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An improved algorithm for the computation of structural invariants of a system pencil and related geometric aspects

C. Oară [a,*,1], P. Van Dooren [b]

[a] DLR – Oberpfaffenhofen, Institute for Robotics and System Dynamics, D-82230 Wessling, Germany
[b] Department of Mathematical Engineering, Université Catholique de Louvain, CESAME, Batiment Euler, 4–6, avenue Georges Lemaitre, B-1348, Louvain-la-Neuve, Belgium

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Abstract

In this paper we propose a new recursive algorithm for computing the staircase form of a matrix pencil, and implicitly its Kronecker structure. The algorithm compares favorably to existing ones in terms of elegance, versatility, and complexity. In particular, the algorithm without any modification yields the structural invariants associated with a generalized state-space system and its system pencil. Two related geometric aspects are also discussed: we show that an appropriate choice of a set of nested spaces related to the pencil leads directly to the staircase form; we extend the notion of deflating subspace to the singular pencil case. © 1997 Elsevier Science B.V.

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1. Introduction

The paper is organized as follows. In this introductory section we briefly recall several notions related to matrix pencils [4] and show how one can easily retrieve the staircase form of an arbitrary pencil [11] by constructing a unitary basis for an appropriate pair of sequences of nested subspaces. Section 2 is dedicated to the natural extension of the notion of deflating subspace to the singular pencil case. It turns out that the deflating subspace contains as special cases both the reducing subspace [12] and the proper deflating subspace [9, 10] previously introduced in connection with various factorization problems [13] and singular Riccati theory [10, 5, 6]. The new algorithm for computing the staircase form of a pencil – called the *system pencil staircase algorithm* – is presented in Section 3. In particular, it applies to the computation of the invariants associated to a system in generalized state-space form. A brief discussion of its complexity and numerical reliability in comparison with existing algorithms is given in Section 4. In Section 5 we draw some conclusions.

Let $A-\lambda E$, with $A, E \in \mathbb{C}^{m\times n}$ be a matrix pencil. If $m = n$ and $\det(A-\lambda E) \not\equiv 0$ the pencil is called regular, otherwise it is called singular. Two matrix pencils $A-\lambda E$ and $\tilde{A}-\lambda\tilde{E}$ are (strictly) equivalent if there exist two invertible constant matrices Q and Z such that

$$Q(A-\lambda E)Z = \tilde{A}-\lambda\tilde{E}. \tag{1}$$

A pencil is equivalent to a (unique) Kronecker canonical form (KCF), i.e. there exist two matrices Q and Z such that the right-hand term in (1) takes the form

$$\tilde{A}-\lambda\tilde{E} = \operatorname{diag}\{L_{\varepsilon_1},\ldots,L_{\varepsilon_{v_r}},L^{\mathrm{T}}_{\eta_1},\ldots,L^{\mathrm{T}}_{\eta_{v_l}},J-\lambda I, I-\lambda M\}, \tag{2}$$

* Corresponding author E-mail: cris@indinf.pub.ro.

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where L_k denotes the bidiagonal $k \times (k+1)$ pencil

$$\begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}.$$

More specifically, the $\varepsilon_i \times \varepsilon_{i+1}$ blocks L_{ε_i}, $i = 1,\dots,\nu_r$, are the right elementary Kronecker blocks, ν_r is the number of right Kronecker blocks and $\varepsilon_i \geqslant 0$ are called the *right* (*or column*) *Kronecker indices*; the $\eta_{j+1} \times \eta_j$ blocks $L_{\eta_j}^{T}$, $j = 1,\dots,\nu_\ell$, are the left elementary Kronecker blocks, ν_ℓ is the number of left Kronecker blocks and $\eta_j \geqslant 0$ are called the *left* (*or row*) *Kronecker indices*; the $n_f \times n_f$ matrix J is in the Jordan canonical form and n_f is the number of finite eigenvalues; the $n_\infty \times n_\infty$ matrix M is a block diagonal nilpotent matrix, each block being an elementary Jordan block (consisting of ones placed on the first upper diagonal and zeros everywhere else), and n_∞ is the number of infinite eigenvalues. The Kronecker indices ε_i and η_j completely characterize the singularity of the pencil. The regular part of the pencil is determined by finite elementary divisors (the elementary Jordan blocks of $\lambda I - J$ which determine the finite spectrum), also called the finite eigenstructure and infinite elementary divisors (the elementary nilpotent blocks of M), also called the infinite eigenstructure. We denote by $\Lambda(E,A)$ the set of finite and infinite eigenvalues of the pencil $A - \lambda E$ and by $\Lambda(A)$ the set of eigenvalues of a square matrix A. With $n_r := \sum_{i=1}^{\nu_r} \varepsilon_i$ and $n_\ell := \sum_{j=1}^{\nu_\ell} \eta_j$ we have that the rank of $A - \lambda E$ seen as a polynomial matrix equals $n_r + n_\ell + n_f + n_\infty \leqslant \min(m,n)$.

From a numerical viewpoint, the computation of the KCF (1) is untractable [1] and one aims to compute, by using unitary transformations Q and Z, a *quasi*-canonical Kronecker form from which all the relevant structural information contained in the KCF can be retrieved. The main step of an algorithm for computing the Kronecker-like form is to bring the pencil to the so-called staircase form [11]. We show below how one can retrieve the staircase form by using a particular sequence of nested spaces defined in terms of image and preimage of A and E.

