CONVERGENCE ANALYSIS OF A RICCATI-BASED STABILIZATION METHOD

X. Rao*, **K. Gallivan***, **P. Van Dooren**†

* Florida State University, Tallahassee, FL 32306, U.S.A.
fax: +1-850-644 0058
e-mail: {rao,gallivan}@cs.fsu.edu

† Université catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium
fax : +32-10-472180
e-mail: vdooren@csam.ucl.ac.be

Keywords: stabilization, linear time-invariant system, Riccati difference equation

Abstract

In this paper we discuss the convergence of a stabilization algorithm based on a singular version of the Discrete Riccati Difference Equation. This method is particularly appealing for large scale linear time invariant dynamical systems since one can nicely exploit the sparsity of such systems in order to reduce the complexity of the algorithm.

1 Introduction

In this paper, we focus on the stabilization of a discrete-time system

$$x_{i+1} = Ax_i + Bu_i, \tag{1}$$

where A and B are $n \times n$ and $n \times p$ real matrices which are known, and x_i and u_i are vectors of dimension n and p respectively. The stabilization of the system requires the computation of a $p \times n$ feedback matrix F such that all eigenvalues of $A - BF$ are inside the unit circle and therefore the system defined by replacing A with $A - BF$ is stable. For small and moderate values of n, F can be computed via pole placement or the solution of a matrix equation, e.g., a Riccati or Lyapunov equation. The computational requirements for standard algorithms for these approaches, however, is prohibitive for large values of n. Fortunately, when n is large and $p << n$, the system matrix A and/or input matrix B are typically very sparse. Algorithms for such problems must therefore exploit this structure in order to efficiently compute a stabilizing feedback.

2 Saad's Approach

An important contribution to solving large scale stabilization problems with a few unstable eigenvalues is Saad's projection method [7]. In this algorithm, stabilization or eigenvalue assignment is only imposed on a small invariant subspace that contains the unstable invariant subspace of A. Such an approach is often effective, but it can have convergence difficulties and the need for a basis of the invariant subspace can cause excess space requirements for very large systems.

In Saad's projection method, a left invariant subspace V^T of A (with presumably small dimension), that contains the left unstable invariant subspace of A is computed. In order to exploit the possible sparsity of the matrix A one often chooses to compute the basis directly by a subspace iteration like method. The low-order projected system (V^TAV, V^TB) is then stabilized and the reduced feedback F_v is lifted back to form a stabilizing feedback $F = F_vV^T$ of the original system (A, B). Subspace iteration like methods as proposed by Saad, generate a sequence of approximations to a particular invariant subspace V starting from an initial subspace V_0. The convergence of such methods depends on the separation between the eigenvalues of A_V, the restriction of A to the invariant subspace V^T, and the remaining eigenvalues of A. This is the so-called *gap* of A with respect to V and if it is too small, one should try to compute a larger space instead (see [6]).

In this paper, we discuss an efficient alternative that addresses this convergence difficulty. We also prove that this algorithm converges under very mild conditions and we show that it avoids the need for an explicitly formed basis of the invariant subspace.

3 Discrete Riccati Equation Stabilization

The major results of this paper are based on the discrete-time Riccati equation (DRE) and the discrete-time Riccati difference equation (DRDE)

$$P = A^T(P - PB(R + B^TPB)^{-1}B^TP)A + Q \tag{2}$$

$$P_{i+1} = A^T(P_i - P_iB(R + B^TP_iB)^{-1}B^TP_i)A + Q \tag{3}$$

where R and Q are $p \times p$ and $n \times n$ non-negative matrices and Q is usually decomposed into $L_Q.L_Q^T$. The most general results about the DRE and DRDE convergence are given in [2]. It is shown there that under the condition of stabilizability of (A, B), a stabilizer and non-negative solution P_s of the DRE (2) exists and a stabilizing feedback F can be computed by

$$F \doteq \hat{R}^{-1}B^TP_sA, \quad \hat{R} \doteq (R + B^TP_sB).$$

Whether the solution of DRDE (3) converges to the stabilizing solution of DRE depends on properties of (A^T, L_Q) and the initial condition P_0. We establish in this paper that this algorithm converges to the stabilizing solution under more general conditions than those reported in [2].

