

An Algorithm for Computing the Staircase Form of a System Pencil and Related Geometric Aspects

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Abstract

In this paper we propose a new recursive algorithm for computing the staircase form of a matrix pencil, and implicitly its Kronecker structure. The algorithm compares favorably to existing ones in terms of elegance, versatility, and complexity. In particular, the algorithm without any modification yields the structural invariants associated with a generalized state-space system and its system pencil. Two related geometric aspects are also discussed: we show that an appropriate choice of a set of nested spaces related to the pencil leads directly to the staircase form; we extend the notion of deflating subspace to the singular pencil case.

1 Introduction

The paper is organized as follows. In this section we briefly recall several notions related to matrix pencils [6] and show how one can easily retrieve the staircase form of an arbitrary pencil [4] by constructing unitary basis for an appropriate pair of sequences of nested subspaces. Section 2 is dedicated to the extension of the notion of deflating subspace to the singular pencil case. It turns out that the deflating subspace contains as special cases both the reducing subspace [3] and the proper deflating subspace [11], [12] previously introduced in connection with various factorization problems [5] and singular Riccati theory [12], [7], [8]. The new algorithm for computing the staircase form of a pencil – called the *system pencil staircase algorithm* – is presented in Section 3. In particular, it applies to the computation of the invariants associated to a system in generalized state-space form. A brief discussion of its advantages over existing algorithms is also given in Section 3. Finally, we draw some conclusions in Section 4.

Let $A - \lambda E$, with $A, E \in \mathbb{C}^{m \times n}$ be a matrix pencil. If $m = n$ and $\det(A - \lambda E) \neq 0$ the pencil is called regular, otherwise it is called singular. Two matrix

pencils $A - \lambda E$ and $\tilde{A} - \lambda \tilde{E}$ are (strictly) equivalent if there exist two invertible constant matrices Q and Z such that

$$Q(A - \lambda E)Z = \tilde{A} - \lambda \tilde{E}. \quad (1)$$

A pencil is equivalent to a (unique) Kronecker canonical form (KCF), i.e. there exist two matrices Q and Z such that the right-hand term in (1) takes the form

$$\begin{aligned} & \tilde{A} - \lambda \tilde{E} \\ & = \text{diag}\{L_{\epsilon_1}, \dots, L_{\epsilon_{\nu_r}}, L_{\eta_1}^T, \dots, L_{\eta_{\nu_l}}^T, J - \lambda I, I - \lambda M\} \end{aligned} \quad (2)$$

where L_k denotes the bidiagonal $k \times (k + 1)$ pencil

$$\begin{bmatrix} -\lambda & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & -\lambda & 1 \end{bmatrix}. \text{ More specifically,}$$

1. The $\epsilon_i \times \epsilon_{i+1}$ blocks L_{ϵ_i} , $i = 1, \dots, \nu_r$, are the right elementary Kronecker blocks, ν_r is the number of right Kronecker blocks and $\epsilon_i \geq 0$ are called the *right (or column) Kronecker indices*.
2. The $\eta_{j+1} \times \eta_j$ blocks $L_{\eta_j}^T$, $j = 1, \dots, \nu_l$, are the left elementary Kronecker blocks, ν_l is the number of left Kronecker blocks and $\eta_j \geq 0$ are called the *left (or row) Kronecker indices*.
3. The $n_f \times n_f$ matrix J is in the Jordan canonical form and n_f is the number of finite eigenvalues.
4. The $n_\infty \times n_\infty$ matrix M is a block diagonal nilpotent matrix, each block being an elementary Jordan block (consisting of ones placed on the first upper diagonal and zeroes everywhere else), and n_∞ is the number of infinite eigenvalues.