For an arbitrary (possibly singular) pencil $A - \lambda E$ consider the following sequence of spaces:

$$\begin{matrix} \mathscr{X}_0 = \{0\} \\ \mathscr{Y}_0 = \{0\} \end{matrix} \quad \begin{cases} \mathscr{X}_i = E^{-1}\mathscr{Y}_{i-1} \\ \mathscr{Y}_i = A\mathscr{X}_i \end{cases} \quad (i = 1,\dots). \tag{3}$$

Here by E^{-1} we denote the preimage of E. We first prove that these spaces are nested and remain invariant after a finite number of steps, more specifically, $\{0\} = \mathscr{X}_0 \subset \mathscr{X}_1 \subset \mathscr{X}_2 \subset \cdots \subset \mathscr{X}_k = \mathscr{X}_{k+1}, \{0\} = \mathscr{Y}_0 \subset \mathscr{Y}_1 \subset \mathscr{Y}_2 \subset \cdots \subset \mathscr{Y}_k = \mathscr{Y}_{k+1}$. The proof is by induction. By definition we have $\mathscr{X}_0 \subset \mathscr{X}_1$ and $\mathscr{Y}_0 \subset \mathscr{Y}_1$. Now, since $\mathscr{X}_2 = E^{-1}\mathscr{Y}_1$ and $\mathscr{X}_1 = E^{-1}\mathscr{Y}_0$ we have $\mathscr{X}_1 \subset \mathscr{X}_2$. Since $\mathscr{Y}_2 = A\mathscr{X}_2$ and $\mathscr{Y}_1 = A\mathscr{X}_1$ we also have $\mathscr{Y}_1 \subset \mathscr{Y}_2$, and so on. This proves the nesting of the $\mathscr{X}_i$ spaces and $\mathscr{Y}_i$ spaces. Let k be the *smallest* index for which $\dim \mathscr{X}_k = \dim \mathscr{X}_{k+1}$ or also $\mathscr{X}_k = \mathscr{X}_{k+1}$. Then, for all $i \geqslant k$, the spaces $\mathscr{X}_i$ and $\mathscr{Y}_i$ are equal. For $i < k$, the $\mathscr{X}_i$ dimensions *strictly increase* and hence k must obviously be a finite number, $k \leqslant n$. Notice that the first equality $\mathscr{Y}_{k'} = \mathscr{Y}_{k'+1}$ may occur for $k' = k - 1$ or for $k' = k$. Define now the index sets $s_i := \dim \mathscr{Y}_i - \dim \mathscr{Y}_{i-1}, t_i := \dim \mathscr{X}_i - \dim \mathscr{X}_{i-1}$ $(i = 1,\dots,k)$ and $s_{k+1} := m - \sum_{i=1}^{k} s_i$, $t_{k+1} := n - \sum_{i=1}^{k} t_i$. Construct the unitary matrices $Q = [Q_1 \mid Q_2 \mid \dots \mid Q_k \mid Q_{k+1}]$, $Z = [Z_1 \mid Z_2 \mid \dots \mid Z_k \mid Z_{k+1}]$ such that $\mathscr{Y}_i \ominus \mathscr{Y}_{i-1} = \operatorname{Im} Q_i, \mathscr{X}_i \ominus \mathscr{X}_{i-1} = \operatorname{Im} Z_i$ $(i = 1,\dots,k)$ and $\mathbb{C}^n \ominus \mathscr{Y}_k = \operatorname{Im} Q_{k+1}$, $\mathbb{C}^n \ominus \mathscr{X}_k = \operatorname{Im} Z_{k+1}$.

Proposition 1. (a) *In the new coordinate system defined by Q and Z the pencil $A - \lambda E$ is in the* staircase form

$$Q^*(A - \lambda E)Z = \begin{bmatrix} A_{\varepsilon,\infty} - \lambda E_{\varepsilon,\infty} & \times \\ O & A_{f,\eta} - \lambda E_{f,\eta} \end{bmatrix}$$

$$= \begin{array}{c} \left[\begin{array}{ccccc} A_{11} & A_{12} - \lambda E_{12} & \cdots & A_{1,k} - \lambda E_{1,k} & A_{1,k+1} - \lambda E_{1,k+1} \\ O & A_{22} & \cdots & A_{2,k} - \lambda E_{2,k} & A_{2,k+1} - \lambda E_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & A_{k,k} & A_{k,k+1} - \lambda E_{k,k+1} \\ O & O & \cdots & O & A_{k+1} - \lambda E_{k+1} \end{array}\right] \begin{array}{l} \}s_1 \\ \}s_2 \\ \\ \}s_k \\ \}s_{k+1} \end{array} \\ \begin{array}{ccccc} \underbrace{\qquad}_{t_1} & \underbrace{\qquad\qquad}_{t_2} & & \underbrace{\qquad\qquad}_{t_k} & \underbrace{\qquad\qquad}_{t_{k+1}} \end{array} \end{array} \tag{4}$$

with $A_{k+1} - \lambda E_{k+1} := A_{f,\eta} - \lambda E_{f,\eta}$ *and where*

(a1) *the blocks* A_{ii} *have full row rank* s_i $(i = 1,\dots,k)$,

(a2) *the blocks* $E_{i-1,i}$ *have full column rank* t_i $(i = 2,\dots,k)$,

(a3) E_{k+1} *has full column rank.*

(b) *The dimension increments* s_i, t_i $(i = 1,\dots,k)$ *satisfy* $t_1 \geqslant s_1 \geqslant t_2 \geqslant s_2 \geqslant \cdots t_k \geqslant s_k \geqslant 0$.