4 Basic properties of the DRDE

The Riccati difference equation (3) has several equivalent formulations. First, one can rewrite it as the Schur complement (with respect to the (1,1) block) of the compound matrix

$$M = \begin{bmatrix} R + B^T P_i B & B^T P_i A \\ A^T P_i B & A^T P_i A + Q \end{bmatrix}. \qquad (4)$$

From this one easily derives a factorized form of the algorithm [4]. One needs to assume that the Cholesky factorizations of the positive semi-definite matrices R, Q and P_i are given :

$$R \doteq L_R.L_R^T, \quad Q \doteq L_Q.L_Q^T, \quad P_i \doteq S_i.S_i^T. \qquad (5)$$

Using these one obtains trivially the following non-square factorization of M :

$$M = \begin{bmatrix} L_R & B^T S_i & 0 \\ 0 & A^T S_i & L_Q \end{bmatrix} \cdot \begin{bmatrix} L_R^T & 0 \\ S_i^T B & S_i^T A \\ 0 & L_Q^T \end{bmatrix}. \qquad (6)$$

The so-called square root form of the Riccati difference iteration is then obtained from a lower triangular reduction of the left factor ([4]) :

$$\begin{bmatrix} L_R & B^T S_i & 0 \\ 0 & A^T S_i & L_Q \end{bmatrix} \cdot U_i = \begin{bmatrix} \hat{L}_i & 0 & 0 \\ \hat{K}_i & S_{i+1} & 0 \end{bmatrix} \qquad (7)$$

where U_i is orthogonal. We will assume in this paper that $R > 0$, which implies that $\hat{R}_i \doteq R + BP_iB^T > 0$ as well. As a consequence, we obtain a decomposition of M :

$$M = \begin{bmatrix} \hat{L}_i & 0 \\ \hat{K}_i & S_{i+1} \end{bmatrix} \cdot \begin{bmatrix} \hat{L}_i^T & \hat{K}_i^T \\ 0 & S_{i+1}^T \end{bmatrix}, \qquad (8)$$

from which it follows that the Schur complement with respect to the (1,1) block equals $P_{i+1} = S_{i+1} \cdot S_{i+1}^T$. Notice that this holds even if P_{i+1} is not of full rank.

Another formulation of (3) follows from the underlying two-point boundary value problem [9, 1] :

$$\begin{bmatrix} A & 0 \\ -Q & I_n \end{bmatrix} \begin{bmatrix} X_{i+1} \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} I_n & BR^{-1}B^T \\ 0 & A^T \end{bmatrix} \begin{bmatrix} X_i \\ Y_i \end{bmatrix},$$

where $P_i = Y_i X_i^{-1}$ implies $P_{i+1} = Y_{i+1} X_{i+1}^{-1}$ and vice versa (this implies of course that both X_i and X_{i+1} must be invertible). We rederive this formulation below in a more explicit form.

Lemma 1
If $R > 0$ the DRDE (3) can be rewritten as follows

$$\begin{bmatrix} A & 0 \\ -Q & I_n \end{bmatrix} \begin{bmatrix} I_n \\ P_{i+1} \end{bmatrix} = \begin{bmatrix} I_n & BR^{-1}B^T \\ 0 & A^T \end{bmatrix} \begin{bmatrix} I_n \\ P_i \end{bmatrix} A_{F_i}, \qquad (9)$$

where

$$A_{F_i} \doteq A - B \cdot F_i, \quad F_i \doteq \hat{R}_i^{-1} B^T P_i A, \quad \hat{R}_i \doteq R + B^T P_i B.$$

Proof : We need to show the following two identities

$$A = (I + BR^{-1}B^T P_i) A_{F_i}, \quad P_{i-1} - Q = A^T P_i A_{F_i}.$$

Using the definition of the matrices involved, the second equation becomes

$$\begin{aligned} P_{i+1} &= A^T P_i A - A^T P_i B F_i + Q \\ &= A^T P_i A - A^T P_i B \hat{R}_i^{-1} B^T P_i A + Q, \end{aligned}$$