The Kronecker indices ϵ_i and η_j completely characterize the singularity of the pencil. The regular part of the pencil is determined by finite elementary divisors (the elementary Jordan blocks of $\lambda I - J$ which determine the finite spectrum), also called the finite eigenstructure and infinite elementary divisors (the elementary

nilpotent blocks of M), also called the infinite eigenstructure. We denote by $\Lambda(E, A)$ the set of finite and infinite eigenvalues of the pencil $A - \lambda E$ and by $\Lambda(A)$ the set of eigenvalues of a square matrix A . With $n_r := \sum_{i=1}^{\nu_r} \epsilon_i$ and $n_l := \sum_{j=1}^{\nu_l} \eta_j$ we have that the rank of $A - \lambda E$ seen as a polynomial matrix equals $n_r + n_l + n_f + n_\infty \leq \min(m, n)$.

From a numerical viewpoint, the computation of the KCF (2) is untractable [1] and one aims to compute, by using unitary transformations Q and Z , a *quasi-canonical* Kronecker form from where all the relevant structural information contained in the KCF can be retrieved. The main step of an algorithm for computing the Kronecker-like form is to bring the pencil to the so-called staircase form [4]. We show below how one can retrieve the staircase form by using a particular sequence of nested spaces defined in terms of image and preimage of A and E .

For an arbitrary (possibly singular) pencil $A - \lambda E$ consider the following sequence of spaces:

$$\begin{aligned} \mathcal{Z}_0 &= \{0\} & \left\{ \begin{array}{l} \mathcal{Z}_i = E^{-1}Q_{i-1}, \\ \mathcal{Q}_i = AZ_i. \end{array} \right. & (i = 1, \dots) \end{aligned} \quad (3)$$

Here by E^{-1} we denote the preimage of E . These spaces are nested and remain invariant after a finite number of steps, more specifically,

$$\begin{aligned} \{0\} &= \mathcal{Z}_0 \subset \mathcal{Z}_1 \subset \mathcal{Z}_2 \subset \dots \subset \mathcal{Z}_k = \mathcal{Z}_{k+1}, \\ \{0\} &= \mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \dots \subset \mathcal{Q}_k = \mathcal{Q}_{k+1}. \end{aligned} \quad (4)$$

where k is the *smallest* index for which $\dim \mathcal{Z}_k = \dim \mathcal{Z}_{k+1}$. The proof follows by induction. Notice that the first equality $\mathcal{Q}_{k'} = \mathcal{Q}_{k'+1}$ may occur for $k' = k-1$ or for $k' = k$. Define now the index sets

$$\begin{aligned} s_i &:= \dim \mathcal{Q}_i - \dim \mathcal{Q}_{i-1}, & (i = 1, \dots, k) \\ t_i &:= \dim \mathcal{Z}_i - \dim \mathcal{Z}_{i-1}, \end{aligned} \quad (5)$$

and

$$\begin{aligned} s_{k+1} &:= m - \sum_{i=1}^k s_i, \\ t_{k+1} &:= n - \sum_{i=1}^k t_i. \end{aligned} \quad (6)$$

Construct the unitary matrices

$$\begin{aligned} Q &= [Q_1 \mid Q_2 \mid \dots \mid Q_k \mid Q_{k+1}], \\ Z &= [Z_1 \mid Z_2 \mid \dots \mid Z_k \mid Z_{k+1}], \end{aligned} \quad (7)$$

such that

$$\begin{aligned} \mathcal{Q}_i \ominus \mathcal{Q}_{i-1} &= \text{Im } Q_i \\ \mathcal{Z}_i \ominus \mathcal{Z}_{i-1} &= \text{Im } Z_i, \end{aligned} \quad (i = 1, \dots, k), \quad (8)$$

and

$$\begin{aligned} \mathbf{C}^n \ominus \mathcal{Q}_k &= \text{Im } Q_{k+1}, \\ \mathbf{C}^n \ominus \mathcal{Z}_k &= \text{Im } Z_{k+1}. \end{aligned} \quad (9)$$

Proposition 1 (a) *The dimension increments s_i, t_i ($i = 1, \dots, k$) satisfy*

$$t_1 \geq s_1 \geq t_2 \geq s_2 \geq \dots \geq t_k \geq s_k \geq 0. \quad (10)$$