(c) *The index sets* $\{s_i\}$ *and* $\{t_i\}$ $(i = 1,\dots,k)$ *completely determine the column Kronecker indices and infinite elementary divisors of the pencil.*

Proof. (a) To show the block structure of (4) it is enough to prove that

$$Q^*_{(i+1)+} A Z_{i-} = 0 \quad (i = 1,\dots,k) \tag{5}$$

and

$$Q^*_{i+} E Z_{i-} = 0 \quad (i = 1,\dots,k), \tag{6}$$

where $Q_{i+} := [Q_i \dots Q_{k+1}]$ and $Z_{i-} := [Z_1 \dots Z_i]$. Since $\operatorname{Im} Q_{(i+1)+} = \mathscr{Q}_i^{\perp}$, $\operatorname{Im} Q_{i+} = \mathscr{Q}_{i-1}^{\perp}$ we have with (3) that $\operatorname{Im} AZ_{i-} = A\mathscr{X}_i = \mathscr{Q}_i$, $\operatorname{Im} EZ_{i-} = E\mathscr{X}_i = Q_{i-1}$, from which (5) and (6) follow immediately.

We prove now the rank properties (a1) and (a2). Let $m_0 = 0$, $m_i := \sum_{j=1}^{i} s_j$, and $n_i := \sum_{j=1}^{i} t_j$, for $1 \leqslant i \leqslant k$. From (3) we get with (4) that

$$\operatorname{Im} \begin{bmatrix} I_{m_i} \\ O \end{bmatrix} = \operatorname{Im} \tilde{A} \begin{bmatrix} I_{n_i} \\ O \end{bmatrix}$$

for which the $m_i \times n_i$ leading diagonal matrix of $\tilde{A}$ should be of full row rank. By induction $(i = 1,\dots,k)$ we get that each A_{ii} is of full row rank and (a1) is proved. Analogously, for (a2) we get

$$\operatorname{Im} \begin{bmatrix} I_{n_i} \\ O \end{bmatrix} = \tilde{E}^{-1} \operatorname{Im} \begin{bmatrix} I_{m_{i-1}} \\ O \end{bmatrix} = \operatorname{Ker}[\, O \quad I_{m-m_{i-1}} \,]\tilde{E}$$

from which we get that the right bottom $(m - m_{i-1}) \times (n - n_i)$ matrix in $\tilde{\mathscr{B}}$ should be of full column rank. Making successively, $i = k,\dots,1$ we obtain (a2).

Finally, (a3) follows from the fact that the nesting of spaces (3) stops exactly after k steps.

(b) Follows from the rank conditions proved at (a).

(c) Follows from [11] by using the particular properties (a1)–(a3) of the pencil in the new coordinate system. □

Remark 2. (a) It is important to notice from (4) that we have separated in the pencil $A - \lambda E$ those elements for which the coefficient matrix of λ has defective column rank. Defining the sequences of spaces

$$\begin{aligned} \mathscr{Q}_0 &= \mathbb{C}^m \\ \mathscr{X}_0 &= \mathbb{C}^n \end{aligned} \quad \begin{cases} \mathscr{Q}_i = E\mathscr{X}_{i-1} \\ \mathscr{X}_i = A^{-1}\mathscr{Q}_i \end{cases} \quad (i = 1,\dots)$$

one can get analogously a form dual to (4) called the *dual staircase form* from which one retrieves the complete information about the row Kronecker indices and infinite elementary divisors.

(b) Notice that for the matrix case ($A = I$, and E square) the two nested sequences (3) satisfy

$$\mathscr{Q}_i = \mathscr{X}_i = \operatorname{Ker}(E^i) \quad (i = 0,\dots). \tag{7}$$

It is well-known [7, 2] that by constructing a unitary basis for the nested kernels of the powers of E one can compute in a numerically sound way the Jordan structure at a certain (known or computed) eigenvalue of E. Therefore, the nesting (3) can be seen as a generalization to the pencil case of the nesting of the kernels (7).

2. Deflating subspaces

We introduce now a novel characterization of *deflating subspace of a matrix pencil* which covers as well the singular case. Moreover, it generalizes in a natural way the notion of invariant subspace of a square matrix. Let

$$\overline{\mathbb{C}} = \mathbb{C}_1 \cup \mathbb{C}_2 \tag{8}$$

be a partition of the closed complex plane in two disjoint sets (however, we admit also the partition $\overline{\mathbb{C}} \cup \emptyset$ or $\emptyset \cup \overline{\mathbb{C}}$).

Definition 1. A subspace $\mathscr{V} \subset \mathbb{C}^n$ of dimension r is called a right deflating subspace if

$$EVS = AVT \tag{9}$$

where $V \in \mathbb{C}^{n\times r}$ is any basis matrix for $\mathscr{V}$, and $S, T \in \mathbb{C}^{r\times r}$ are two appropriate matrices such that the pencil $S - \lambda T$ is regular. The subspace $\mathscr{V}$ is called a $\mathbb{C}_1$ right deflating subspace if in addition $\Lambda(T, S) \subset \mathbb{C}_1$. Dual definitions hold for left deflating subspaces.

Since all the results for left or right deflating subspaces are similar, we only treat hereafter the case of right deflating subspaces and call them briefly deflating subspaces.

Remark 3. (a) For the matrix case, i.e. $E = I$ and A square, (9) reduces to $VS = AVT$. In this case one can easily prove that since $S-\lambda T$ is regular, T is invertible and (9) can be written as $AV=V\tilde{S}$, where $\tilde{S}:=ST^{-1}$, which is precisely the definition of invariant subspace of a matrix A. In this case we have $\Lambda(S)\subset\Lambda(A)$.

(b) For the regular pencil case, i.e. $\det(A-\lambda E)\not\equiv 0$, one can prove that automatically $\Lambda(T,S)\subset\Lambda(E,A)$.

The following proposition gives a complete characterization of deflating subspaces.