which is the DRDE. The first equation becomes

$$A = A + BR^{-1}B^T P_i A - BF_i - BR^{-1}B^T P_i B F_i$$

which is equivalent to

$$0 = B[R^{-1}\hat{R}_i - I - R^{-1}B^T P_i B] F_i$$

and is clearly an identity. ■

5 Convergence of the DRDE

If one wants to study the convergence of the DRDE, the above lemma plays a crucial role. It is clear from (9) that the generalized eigenvalue problem

$$\lambda M_1 - M_2 \doteq \lambda \begin{bmatrix} A & 0 \\ -Q & I_n \end{bmatrix} - \begin{bmatrix} I_n & BR^{-1}B^T \\ 0 & A^T \end{bmatrix}$$

determines the convergence of the DRDE. Let us first assume A to be invertible (we will show that this assumption does not affect our results). Iteration (9) is then a subspace iteration with a space of dimension n :

$$\begin{bmatrix} X_{i+1} \\ Y_{i+1} \end{bmatrix} = M_1^{-1} M_2 \begin{bmatrix} X_i \\ Y_i \end{bmatrix}.$$

Let λ_i be an eigenvalue of $M_1^{-1}M_2$ and assume they are ordered by decreasing magnitude $|\lambda_i|$. If $|\lambda_n|$ is strictly larger than $|\lambda_{n+1}|$ then the above recurrence is known to converge for almost all initial conditions X_0, Y_0, to the so-called dominant invariant subspace of $M_1^{-1}M_2$. If, on the other hand, $|\lambda_n| = |\lambda_{n+1}|$ then the iteration almost never converges : there exist fixed points but they correspond to very special initial conditions [3]. It turns out that $M_1^{-1}M_2$ is simplectic and therefore has a special eigenvalue pattern : the eigenvalues which are not on the unit circle come in pairs that are mirror images of each other with respect to the unit circle. Therefore the condition $|\lambda_n| > |\lambda_{n+1}|$ is satisfied iff $M_1^{-1}M_2$ has no eigenvalues on the unit circle. We make this assumption in the rest of the paper. This is a classical assumption in the RDE literature since it is closely linked to the existence of stabilizing solutions of the corresponding feedback problem [2]. We recall in this context the following results proved in [2].

Theorem 2
A stabilizing solution P_s of the DRE exists and is unique if and only if either of the following two conditions is satisfied

1. (A, B) *is stabilizable and* (A^T, Q) *has no unobservable eigenvalues on the unit circle,*

2. (A, B) *is stabilizable and the pencil* $\lambda M_1 - M_2$ *has no generalized eigenvalues on the unit circle.*

The simplectic structure of the pencil implies that all eigenvalues are then mirror images of each other with respect to the unit circle, and the following result then holds ([2, 3]).

Theorem 3
Let the simplectic pencil $\lambda M_1 - M_2$ have no generalized eigenvalues on the unit circle. Then there exist invertible matrices S and T such that

$$\lambda M_1 - M_2 = T \begin{bmatrix} \lambda A_F - I & 0 \\ 0 & \lambda I - A_F^T \end{bmatrix} S,$$

where A_F is stable and depends on the stabilizing solution P_s as follows :

$$A_F \doteq A - B \cdot F, \quad F \doteq \hat{R}^{-1} B^T P_s A, \quad \hat{R} \doteq R + B^T P_s B.$$

Under these conditions, the power method thus converges, provided the initial matrix $\begin{bmatrix} I_n \\ P_0 \end{bmatrix}$ has a "non-degenerate" component in the direction of the invariant subspace $\begin{bmatrix} I_n \\ P_s \end{bmatrix}$. When expressing the initial matrix as a linear combination of both invariant spaces (spanned by the block columns of S^{-1}) :

$$\begin{bmatrix} I_n \\ P_0 \end{bmatrix} = S^{-1} \begin{bmatrix} V \\ W \end{bmatrix},$$

the non-degeneracy implies that V must be invertible. Since

$$V = \begin{bmatrix} I_n & 0_n \end{bmatrix} S \begin{bmatrix} I_n \\ P_0 \end{bmatrix} = S_{11} + S_{12} P_0, \qquad (10)$$

it is easy to see that for a random initial matrix P_0 the matrix V is generically invertible. The DRDE thus almost always converges to the stabilizing solution of the RDE the corresponding simplectic pencil $\lambda M_1 - M_2$ has no unit circle eigenvalues.