(b) *In the new coordinate system defined by Q and Z the pencil $A - \lambda E$ is in the staircase form*

$$\begin{aligned} Q^*(A - \lambda E)Z &= \begin{bmatrix} A_{\epsilon, \infty} - \lambda E_{\epsilon, \infty} & \times \\ O & A_{f, \eta} - \lambda E_{f, \eta} \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} A_{11} & \dots & A_{1,k} - \lambda E_{1,k} & A_{1,k+1} - \lambda E_{1,k+1} \\ O & \dots & A_{2,k} - \lambda E_{2,k} & A_{2,k+1} - \lambda E_{2,k+1} \\ \vdots & & \vdots & \vdots \\ O & \dots & A_{k,k} & A_{k,k+1} - \lambda E_{k,k+1} \\ \vdots & & \vdots & \vdots \\ O & \dots & O & A_{k+1} - \lambda E_{k+1} \end{bmatrix}}_{\substack{\}s_1 \\ \}s_2 \\ \} \\ \}s_k \\ \}s_{k+1}} \end{aligned} \quad (11)$$

with $A_{k+1} - \lambda E_{k+1} := A_{f, \eta} - \lambda E_{f, \eta}$ and where

1. E_{k+1} has full column rank,
2. the blocks A_{ii} have full row rank s_i , ($i = 1, \dots, k$),
3. the blocks $E_{i-1,i}$ have full column rank t_i , ($i = 2, \dots, k$).

(c) The index sets $\{s_i\}$ and $\{t_i\}$, ($i = 1, \dots, k$), completely determine the column Kronecker indices and infinite elementary divisors of the pencil.

2 Deflating subspaces

We introduce now a novel characterization of *deflating subspace* which extends as well to the singular case. Moreover, it generalizes the notion of invariant subspace of a square matrix. Let

$$\overline{\mathbf{C}} = \mathbf{C}_1 \cup \mathbf{C}_2 \quad (12)$$

be a partition of the closed complex plane in two disjoint sets (however, we admit also the partition $\overline{\mathbf{C}} \cup \emptyset$ or $\emptyset \cup \overline{\mathbf{C}}$).

Definition 2 *A subspace $\mathcal{V} \subset \mathbf{C}^n$ of dimension r is called a right deflating subspace if*

$$EVS = AVT \quad (13)$$

where $V \in \mathbf{C}^{n \times r}$ is any basis matrix for \mathcal{V} , and $S, T \in \mathbf{C}^{r \times r}$ are two appropriate matrices such that the pencil $S - \lambda T$ is regular. The subspace \mathcal{V} is called a \mathbf{C}_1 right deflating subspace if in addition $\Lambda(T, S) \subset \mathbf{C}_1$. Similar definitions hold for left deflating subspaces.

Since all the results for left or right deflating subspaces are similar, we only treat hereafter the case of right deflating subspaces and call them briefly deflating subspaces.

The following Proposition gives a complete characterization of deflating subspaces.

Proposition 3 Let $\Lambda(E, A) = \Lambda_1 \cup \Lambda_2$ be a split of the spectrum of the pencil according to (12), i.e. $\Lambda_1 \subset \mathbf{C}_1$ and $\Lambda_2 \subset \mathbf{C}_2$. Let n_1 and n_2 be the number of elements (multiplicity counted) in Λ_1 and Λ_2 , respectively. Then:

(a) The dimension of a \mathbf{C}_1 deflating subspace satisfies

$$\dim \mathcal{V} \leq n_r + \nu_r + n_1 =: r_{\mathbf{C}_1}. \quad (14)$$

(b) \mathbf{C}_1 deflating subspaces are closed under addition and there exists a unique maximal \mathbf{C}_1 deflating subspace of dimension $r_{\mathbf{C}_1}$.