Proposition 4. *Let $\Lambda(E,A)=\Lambda_1\cup\Lambda_2$ be a split of the spectrum of the pencil according to* (8), *i.e. $\Lambda_1\subset\mathbb{C}_1$ and $\Lambda_2\subset\mathbb{C}_2$. Let n_1 and n_2 be the number of elements* (*multiplicity counted*) *in Λ_1 and Λ_2, respectively. Then*:

(a) *The dimension of a $\mathbb{C}_1$ deflating subspace satisfies*

$$\dim \mathscr{V}\leqslant n_r+\nu_r+n_1=:r_{\mathbb{C}_1}. \tag{10}$$

(b) *$\mathbb{C}_1$ deflating subspaces are closed under addition and there exists a unique maximal $\mathbb{C}_1$ deflating subspace of dimension $r_{\mathbb{C}_1}$.*

(c) *For a maximal $\mathbb{C}_1$ deflating subspace $\Lambda(T,S)$ is such that n_1 elements coincide with the elements of Λ_1 and the rest of ν_r+n_r elements can assume arbitrary values* (*in* $\mathbb{C}_1$).

The proof strongly relies on a technical lemma given below.

Lemma 5. *Consider the equality*

$$EVS=AVT, \tag{11}$$

where $S-\lambda T$ is regular.

(a) *If $A-\lambda E$ is regular and $\Lambda(E,A)\cap\Lambda(T,S)=\emptyset$ then* (11) *has only the trivial solution $V=0$.*

(b) *If $A-\lambda E:=S-\lambda T$ then* (11) *has an invertible solution V.*

(c) *Suppose $A-\lambda E$ is right invertible for all $\lambda\in\overline{\mathbb{C}}$, i.e. the Kronecker structure of the pencil $A-\lambda E$ consists only of right elementary Kronecker blocks. Then $S-\lambda T$ can be chosen to be regular with all eigenvalues at one point* (*finite or infinite*) *and such that* (11) *is satisfied for an invertible matrix V.*

Proof. Without restricting generality we assume that both pencils $A-\lambda E$ and $S-\lambda T$ in (11) are in the Kronecker canonical form. This can be achieved by multiplying (11) on the left and right with invertible transformations and an appropriate update of V.

(a) Since $A-\lambda E$ and $S-\lambda T$ have disjoint spectra, not both can have infinite eigenvalues and therefore we assume hereafter that $S-\lambda T$ has no infinite eigenvalues. Then (11) reduces to

$$\begin{bmatrix} I & O \\ O & M \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}J_S=\begin{bmatrix} J_A & O \\ O & I \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{12}$$

where $\Lambda(J_S)\cap\Lambda(J_A)=\emptyset$ and M is nilpotent. From the first equation in (12) we get automatically $V_1=0$ while the second can be written explicitly as

$$MV_2J_S=V_2. \tag{13}$$

From (13) we shall deduce that $V_2=0$. For this purpose remember that M is nilpotent and let $k\in\mathbb{N}$ such that $M^k=0$. Multiplying (13) to the left with M^{k-1}, it follows that $M^{k-1}V_2=0$. Iterating this we end with $MV_2=0$ and (13) gives $V_2=0$. Thus

$$V=\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}=0$$

and the proof of (a) ends.

(b) Since now

$$A-\lambda E:=S-\lambda T=\begin{bmatrix} J_S & O \\ O & I \end{bmatrix}-\lambda\begin{bmatrix} I & O \\ O & M \end{bmatrix}$$

we get that (11) is satisfied for the invertible matrix $V=I$.

(c) We shall deal separately with the cases where $S-\lambda T$ has all the eigenvalues at a finite or at an infinite point.

Infinite eigenvalues: Since $A-\lambda E$ has only right Kronecker blocks it takes the form $A-\lambda E=A_r-\lambda E_r=\operatorname{diag}\{L_{\varepsilon_i}\}_{i=1}^{\nu_r}$ where L_{ε_i} are the right elementary Kronecker blocks. Choose $S-\lambda T:=\operatorname{diag}\{M_i\}_{i=1}^{\nu_r}$ where $M_i:=I_{\varepsilon_i+1}+\lambda J_{\varepsilon_i+1}$ and J_{ε_i+1} is a Jordan block of dimension $(\varepsilon_i+1)\times(\varepsilon_i+1)$. A direct check shows that (11) is satisfied for the invertible matrix $V=\operatorname{diag}\{\hat{I}_{\varepsilon_i+1}\}$ where $\hat{I}_n$ denotes an $n\times n$ matrix with units placed on the antidiagonal and zeros in the rest.

Finite eigenvalues: It can be easily seen that by left and right invertible transformations $A-\lambda E$ can be put in the form

$$A-\lambda E=\begin{bmatrix} \tilde{A}-\lambda I & \tilde{B} \end{bmatrix}$$

where the pair $(\tilde{A},\tilde{B})$ is controllable. Thus, there exists a feedback matrix F such that $\tilde{A}_F:=\tilde{A}+\tilde{B}F$ has all the eigenvalues at a certain prescribed point λ_0.

Therefore, without restricting generality, we may assume that $A - \lambda E$ takes the form

$$A - \lambda E = \left[\tilde{A}_{\mathrm{F}} - \lambda I \ \tilde{B} \right].$$

Take

$$S - \lambda T := \begin{bmatrix} \tilde{A}_{\mathrm{F}} & \tilde{B} \\ O & A_{\mathrm{l}} \end{bmatrix} - \lambda \begin{bmatrix} I & O \\ O & I \end{bmatrix},$$

where A_{l} has all the eigenvalues at λ_0. A direct check shows that (11) is fulfilled for the invertible matrix $V = I$ and the proof ends. □