Theorem 4
Let the simplectic pencil $\lambda M_1 - M_2$ have no generalized eigenvalues on the unit circle and let the initial matrix P_0 satisfy the non-degeneracy condition rank$(S_{11} + S_{12} P_0) = n$. Then the iterates P_i converge linearly to the stabilizing solution P_s of the RDE :

$$\lim_{i \to \infty} P_i = P_s, \quad \lim_{i \to \infty} \|P_{i+1} - P_s\| / \|P_i - P_s\| = c < 1.$$

We now return to the case where A is singular. If this is the case we consider a perturbed matrix $A_\epsilon \doteq A - \epsilon E$ which has the same eigenvectors as A and the same eigenvalues, except for the zero eigenvalue of A which now gets perturbed to ϵ. The assumptions of Theorem 2 are clearly not affected by this since stabilizability of (A_ϵ, B) and (A, B) are equivalent and (A_ϵ, Q) has no unobservable modes on the unit circle provided ϵ is sufficiently small. The stabilizing solution $P_{S,\epsilon}$ of the corresponding perturbed RDE is then well defined. Moreover

$$\lim_{\epsilon \to 0} P_{S,\epsilon} = P_S$$

since the corresponding invariant subspaces

$$\begin{bmatrix} I_n \\ P_{S,\epsilon} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} I_n \\ P_S \end{bmatrix}$$

are well defined and ϵ close to each other [3]. By continuity arguments, one then sees that the invertibility of A is not needed to prove the convergence of the RDE.

Remark The result of the above Theorem 4 relaxes the assumptions that were needed to prove convergence of the RDE so far. In [2] it is shown that under the assumption of Theorem 2, the RDE converges for any initial condition P_0 which is either positive definite (i.e. $P_0 > 0$), or larger than the stabilizing solution (i.e. $P_0 > P_S$). The economical SQR algorithm described in section 7 requires a *singular* matrix P_0 of rank larger or equal to P_S. Both assumption required in [2] therefore do not hold then. This is why the above theorem is so crucial for the rest of this paper.

6 Convergence to P_s

We already know that the invariant subspace computed at each iteration i converges to the stable invariant subspace we are interested in, but one typically wants to know this in terms of the matrix P_i as well. Although it is normal to expect linear convergence here as well, we analyze this in more detail in this section.

The following simple lemma follows by straightforward error analysis of the inverse of a matrix and can be found in slightly modified form in [8].

Lemma 5 Let A be a square invertible matrix with smallest singular value σ_{min} and let E be a perturbation of norm smaller than this :

$$\|E\|_2 \doteq \delta < \sigma_{min}.$$

Then

$$(A + E)^{-1} = A^{-1} - A^{-1} E A^{-1} + \Delta$$

$$\Delta = (A + E)^{-1} E A^{-1} E A^{-1} = A^{-1} E A^{-1} E (A + E)^{-1},$$

$$\|\Delta\|_2 \approx \|A^{-1} E A^{-1} E A^{-1}\|_2 < \delta^2 / \sigma_{min}^3.$$

■

Defining the convergence error as follows

$$E_i \doteq P_i - P_s,$$

and applying the above lemma to the expressions

$$\hat{R}_i^{-1} = (R + B^T P_i B)^{-1},$$

$$P_{i+1} = A^T[P_i - P_i B \hat{R}_i^{-1} B^T P_i]A + Q,$$

we obtain

$$\hat{R}_i^{-1} = \hat{R}_i^{-1} - \hat{R}_i^{-1} B^T E_i B \hat{R}_i^{-1} + O(\|E_i\|_2^2)$$

and

$$\begin{aligned} P_{i+1} &= A^T[E_i - E_i B \hat{R}_i^{-1} B^T P_i - P_i B \hat{R}_i^{-1} B^T E_i \\ &+ P_i B \hat{R}_i^{-1} B^T E_i B \hat{R}_i^{-1} B^T P_i]A + O(\|E_i\|_2^2), \\ &= (A - BF_i)^T E_i (A - BF_i) + O(\|E_i\|_2^2), \end{aligned}$$

where $F_i \doteq \hat{R}_i^{-1} B^T P_i A$.