(c) For a maximal \mathbf{C}_1 deflating subspace $\Lambda(T, S)$ is such that n_1 elements coincide with the elements of Λ_1 and the rest of $\nu_r + n_r$ elements can assume arbitrary values (in \mathbf{C}_1).

Remark 4 (a) Notice the two extreme cases:

(i) $\mathbf{C}_1 = \emptyset$ and $n_1 = 0$,

(ii) $\mathbf{C}_1 = \overline{\mathbf{C}}$ and $n_1 = n_f + n_\infty$.

(b) Let \mathcal{V} be a maximal \mathbf{C}_1 deflating subspaces and define $\mathcal{X} := \mathcal{V}$ and $\mathcal{Y} := E\mathcal{X} + A\mathcal{X}$. One can prove that $(\mathcal{X}, \mathcal{Y})$ is the (unique) pair of reducing subspaces (for reducing subspaces see [3]) that induce the split $\Lambda_1 \cup \Lambda_2$ in the spectrum of $A - \lambda E$. Notice that in the two extreme cases indicated at (a) one gets the minimal and maximal reducing subspace, respectively.

(c) In particular, for $T = I$ and imposing the additional constraint to EV to be of full column rank one retrieves for $\mathcal{V} = \text{Im } V$ the definition of \mathbf{C}_1 proper deflating subspace [11] that plays an instrumental role in the singular Riccati theory [12], [7], [8]. In this case one can easily see that not even the maximal \mathbf{C}_1 proper deflating subspace \mathcal{V}_M is unique, yet EV_M is. In fact one shows (see [12]) that $\mathcal{V}_M \subset \mathcal{X}$ and $EV_M = \mathcal{Y}$, where $(\mathcal{X}, \mathcal{Y})$ is the (unique) pair of reducing subspaces that induce the split $\Lambda_1 \cup \Lambda_2$ in the spectrum of the pencil.

3 The system-pencil staircase algorithm

In this section we describe the new system-pencil staircase algorithm which efficiently reduces an arbitrary pencil to the staircase form (11). Our starting pencil is a system pencil

$$S(\lambda) := A - \lambda E = \left[\begin{array}{c|c} A_{11} & A_{12} - \lambda E_{12} \\ \hline A_{21} & A_{22} \end{array} \right] \begin{array}{l} \} \rho_E \\ \} m - \rho_E \end{array} \quad (15)$$

where E_{12} is square and invertible. However, our algorithm applies as well for a general system pencil (where E_{12} is not invertible but $A_{12} - \lambda E_{12}$ is regular [10]), or even to an arbitrary pencil $A_0 - \lambda E_0$ which is first

brought to the form (15) by a two sided rank revealing decomposition of E_0 , such that the resulting E_{12} is square and invertible. As was proved in [10], there is a one to one correspondence between different structural invariants of a system in generalized state-space form and the Kronecker structure of the system pencil (or some of the subpencils in the system pencil). Therefore, we focus hereafter on constructing the staircase form of a system pencil from where the information about the Kronecker structure can be retrieved.

Below we show how we can efficiently determine unitary left and right transformations such that the staircase form (11) is recursively constructed and at each step the system pencil form (15) is preserved. As was indicated in [10], preserving at each step the form (15) is instrumental for keeping the algorithm complexity to $\mathcal{O}(n^3)$. More precisely, we indicate below how the unitary matrices Q and Z can be efficiently constructed such that (we reuse block names)

$$Q^* S(\lambda) Z = \left[\begin{array}{c|cc} A_{11} & A_{12} - \lambda E_{12} & A_{13} - \lambda E_{13} \\ \hline O & A_{22} & A_{23} - \lambda E_{23} \\ O & A_{32} & A_{33} \end{array} \right] \begin{array}{l} \} \rho_{A_{\bullet 1}} \\ \} m - \rho_{A_{\bullet 1}} \end{array} \quad (16)$$