Proof of Proposition 4. It is easy to see that by appropriate row and column permutations we can bring a pencil which is in the KCF to look like

$$A - \lambda E = \mathrm{diag}\left\{ [A_{\mathrm{r}} - \lambda I_{\mathrm{r}} \ B], A_1 - \lambda E_1, \right. \\ \left. A_2 - \lambda E_2, \begin{bmatrix} A_{\ell} - \lambda I \\ C \end{bmatrix} \right\}, \quad (14)$$

where (A_{r}, B) is a reachable pair, $\Lambda(E_1, A_1) \subset \Lambda_1$, $\Lambda(E_2, A_2) \subset \Lambda_2$, and (C, A_{ℓ}) is an observable pair. Without restricting generality we assume directly that $A - \lambda E$ is in the form (14). From (9), with $V = [V_1^{\mathrm{T}} \ V_2^{\mathrm{T}} \ V_3^{\mathrm{T}} \ V_4^{\mathrm{T}} \ V_5^{\mathrm{T}}]^{\mathrm{T}}$ partitioned accordingly to (14) we get an almost decoupled system of equations in unknowns V_i $(i = 1, \ldots, 5)$, from which using (a) of Lemma 5 and the observability of the pair (C, A_{ℓ}) we get $V_4 = 0$ and $V_5 = 0$ and (a) is proved.

For (b) and (c), we show first the existence of a $\mathbb{C}_1$ deflating subspace $\mathscr{V}_M$ of maximal dimension $r_{\mathbb{C}_1}$ which coincides with

$$\mathrm{Im} \begin{bmatrix} I_{r_{\mathbb{C}_1}} \\ O \end{bmatrix}.$$

This follows directly from (b) and (c) of Lemma 5. It turns out that any $\mathbb{C}_1$ deflating subspace is included in $\mathscr{V}_M$ from where (b) follows. Finally, (c) follows from the construction performed at (b). □

Remark 6. (a) Notice the two extreme cases: (i) $\mathbb{C}_1 = \emptyset$ and $n_1 = 0$, (ii) $\mathbb{C}_1 = \overline{\mathbb{C}}$ and $n_1 = n_{\mathrm{f}} + n_{\infty}$.

(b) Let $\mathscr{V}$ be a maximal $\mathbb{C}_1$ deflating subspaces and define $\mathscr{X} := \mathscr{V}$ and $\mathscr{Y} := E\mathscr{X} + A\mathscr{X}$. One can prove that $(\mathscr{X}, \mathscr{Y})$ is the (unique) pair of reducing subspaces (for reducing subspaces see [12]) that induce the split $\Lambda_1 \cup \Lambda_2$ in the spectrum of $A - \lambda E$. Notice that in the two extreme cases indicated at (a) one gets the minimal and maximal reducing subspace, respectively.

(c) In particular, for $T = I$ and imposing the additional constraint to EV to be of full column rank one retrieves for $\mathscr{V} = \mathrm{Im}\, V$ the definition of $\mathbb{C}_1$ proper deflating subspace [9] that plays an instrumental role in the singular Riccati theory [10, 5, 6]. In this case one can easily see that not even the maximal $\mathbb{C}_1$ proper deflating subspace $\mathscr{V}_M$ is unique, yet $E\mathscr{V}_M$ is. In fact, one shows (see [10]) that $\mathscr{V}_M \subset \mathscr{X}$ and $E\mathscr{V}_M = \mathscr{Y}$, where $(\mathscr{X}, \mathscr{Y})$ is the (unique) pair of reducing subspaces that induce split $\Lambda_1 \cup \Lambda_2$ in the spectrum of the pencil.

(d) Deflating subspaces defined for a non-disjoint split of the closed plane can be studied in a similar way, and contain as a particular case reducing subspaces which are non-unique and split the spectrum of the pencil in two non-disjoint sets.

3. The system-pencil staircase algorithm

In this section we describe the new system-pencil staircase algorithm which efficiently reduces an arbitrary pencil to the staircase form (4). Our starting pencil is a system pencil

$$S(\lambda) := A - \lambda E = \left[\begin{array}{c|c} A_{11} & A_{12} - \lambda E_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{array}{l} \}\rho_E \\ \}m - \rho_E \end{array} \quad (15)$$

where E_{12} is square and invertible. However, our algorithm applies as well for a general system pencil (where E_{12} is not invertible but $A_{12} - \lambda E_{12}$ is regular [8]), or even to an arbitrary pencil $A_0 - \lambda E_0$ which is first brought to the form (15) by a two-sided rank revealing decomposition of E_0, such that the resulting E_{12} is square and invertible. As was proved in [8], there is a one-to-one correspondence between different structural invariants of a system in generalized state-space form and the Kronecker structure of the system pencil (or some of the subpencils in the system pencil). Therefore, we focus hereafter on constructing the staircase form of a system pencil from which the information about the Kronecker structure can be retrieved.

Below we show how we can efficiently determine unitary left and right transformations such that the staircase form (4) is recursively constructed and at each step the system pencil form (15) is preserved. As was indicated in [8], preserving at each step the form (15) is instrumental for keeping the algorithm complexity to $\mathcal{O}(n^3)$. More precisely, we indicate below how the unitary matrices Q and Z can be efficiently constructed such that (we reuse block names)

$$Q^*S(\lambda)Z$$

$$= \begin{array}{c} \left[\begin{array}{c|cc} A_{11} & A_{12}-\lambda E_{12} & A_{13}-\lambda E_{13} \\ \hline O & A_{22} & A_{23}-\lambda E_{23} \\ O & A_{32} & A_{33} \end{array}\right] \begin{array}{l} \}\rho_{A_{\bullet 1}} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} m-\rho_{A_{\bullet 1}} \end{array} \\ \begin{array}{cc} \underbrace{\qquad}_{n-\rho_E} & \underbrace{\qquad\qquad\qquad\qquad\qquad}_{\rho_E} \end{array} \end{array}, \tag{16}$$

where A_{11} is row compressed, E_{12} is full column rank, E_{23} is upper triangular and invertible and

