} 0 & -d_2 I_n \\ 0 & -F_2 \end{bmatrix}, \text{ and } \widehat{C} = \begin{bmatrix} d_0 I_n & d_1 I_n \\ F_0 & F_1 \end{bmatrix}.$$

Using the same methodology as before, by updating formula (11) to (28) and by invoking Remark 4, one obtains the equivalent problem

$$\inf_{\gamma>0,\, Z=Z^*} \gamma \text{ subject to } Z > 0 \text{ and } D(Z,F) < 0. \quad (30)$$

Clearly, $D(Z,F)$ is $D(Y)$ in (11) updated to (29) and Z. Let us pre and post-multiply now $D(Z,F)$ by

$$T := \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & B \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \quad \text{and} \quad T^*.$$

Then $D(Z,F) < 0$ is equivalent to

$$D(Z,F_0,F_1,F_2) := \begin{bmatrix} 0 \\ I_n \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & I_n & 0 & 0 \end{bmatrix} +$$

$$- \begin{bmatrix} 0 & 0 \\ 0 & B \\ I_n & 0 \\ 0 & I_m \end{bmatrix} \gamma^2 \begin{bmatrix} 0 & 0 & I_n & 0 \\ 0 & B^* & 0 & I_m \end{bmatrix}$$

$$+ \left[\begin{array}{cc} 0 & I_n \\ -A_0 & -A_1 \\ d_0 I_n & d_1 I_n \\ \hline F_0 & F_1 \end{array}\right] \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix} \times \left[\begin{array}{ccc|c} I_n & 0 & 0 & 0 \\ 0 & A_2^* & -\bar{d}_2 I_n & -F_2^* \end{array}\right]$$

$$+ \left[\begin{array}{cc} I_n & 0 \\ 0 & A_2 \\ 0 & -d_2 I_n \\ \hline 0 & -F_2 \end{array}\right] \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix} \times \left[\begin{array}{ccc|c} 0 & -A_0^* & \bar{d}_0 I_n & F_0^* \\ I_n & -A_1^* & \bar{d}_1 I_n & F_1^* \end{array}\right] < 0. \quad (31)$$

One can recognize now (31) as a "second order" counterpart of inequality (15). While the role of F in (15) is taken over by the block matrix $\begin{bmatrix} F_0 & F_1 \end{bmatrix}$ in (31), K is retrieved in $\begin{bmatrix} 0 & F_2 \end{bmatrix}$. If $F_2 = 0$ (which means that we restrict our analysis to control laws of the form $u = -F_1\dot{x} - F_0 x + v$), then one can follow the methodology proposed in Section 3. If $F_2 \neq 0$, an additional constraint must be imposed on K, and hence on its associated variable Q. This problem is object of further research and will be addressed separately.

5 Final remarks

Several aspects are worthwhile to be emphasized. Maximizing the *complex* stability radius via state feedback is equivalent solving an appropriate state-feedback H^∞ control problem. Such problems have been solved for instance in [1] (LMI approach) or in [4] (Riccati equation approach). A similar development can be employed to treat the discrete-time case as well.

To our knowledge, a problem which is still open is to maximize the *real* stability radius. We also mention that the related numerical aspects concerning the solution of the above mentioned (linear) matrix inequalities are under investigation.

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