$\underbrace{\hspace{10em}}_{n - \rho_E} \quad \underbrace{\hspace{10em}}_{\rho_E}$

where A_{11} is row compressed, E_{12} is full column rank, E_{23} is upper triangular and invertible and $\rho_{A_{\bullet 1}} := \text{rank} \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}$ in (15). Notice the new block row partition of the pencil in (16) and the fact that the resulting subpencil in (16) $A_2 - \lambda E_2 := \begin{bmatrix} A_{22} & A_{23} - \lambda E_{23} \\ A_{32} & A_{33} \end{bmatrix}$ exhibits the same structure and the constitutive blocks have the same properties as the starting pencil $S(\lambda)$. The algorithm continues further on the subpencil $A_2 - \lambda E_2$. The transformations Q and Z are constructed such that the first $n - \rho_E$ columns of A in (15) are row compressed while keeping E_{12} upper trapezoidal in an economical manner. This is explained in detail below. The novelty of our staircase algorithm consists in the efficient reduction of a pencil of form (15) to (16) and we shall describe this reduction – called the basic step reduction – only. Notice that at each step i , ($i = 1, \dots, k$) one retrieves a pair of indices s_i, t_i and basis matrices for the spaces \mathcal{Q}_i and \mathcal{Z}_i are implicitly constructed. For example, after the first step has been performed we have from the pencil in (16) that $s_1 = \rho_{A_{\bullet 1}}$ and $t_1 = n - \rho_E$.

Before going into fine details, we state the following lemma that will be used further on.

Lemma 5 Let M be a square matrix partitioned as fol-

lows

$$M = \left[\begin{array}{cc|c} M_{11} & M_{12} & \\ M_{21} & O & \end{array} \right] \begin{array}{l} \} n_2 \\ \} n_1 \end{array} \quad (17)$$

with M_{12} and M_{21} invertible, and let U be an invertible transformation which compresses the first n_1 columns of M . Then

a)

$$\tilde{M} = UM = \left[\begin{array}{cc|c} \tilde{M}_{11} & \tilde{M}_{12} & \\ O & \tilde{M}_{22} & \end{array} \right] \begin{array}{l} \} n_1 \\ \} n_2 \end{array} \quad (18)$$

and \tilde{M}_{11} and \tilde{M}_{22} are invertible, too.

b) If M_{21} and M_{12} are upper triangular U can be efficiently constructed as a sequence of Givens rotations such that the resulting \tilde{M}_{11} and \tilde{M}_{22} are upper triangular. This is described below.

Proof. a) Trivial.

b) We illustrate by means of an example how U can be efficiently constructed such that \tilde{M}_{11} and \tilde{M}_{22} are upper triangular. Let $n_1 = 3$, $n_2 = 4$. Then M takes the form

$$M := \left[\begin{array}{cc|c} M_{11} & M_{12} & \\ M_{21} & O & \end{array} \right] = \left[\begin{array}{ccc|cccc} \times & \times & \times & \mathbf{x} & \times & \times & \times \\ \otimes_4 & \times & \times & & \mathbf{x} & \times & \times \\ \otimes_3 & \times & \times & & & \mathbf{x} & \times \\ \otimes_2 & \times & \times & & & & \mathbf{x} \\ \otimes_1 & \times & \times & & & & \\ & \mathbf{x} & \times & & & & \\ & & \mathbf{x} & & & & \end{array} \right] \quad (19)$$

The bold "x" denotes nonzero entries. Clearly \otimes_1 is nonzero as well. We determine first a sequence of Givens rotations $G_{j+1,j}$ between adjacent rows j and $j+1$ ($j = 4, \dots, 1$) such that elements \otimes_i ($i = 1, \dots, 4$) are successively annihilated in the first column of M . For $Q_b^* := G_{21}G_{32}G_{43}G_{54}$ we get

$$Q_b^*M = \left[\begin{array}{ccc|cccc} \mathbf{x} & \times & \times & \times & \times & \times & \times \\ & \times & \times & \mathbf{x} & \times & \times & \times \\ & \times & \times & & \mathbf{x} & \times & \times \\ & \times & \times & & & \mathbf{x} & \times \\ \hline & \times & \times & & & & \mathbf{x} \\ & \mathbf{x} & \times & & & & \\ & & \mathbf{x} & & & & \end{array} \right] \quad (20)$$