$$\rho_{A_{\bullet 1}} := \operatorname{rank}\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}$$

in (15). Notice the new block row partition of the pencil in (16) and the fact that the resulting subpencil in (16),

$$A_2 - \lambda E_2 := \begin{bmatrix} A_{22} & A_{23} - \lambda E_{23} \\ A_{32} & A_{33} \end{bmatrix}$$

exhibits the same structure and the constitutive blocks have the same properties as the starting pencil $S(\lambda)$. The algorithm continues further on the subpencil $A_2 - \lambda E_2$. The transformations Q and Z are constructed such that the first $n - \rho_E$ columns of A in (15) are row compressed while keeping E_{12} upper trapezoidal in an economical manner. This is explained in detail below. The novelty of our staircase algorithm consists in the efficient reduction of a pencil of form (15) to (16) and we shall describe this reduction – called the basic step reduction – only. Notice that at each step i $(i = 1,\dots,k)$ one retrieves a pair of indices s_i, t_i and basis matrices for the spaces $\mathscr{Q}_i$ and $\mathscr{X}_i$ are implicitly constructed. For example, after the first step has been performed we have from the pencil in (16) that $s_1 = \rho_{A_{\bullet 1}}$ and $t_1 = n - \rho_E$.

Before going into fine details, we state the following lemma that will be used further on.

Lemma 7. *Let M be a square matrix partitioned as follows*:

$$M = \begin{array}{c} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & O \end{bmatrix} \begin{array}{l} \}n_2 \\ \}n_1 \end{array} \\ \begin{array}{cc} \underbrace{\qquad}_{n_1} & \underbrace{\qquad}_{n_2} \end{array} \end{array}$$

with M_{12} and M_{21} invertible, and let U be an invertible transformation which compresses the first n_1 columns of M. Then

(a)

$$\tilde{M} = UM = \begin{array}{c} \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ O & \tilde{M}_{22} \end{bmatrix} \begin{array}{l} \}n_1 \\ \}n_2 \end{array} \\ \begin{array}{cc} \underbrace{\qquad}_{n_1} & \underbrace{\qquad}_{n_2} \end{array} \end{array}$$

and $\tilde{M}_{11}$ and $\tilde{M}_{22}$ are invertible, too.

(b) *If M_{21} and M_{12} are upper triangular U can be efficiently constructed as a sequence of Givens rotations such that the resulting $\tilde{M}_{11}$ and $\tilde{M}_{22}$ are upper triangular. This is described below.*

Proof. (a) Trivial.

(b) We illustrate by means of an example how U can be efficiently constructed such that $\tilde{M}_{11}$ and $\tilde{M}_{22}$ are upper triangular. Let $n_1 = 3$, $n_2 = 4$. Then M takes the form

$$M := \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & O \end{array}\right]$$

$$= \left[\begin{array}{ccc|cccc} \times & \times & \times & \mathbf{x} & \times & \times & \times \\ \otimes_4 & \times & \times & & \mathbf{x} & \times & \times \\ \otimes_3 & \times & \times & & & \mathbf{x} & \times \\ \otimes_2 & \times & \times & & & & \mathbf{x} \\ \hline \otimes_1 & \times & \times & & & & \\ & \mathbf{x} & \times & & & & \\ & & \mathbf{x} & & & & \end{array}\right]. \tag{17}$$

The bold "x" denotes nonzero entries. Clearly, $\otimes_1$ is nonzero as well. We determine first a sequence of Givens rotations $G_{j+1,j}$ between adjacent rows j and $j+1$ $(j = 4,\dots,1)$ such that elements $\otimes_i$ $(i = 1,\dots,4)$ are successively annihilated in the first column of M. For $Q_b^* := G_{21}G_{32}G_{43}G_{54}$ we get

$$Q_b^*M = \left[\begin{array}{ccc|cccc} \mathbf{x} & \times & \times & \times & \times & \times & \times \\ & \times & \times & \mathbf{x} & \times & \times & \times \\ & \times & \times & & \mathbf{x} & \times & \times \\ & \times & \times & & & \mathbf{x} & \times \\ \hline & \times & \times & & & & \mathbf{x} \\ & \mathbf{x} & \times & & & & \\ & & \mathbf{x} & & & & \end{array}\right]. \tag{18}$$

Notice that the non-singularity of M guarantees that the bold entries in (18) are nonzero. We proceed similarly with columns $j = 2, 3$, finally obtaining after accumulating in Q_b^* all Givens rotations

$$Q_b^* M$$
$$= \left[\begin{array}{ccc|cccc} \mathbf{x} & \times & \times & \times & \times & \times & \times \\ & \mathbf{x} & \times & \times & \times & \times & \times \\ & & \mathbf{x} & \times & \times & \times & \times \\ \hline & & & \mathbf{x} & \times & \times & \times \\ & & & & \mathbf{x} & \times & \times \\ & & & & & \mathbf{x} & \times \\ & & & & & & \mathbf{x} \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}x\\x\\x\end{array}}\right\} n_1 \\ \left.\vphantom{\begin{array}{c}x\\x\\x\\x\end{array}}\right\} n_2 \end{array}.$$

Notice that this is in fact equivalent to updating the QR decomposition of a matrix whose columns have been permuted [3]. □

3.1. The basic reduction step

In order to perform a reduction of the pencil (15) to the form (16) three steps are taken.