Corollary 6 Let A_{F_i} be the closed loop matrix $A + BF_i$ and let the error $E_i \doteq P_i - P_s$ between the i-th iterate of the DRDE and its steady state value P_s be small, then this error converges linearly and is in first order equal to

$$E_{i+1} = A_{F_i}^T E_i A_{F_i} + O(\|E_i\|_2^2).$$

■

Remark The convergence ratio c of Theorem 4 is therefore approximately equal to $\rho(A_F)^2$ (the square of the spectral radius of A_F), since A_{F_i} tends to A_F. Notice that this is smaller than 1 since A_F is the stabilized closed loop matrix.

7 The singular SQR algorithm

The square root algorithm (SQR) of this paper is based on the DRDE with $Q = 0$. In the previous section we showed that the DRDE equation converges under very mild conditions to the stabilizing solution provided the corresponding pencil $\lambda M_1 - M_2$ has no unit circle eigenvalues. For $Q = 0$ this pencil has a spectrum that is the union of the spectrum of A and that of A^{-1} since

$$\lambda M_1 - M_2 = \lambda \begin{bmatrix} A & 0 \\ 0 & I_n \end{bmatrix} - \begin{bmatrix} I_n & BR^{-1}B^T \\ 0 & A^T \end{bmatrix}.$$

Therefore the feedback F generated in the limit moves the unstable eigenvalues of A, λ to their unit circle mirror images, $1/\overline{\lambda}$, and leaves the stable eigenvalues unchanged. As a special case of the square root form of DRDE, the SQR stabilization algorithm (developed in [6]) has the form

$$\begin{bmatrix} L_R & B^T S_i \\ 0 & A^T S_i \end{bmatrix} U_i = \begin{bmatrix} \hat{L}_i & 0 \\ \hat{K}_i & S_{i+1} \end{bmatrix} \qquad (11)$$

where U_i is orthogonal and the dimension of S_i is $n \times l$, the same as S_0. Note that the QR decomposition is computed for a small matrix with size $(p + l) \times p$ (the first row of (7)) and feedback F_i can be computed from $\hat{L}_i$ and $\hat{K}_i$ as follows :

$$F_i = \hat{L}_i^{-T} \hat{K}_i^T.$$

Moreover, if A and B are sparse, the construction of the left factor in the left hand side of (11) is cheap as well (see [6]).

The SQR iteration can produce the same sequence of subspaces as Saad's subspace iteration method with only an additional economical QR decomposition of S_i since the updating of S_i has the form $S_{i+1} = A^T S_i U_i^{22}$. If S_0 is taken to be the same initial subspace basis as used for Saad's method, SQR will converge. Moreover convergence is easier to check as was pointed out in [6].

It is also useful to point out that for $Q = 0$ the DRDE can be rewritten in a very compact manner :

$$P_{i+1} = A^T P_i A_{F_i},$$

or equivalently

$$S_{i+1} S_{i+1}^T = A^T S_i S_i^T A_{F_i}. \qquad (12)$$

In the limit we also have that P_s satisfies the discrete-time Sylvester equation

$$P_s = A^T P_s A_F.$$

8 Comparing Saad's method and SQR

Saad's subspace iteration method essentially performs the QR factorization of $A^T V_i$ where V_i is the previously computed orthogonal basis :

$$A^T V_i = V_{i+1} R_{i+1}. \qquad (13)$$

Comparing this with

$$A^T S_i U_i^{22} = S_{i+1}, \qquad (14)$$

it is obvious that both methods compute the same spaces. Because of (13,14),

$$\mathrm{Im} V_0 = \mathrm{Im} S_0 \Longrightarrow \mathrm{Im} V_i = \mathrm{Im} S_i \quad \forall i,$$

as long as U_i^{22} and R_{i+1} are invertible. Multiplying (12) by the right inverse $(S_{i+1}^T)^+$ of S_{i+1}^T we obtain :

$$U_i^{22} = S_i^T A_{F_i} (S_{i+1}^T)^+.$$

Upon convergence, S_{i+1} and S_i are close to each other, and $U_i^{22} = S_i^T A_{F_i} (S_{i+1}^T)^+$ tends thus to a constant matrix whose eigenvalues are smaller or equal to 1 since $\|U_i^{22}\|_2 \leq 1$. The effect of such a multiplication is to dampen out the components along the smallest eigenvalues of $S^T A_{F_i} (S^T)^+$, and the iterates S_i may converge to a smaller rank matrix. This is actually what happens in practice if S_0 has dimension larger than the number of unstable eigenvalues of A.