Notice that the non singularity of M guarantees that the bold entries in (20) are nonzero. We proceed similarly with columns $j = 2, 3$, finally obtaining after ac-

cumulating in Q_b^* all Givens rotations

$$Q_b^*M = \left[\begin{array}{ccc|cccc} \mathbf{x} & \times & \times & \times & \times & \times & \times \\ & \mathbf{x} & \times & \times & \times & \times & \times \\ & & \mathbf{x} & \times & \times & \times & \times \\ \hline & & & \mathbf{x} & \times & \times & \times \\ & & & & \mathbf{x} & \times & \times \\ & & & & & \mathbf{x} & \times \end{array} \right] \begin{array}{l} \} n_1 \\ \} n_2 \end{array} \quad (21)$$

We describe now the *basic reduction step*. In order to perform a reduction of pencil (15) to the form (16) three steps are taken.

Step a. We compress by rows A_{21} (using for example a QR algorithm with pivoting) by constructing unitary Q_a and permutation P_a such that

$$Q_a^*A_{21}P_a = \left[\begin{array}{cc} A_{21}^1 & A_{21}^2 \\ O & O \end{array} \right] \rho_{A_{21}} \quad (22)$$

where A_{21}^1 is square, upper triangular and invertible. Defining

$$Q \leftarrow \left[\begin{array}{c} I \\ Q_a \end{array} \right], \quad Z \leftarrow \left[\begin{array}{cc} P_a & \\ & I \end{array} \right]$$

we get at the end of this step

$$A_a - \lambda E_a := Q^*(A - \lambda E)Z = \left[\begin{array}{cc|c} A_{11}^1 & A_{11}^2 & A_{12} - \lambda E_{12} \\ A_{21}^1 & A_{21}^2 & A_{22}^1 \\ O & O & A_{22}^2 \end{array} \right] \begin{array}{l} \} \rho_E \\ \} \rho_{A_{21}} \end{array} \quad (23)$$

where $\left[\begin{array}{c} A_{11}^1 \\ A_{22}^2 \end{array} \right] := Q_a^*A_{22}$ and $\left[\begin{array}{cc} A_{11}^1 & A_{11}^2 \end{array} \right] := A_{11}P_a$ have been adequately partitioned. Notice that at this step the subpencil $A_{12} - \lambda E_{12}$ is not affected.

Step b. We focus now on the subpencil

$$A_s - \lambda E_s := \left[\begin{array}{c|c} A_{11}^1 & A_{12} - \lambda E_{12} \\ A_{21}^1 & A_{22}^2 \end{array} \right] \begin{array}{l} \} \rho_E \\ \} \rho_{A_{21}} \end{array} \quad (24)$$

where A_{21}^1 and E_{12} are square, upper triangular and invertible. We construct a unitary left transformation Q_b as a sequence of row Givens rotations such that the first block column of A_s is row compressed while A_s is preserved in upper trapezoidal form. We obtain (after reusing block names)

$$Q_b^*(A_s - \lambda E_s) = \left[\begin{array}{cc} A_{11}^1 & A_{12} - \lambda E_{12} \\ 0 & A_{22}^2 - \lambda E_{22}^1 \end{array} \right] \begin{array}{l} \} \rho_{A_{21}} \\ \} \rho_E \end{array} \quad (25)$$

where A_{11}^1 and E_{22}^1 are invertible and also upper triangular. Q_b is constructed according to Lemma 5, where we take $M := \left[\begin{array}{c|c} A_{11}^1 & E_{12} \\ A_{21}^1 & 0 \end{array} \right]$.