Step a. We compress by rows A_{21} (using for example a QR algorithm with column pivoting) by constructing unitary Q_a and permutation P_a such that

$$Q_a^* A_{21} P_a = \begin{bmatrix} A_{21}^1 & A_{21}^2 \\ O & O \end{bmatrix} \begin{array}{l} \}\rho_{A_{21}} \\ \\ \end{array}, \tag{19}$$

where A_{21}^1 is square, upper triangular and invertible. Defining

$$Q \leftarrow \begin{bmatrix} I & \\ & Q_a \end{bmatrix}, \qquad Z \leftarrow \begin{bmatrix} P_a & \\ & I \end{bmatrix}$$

we get at the end of this step

$$A_a - \lambda E_a := Q^*(A - \lambda E)Z$$
$$= \left[\begin{array}{cc|c} A_{11}^1 & A_{11}^2 & A_{12} - \lambda E_{12} \\ \hline A_{21}^1 & A_{21}^2 & A_{22}^1 \\ O & O & A_{22}^2 \end{array}\right] \begin{array}{l} \}\rho_E \\ \}\rho_{A_{21}} \\ \\ \end{array}$$
$$\underbrace{\qquad\qquad}_{\rho_{A_{21}}}$$

where

$$\begin{bmatrix} A_{22}^1 \\ A_{22}^2 \end{bmatrix} := Q_a^* A_{22} \quad \text{and} \quad \begin{bmatrix} A_{11}^1 & A_{11}^2 \end{bmatrix} := A_{11} P_a$$

have been adequately partitioned. Notice that at this step the subpencil $A_{12} - \lambda E_{12}$ is not affected.

Step b. We focus now on the subpencil

$$A_s - \lambda E_s := \left[\begin{array}{c|c} A_{11}^1 & A_{12} - \lambda E_{12} \\ A_{21}^1 & A_{22}^1 \end{array}\right] \begin{array}{l} \}\rho_E \\ \}\rho_{A_{21}} \end{array},$$

where A_{21}^1 and E_{12} are square, upper triangular and invertible. We construct a unitary left transformation Q_b as a sequence of row Givens rotations such that the first block column of A_s is row compressed while A_s is preserved in upper trapezoidal form. We obtain (after reusing block names)

$$Q_b^*(A_s - \lambda E_s) = \begin{bmatrix} A_{11}^1 & A_{12} - \lambda E_{12} \\ O & A_{22}^1 - \lambda E_{22}^1 \end{bmatrix} \begin{array}{l} \}\rho_{A_{21}} \\ \}\rho_E \end{array},$$

where A_{11}^1 and E_{22}^1 are invertible and also upper triangular. Q_b is constructed according to Lemma 7, where we take

$$M := \left[\begin{array}{c|c} A_{11}^1 & E_{12} \\ \hline A_{21}^1 & O \end{array}\right].$$

At the end of this step we obtain (after reusing block names)

$$A_b - \lambda E_b = Q^*(A - \lambda E)$$
$$= \left[\begin{array}{cc|c} A_{11}^1 & A_{11}^2 & A_{12} - \lambda E_{12} \\ \hline O & A_{21}^2 & A_{22}^1 - \lambda E_{22}^1 \\ O & O & A_{22}^2 \end{array}\right] \begin{array}{l} \}\rho_{A_{21}} \\ \}\rho_E \\ \}m - \rho_E - \rho_{A_{21}} \end{array}$$
$$\underbrace{\qquad}_{\rho_{A_{21}}}$$

where now A_{11}^1 and E_{22}^1 are upper triangular and invertible.

Step c. We compress by rows A_{21}^2 while keeping E_{22}^1 in upper triangular form. This is done by using an appropriate sequence of row and column Givens rotations. At this step we focus only on transformations of the blocks A_{21}^2 and E_{22}^1 and track therefore the matrix $N := [A_{21}^2 | E_{22}^1]$ where E_{22}^1 is invertible and upper triangular. We demonstrate again the algorithm by means of an illustrative example for which N outlines the following structure, i.e.

$$N = \left[\begin{array}{cc|cccc} \times & \times & \mathbf{x} & \times & \times & \times \\ \times & \times & & \mathbf{x} & \times & \times \\ \times & \times & & & \mathbf{x} & \times \\ \times & \times & & & & \mathbf{x} \end{array}\right].$$

We first compress by rows the first column by using Givens rotations $G_{i+1,i}$ between adjacent rows $i+1$ and i $(i = 3, \ldots, 1)$ such that

$$G_{21} G_{32} G_{43} N = \left[\begin{array}{cc|cccc} \mathbf{x} & \times & \times & \times & \times & \times \\ & \times & \otimes_3 & \times & \times & \times \\ & \times & & \otimes_2 & \times & \times \\ & \times & & & \otimes_1 & \times \end{array}\right]$$

(a permutation P_c of columns in A_{21}^2 is implicitly assumed in order to increase the reliability of rank decisions).

The elements $\otimes_i$ introduced by the row Givens rotations are now annihilated by using a sequence of column Givens rotations, i.e. we determine $G_{j,j+1}$ ($j = 5,\ldots,3$), to annihilate successively elements $\otimes_j$ ($j = 3,\ldots,1$). We proceed similarly on next columns of A_{21}^2. If the rank of A_{21}^2 is lower than its number of columns the procedure terminates earlier but remains essentially the same. All row and column Givens rotations are accumulated in Q_c and Z_c, respectively.

Overall, we conclude that A_{21}^2 can be row compressed while keeping E_{22}^1 in upper triangular form in an economical manner. Set

$$Q \leftarrow Q\begin{bmatrix} I & & \\ & Q_c & \\ & & I \end{bmatrix}, \qquad Z = \begin{bmatrix} I & & \\ & P_c & \\ & & Z_c \end{bmatrix}.$$

At the end of this last step we get (after reusing block names) the pencil

$$\begin{aligned} A_c - \lambda E_c &:= Q^*(A - \lambda B)Z \\ &= \left[\begin{array}{cc|cc} A_{11}^1 & A_{11}^2 & A_{12}^1 - \lambda E_{12}^1 & A_{12}^2 - \lambda E_{12}^2 \\ O & A_{21}^2 & A_{22}^1 - \lambda E_{22}^1 & A_{22}^2 - \lambda E_{22}^2 \\ \hline O & O & A_{22}^4 & A_{22}^3 - \lambda E_{22}^3 \\ O & O & A_{22}^5 & A_{22}^6 \end{array}\right], \end{aligned} \tag{20}$$

where A_{11}^1 and A_{21}^2 are row compressed, E_{22}^1 and E_{22}^3 are square, invertible and upper triangular.