In order to analyze this we put ourselves in a special coordinate system, where

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} is unstable and A_{22} is stable.

Theorem 7 *Let A be in the coordinate system described above. Then the solution to the DRE has rank equal to the dimension of the unstable subspace of A. The invariant subspace satisfies*

$$\begin{bmatrix} A_{11} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ P_{11} & 0 \\ 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} I & 0 & W_{11} & W_{12} \\ 0 & I & W_{21} & W_{22} \\ 0 & 0 & A_{11}^T & A_{21}^T \\ 0 & 0 & 0 & A_{22}^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ P_{11} & 0 \\ 0 & 0 \end{bmatrix} A_F,$$

where

$$\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \doteq \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} R^{-1} \begin{bmatrix} B_1^T & B_2^T \end{bmatrix}.$$

The matrices P_s and A_F in this coordinate system are given by

$$P_s \doteq \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix}, A_F \doteq \begin{bmatrix} A_{11} - B_1 \hat{R}^{-1} B_1^T P_{11} A_{11} & 0 \\ A_{21} - B_2 \hat{R}^{-1} B_1^T P_{11} A_{11} & A_{22} \end{bmatrix}.$$

Moreover, P_{11} has rank equal to the number of unstable eigenvalues of A.

Proof : Let P_{11} solve the following RDE of smaller dimension :

$$P_{11} = A_{11}^T (P_{11} - P_{11} B_1 (R + B_1^T P_{11} B_1)^{-1} B_1^T P_{11}) A_{11},$$

then it is easy to see that P_s given above solves the larger RDE, which can e.g. be written as

$$P_s = A^T P_s A_F.$$

Moreover A_F given above is stable since A_{22} is already stable. Since there is a unique stabilizing solution to the RDE, P_s must be that solution. ∎

This theorem implies that the image of P_s is also the desired unstable left invariant subspace of A, which explains that when S_0 has rank larger than the number of unstable eigenvalues of A, some components of S_i have to be damped out in the iteration. When we overestimate the dimension of the unstable invariant subspace, we therefore nevertheless converge to a subspace of correct dimension. Moreover, Corollary 6 implies that the spectrum of A_F determines the convergence ratio of P_i towards the stabilizing solution P_s. Convergence will occur provided the initial matrix P_0 satisfies the non-degeneracy condition (10). Tests for checking whether convergence has occurred and extensive numerical experiments are reported in [5],[6].

9 Numerical experiments

The results of this paper give a theoretical explanation of the convergence behavior observed in [6]. The analysis also give a proof that the DRDE converges to a stabilizing solution of the DRE under milder conditions than those of [2], provided an asymptotically stabilizing solution exists. The number of iteration steps needed to obtain a stabilized system will depend on several factors and it does not seem possible to give upper bounds on this. Results are nevertheless encouraging, in the sense that one can expect stabilization in very few steps. We quote an example from [6] to exemplify this.

The system matrix A is constructed by randomly generating a 100×100 matrix with MATLAB RANDN and scaling it to $\tilde{A}$ so that the spectral radius of $\tilde{A}$ is 0.9. The 100×1 matrix B is generated randomly with RAND as is a 1×100 matrix $\tilde{F}$. Construct $A = \tilde{A} + B\tilde{F}$. All eigenvalues of A (dots in Figure 1) are well-separated from the unit circle and only two are unstable. The norm $\|\tilde{F}\|_2 = 9.9734$ and eigen-condition number of $\tilde{A}$ is $\kappa_2(X, \tilde{A}) = 110.8009$. So the system (A, B) should be well-conditioned and easy to stabilize. Figure 2 show the results of rank 2 SQR. Figure 3 show the result of rank 3 SQR. $P_0^{1/2}$ is randomly generated with RAND. The spectral radius of $A - BF_i$ from both rank 2 and rank 3 SQR converges within 7 iterations, which is shown in Figure 2 and Figure 3. Furthermore, Figure 1 shows the spectrum of $A - BF_5$ (the $+$ symbols) converges to a stable configuration in 5 iterations. We also see that the only eigenvalues moved are the two unstable ones of A.