At the end of this step we obtain (after reusing block names)

$$A_b - \lambda E_b = Q^*(A - \lambda E)$$

$$= \underbrace{\left[\begin{array}{c|c|c} A_{11}^1 & A_{11}^2 & A_{12} - \lambda E_{12} \\ O & A_{21}^2 & A_{22}^1 - \lambda E_{22}^1 \\ O & O & A_{22}^2 \end{array} \right]}_{\rho_{A_{21}}} \left. \begin{array}{l} \} \rho_{A_{21}} \\ \} \rho_E \\ \} m - \rho_E - \rho_{A_{21}} \end{array} \right\} \quad (26)$$

where now A_{11}^1 and E_{22}^1 are upper triangular and invertible.

Step c. We compress by rows A_{21}^2 while keeping E_{22}^1 in upper triangular form. This is done by using an appropriate sequence of row and column Givens rotations. At this step we focus only on transformations of the blocks A_{21}^2 and E_{22}^1 and track therefore the matrix

$$N := [A_{21}^2 \mid E_{22}^1] \quad (27)$$

where E_{22}^1 is invertible and upper triangular. We demonstrate again the algorithm by means of an illustrative example for which N outlines the following structure, i.e.

$$N = \left[\begin{array}{c|ccc} \times & \times & \mathbf{x} & \times & \times & \times \\ \times & \times & & \mathbf{x} & \times & \times \\ \times & \times & & & \mathbf{x} & \times \\ \times & \times & & & & \mathbf{x} \end{array} \right] \quad (28)$$

We first compress by rows the first column by using Givens rotations $G_{i+1,i}$ between adjacent rows $i+1$ and i ($i = 3, \dots, 1$) such that

$$G_{21}G_{32}G_{43}N = \left[\begin{array}{c|ccc} \mathbf{x} & \times & \times & \times & \times \\ & \times & \otimes_3 & \times & \times \\ & \times & & \otimes_2 & \times \\ & \times & & & \otimes_1 \end{array} \right] \quad (29)$$

(a permutation P_c of columns in A_{21}^2 is implicitly assumed in order to increase the reliability of rank decisions).

The elements \otimes_i introduced by the row Givens rotations are now annihilated by using a sequence of column Givens rotations, i.e. we determine $G_{j,j+1}$ ($j = 5, \dots, 3$), to annihilate successively elements \otimes_j ($j = 3, \dots, 1$). We proceed similarly on next columns of A_{21}^2 . If the rank of A_{21}^2 is lower than its number of columns the procedure terminates earlier but remains essentially the same. All row and column Givens rotations are accumulated in Q_c and Z_c , respectively.

Overall, we conclude that A_{21}^2 can be row compressed while keeping E_{22}^1 in upper triangular form in an economical manner. Set

$$Q \leftarrow Q \begin{bmatrix} I & & \\ & Q_c & \\ & & I \end{bmatrix}, \quad Z = \begin{bmatrix} I & & \\ & P_c & \\ & & Z_c \end{bmatrix}.$$

At the end of this last step we get (after reusing block names) the pencil

$$A_c - \lambda E_c := Q^*(A - \lambda B)Z$$

$$= \left[\begin{array}{c|c|c} A_{11}^1 & A_{11}^2 & A_{12}^1 - \lambda E_{12}^1 & A_{12}^2 - \lambda E_{12}^2 \\ O & A_{21}^2 & A_{22}^1 - \lambda E_{22}^1 & A_{22}^2 - \lambda E_{22}^2 \\ O & O & A_{22}^4 & A_{22}^3 - \lambda E_{22}^3 \\ O & O & A_{22}^5 & A_{22}^6 \end{array} \right] \quad (30)$$

where A_{11}^1 and A_{21}^2 are row compressed, E_{22}^1 and E_{22}^3 are square, invertible and upper triangular.

By comparing (30) with (16) it is easy to see that our basic reduction step produces the desired effect and this is done in an efficient way. The operation count for this algorithm shows that its complexity is $\mathcal{O}(n^3)$.