By comparing (20) with (16) it is easy to see that our basic reduction step produces the desired effect and this is done in an efficient way. The operation count for this algorithm shows that its complexity is $\mathcal{O}(n^3)$. The details are omitted for brevity.

Remark 8. (a) The algorithm stops when at a certain step $j = k + 1$ the resulting subpencil to be further reduced has a matrix E of full column rank. In the resulting staircase form (4) the subpencil $A_{k+1} - \lambda E_{k+1}$ then has the form

$$A_{k+1} - \lambda B_{k+1} = \begin{bmatrix} \hat{A}_1 - \lambda \hat{E}_1 \\ \hat{A}_2 \end{bmatrix}, \tag{21}$$

where $\hat{E}_1$ is square, upper triangular, and invertible.

(b) Notice that at each step the relevant subpencils outline a "generalized state-space system representation" form. More specifically, if we denote the partitioned pencil (15),

$$S(\lambda) = \left[\begin{array}{c|c} A_{11} & A_{12} - \lambda E_{12} \\ \hline A_{21} & A_{22} \end{array}\right] := \left[\begin{array}{c|c} -B & A - \lambda E \\ \hline -D & -C \end{array}\right]$$

we get that $S(\lambda)$ is the transmission matrix (or pencil) of a system in generalized state-space representation

$$\begin{aligned} \lambda Ex &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{22}$$

where moreover E is invertible. Here λ stands for the differential operator or for the unit shift.

(c) If a dual version of the new staircase algorithm is applied to the resulting subpencil (21) one retrieves the row Kronecker structure of the original pencil and at the end of this reduction the resulting subpencil (21) will be square (will have a void matrix $\hat{A}_2$), regular, with invertible $\hat{E}_1$ matrix. Notice that for this dual staircase algorithm applied to (21) at each basic reduction step only substep (c) is needed.

4. Numerical aspects

In this section we discuss aspects of complexity and numerical reliability and compare our algorithm with existing ones.

There are mainly two other methods for performing a reduction step leading to an overall $\mathcal{O}(n^3)$ complexity algorithm for computing the Kronecker-like form. The first $\mathcal{O}(n^3)$ complexity algorithm, called the *echelon staircase algorithm*, which computes precisely the staircase form (4) was proposed in [1]. For this algorithm, E is first compressed to a column echelon form that is preserved during further steps. As noted in [8], the main drawback of this algorithm is the alternative rank decision made at each step on both the intervening E and A matrices, one of which without pivoting leading thus to a potential unreliable algorithm. Moreover, the algorithm in [1] does not preserve a generalized state-space-like form at each step making the analysis of the structural invariants of generalized state-space systems somehow more intricate. More recently, an algorithm performing a decomposition from which the structural invariants of a generalized state-space system can be retrieved and which could be applied as well for computing the

Kronecker structure of an arbitrary pencil was proposed in [8]. The main difference between this algorithm and ours is that the first one uses at each reduction step a column and a row compression with pivoting of the intervening A matrix while keeping E in upper triangular form. This algorithm preserves also at each reduction step a system-pencil-like form leading to an overall $\mathcal{O}(n^3)$ algorithm. However, this algorithm lacks symmetry in revealing the structure at infinity (it needs inelegant extra reduction steps for computing the infinite elementary divisors), does not compute exactly the staircase like form (and therefore handles with difficulty the computation of deflating subspaces), and has the same complexity for the further reduction of the subpencil (21) (see also Remark 8c). Actually, the present new staircase algorithm combines the advantages of the two methods in [1, 8].

It is important to point out here that all three algorithms are in fact backward stable, since they perform only stable (orthogonal) transformations directly on the original pencil (see the detailed analysis in [1]). On the other hand, this does *not* imply that all rank decisions are made in a reliable manner. It is well known that the *singular values* of a matrix are the most reliable measure to determine its rank. However, the subblocks whose ranks have to be determined depend here on transformations (i.e. *singular vectors*) computed in previous steps of the algorithm, and this is inherent to the recursivity of *any* staircase reduction. A small numerical example where the effect is apparent, is given in [2]. When small singular values (or pivots) are obtained in one step, it will affect later blocks and further rank decisions become very delicate.

Nevertheless, it is clear that rank decisions based on singular values are most reliable and therefore in our algorithm one could use the singular value decomposition on the matrices A_{21} and A_{21}^2 of steps (a) and (c), respectively. This would slightly increase the complexity of the overall algorithm but would still keep it $\mathcal{O}(n^3)$.

5. Conclusions

A new algorithm for computing the staircase form of a pencil, which preserves at each step a system-pencil-like form, was proposed. The algorithm combines advantages of previously proposed staircase-like algorithms. The concept of deflating subspace of a singular pencil which unifies the notions of reducing and proper deflating subspaces has been introduced. Our characterization of deflating subspaces – and in particular of the reduction subspaces – in terms of associated basis matrices is effective both from a theoretical and numerical viewpoint. The proposed algorithm in conjunction with a pole placement algorithm for systems in generalized state-space form [14] can be used for computing deflating subspaces with specified spectrum [9]. Interpretations of different geometric spaces associated to a generalized state-space system in terms of deflating subspaces of the associated system pencil will be discussed in a future paper.

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