Remark 6 (a) The algorithm stops when at a certain step $j = k+1$ the resulting subpencil to be further reduced has a matrix E of full column rank. In the resulting staircase form (11) the subpencil $A_{k+1} - \lambda E_{k+1}$ then has the form

$$A_{k+1} - \lambda B_{k+1} = \begin{bmatrix} \hat{A}_1 - \lambda \hat{E}_1 \\ \hat{A}_2 \end{bmatrix} \quad (31)$$

where \hat{E}_1 is square, upper triangular, and invertible.

(b) Notice that at each step the relevant subpencils outline a “generalized state–space system representation” form. More specifically, if we denote the partitioned pencil (15)

$$S(\lambda) = \left[\begin{array}{c|c} A_{11} & A_{12} - \lambda E_{12} \\ \hline A_{21} & A_{22} \end{array} \right] := \left[\begin{array}{c|c} -B & A - \lambda E \\ \hline -D & -C \end{array} \right]$$

we get that $S(\lambda)$ is the transmission matrix (or pencil) of a system in generalized state–space representation

$$\begin{cases} \lambda E x = A x + B u \\ y = C x + D u, \end{cases} \quad (32)$$

where moreover E is invertible. Here λ stands for the differential operator or for the unit shift.

(c) If a dual version of the new staircase algorithm is applied to the resulting subpencil (31) one retrieves the row Kronecker structure of the original pencil and at the end of this reduction the resulting subpencil (31) will be square (will have a void matrix \hat{A}_2), regular, with invertible \hat{E}_1 matrix. Notice that for this dual staircase algorithm applied to (31) at each basic reduction step only substep (c) is needed.

Finally, we discuss advantages of our algorithm over existing ones. There are mainly two other methods for performing a reduction step leading to an overall $\mathcal{O}(n^3)$ complexity algorithm for computing the Kronecker–like

form. The first $\mathcal{O}(n^3)$ complexity algorithm, called the echelon staircase algorithm, which computes precisely the staircase form (11) was proposed in [2]. For this algorithm, E is initially compressed to a column echelon form that is preserved during further steps. As noted in [10], the main drawback of this algorithm is the alternative rank decision made at each step on both the intervening E and A matrices, one of which without pivoting leading thus to a potential unreliable algorithm. Moreover, the algorithm in [2] does not preserve a generalized state-space like form at each step making the analysis of the structural invariants of generalized state-space systems somehow more intricate. More recently, an algorithm performing a decomposition from where the structural invariants of a generalized state-space system can be retrieved and which could be applied as well for computing the Kronecker structure of an arbitrary pencil was proposed in [10]. The main difference between this algorithm and ours is that the first one uses at each reduction step a column and a row compression with pivoting of the intervening A matrix while keeping E in upper triangular form. This algorithm preserves also at each reduction step a system pencil like form leading to an overall $\mathcal{O}(n^3)$ algorithm. However, this algorithm lacks symmetry in revealing the structure at infinity (it needs inelegant extra reduction steps for computing the infinite elementary divisors), does not compute exactly the staircase like form (and therefore handles with difficulty the computation of deflating subspaces), and has the same complexity for the further reduction of the subpencil (31) (see also Remark 6 c). Actually, our new staircase algorithm combines the advantages of the two methods in [2] and [10].

4 Conclusions

A new algorithm for computing the staircase form of a pencil, which preserves at each step a system pencil like form, was proposed. The algorithm combines advantages of previously proposed staircase-like algorithms. The concept of deflating subspace of a singular pencil which unifies the notions of reducing and proper deflating subspaces has been introduced. Our characterization of deflating subspaces – and in particular of the reduction subspaces – in terms of associated basis matrices is effective both from a theoretical and numerical viewpoint. The proposed algorithm in conjunction with a pole placement algorithm for systems in generalized state-space form [13] can be used for computing deflating subspaces with specified spectrum [11]. Interpretations of different geometric spaces associated to a generalized state-space system in terms of deflating subspaces of the associated system pencil will be discussed in a future paper.

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