This example illustrates that for a well-conditioned stabilization problem where A has only few unstable eigenvalues and all eigenvalues of A are well-separated from the unit circle, SQR is very efficient and stabilization is reached within only a few steps. We can relax the condition that all eigenvalues of A are well-separated from the unit circle to that all unstable eigenvalues of A are well-separated from all stable eigenvalues of A. If the unstable eigenvalues of A are well-separated from the unit circle, fast stabilization with SQR is expected and feedback convergence with SQR depends on the choice of the rank of $P_0^{1/2}$, with the worst case when some stable eigenvalues of A are very close to the unit circle and we choose an incorrect rank of $P_0^{1/2}$ (larger than the number of unstable eigenvalues of A). In this case, we can monitor the eigenvalue convergence of $V_i'(A - BF_i)V_i$ to catch the stability of $A - BF_i$ or modify the rank of $P_i^{1/2}$ during the iteration. If some unstable eigenvalues of A are very close to the unit circle and stable eigenvalues of A are well-separated from the unit circle, some scaling on A can help to accelerate both stabilization and feedback convergence (see [5], [6] for more details).

Acknowledgments

This work was supported by the National Science Foundation under Grant No. CCR-97-96315 and by the Belgian Programme on Inter-university Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture.

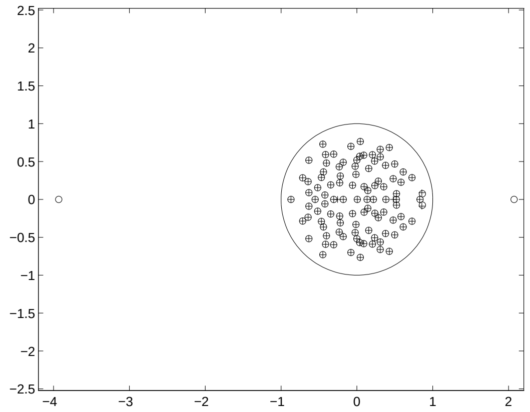

Figure 1: SQR: Spectrum

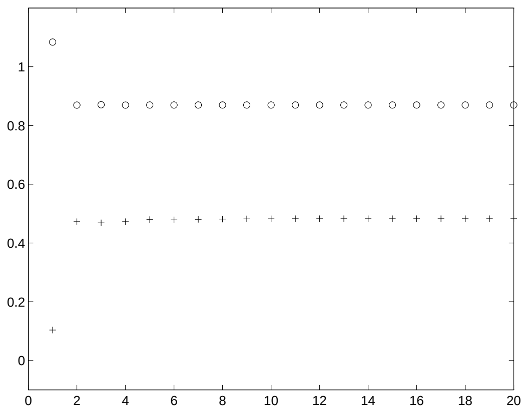

Figure 2: SQR: $A - BF_i$ (o) and $V_i'(A - BF_i)V_i$ (+)

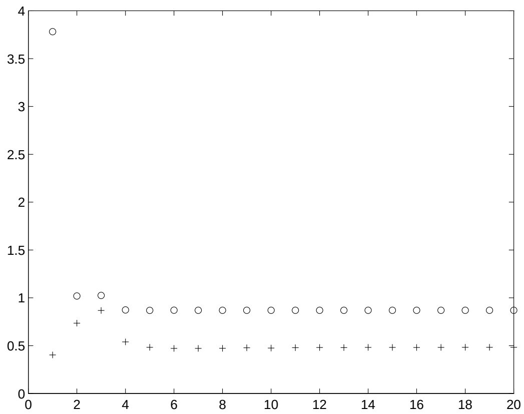

Figure 3: SQR: $A - BF_i$ (o) and $V_i'(A - BF_i)V_i$ (+)

References

[1] B.D.O. Anderson, Second order convergent algorithms for the steady-state Riccati equation, *Int. J. Control*, **28**, pp. 295-306, 1978.

[2] C.E. De Souza, M.R. Gevers and G.C. Goodwin, Riccati Equations in optimal filtering of non-stabilizable systems having singular transition matrices, *IEEE Trans. Aut. Contr.*, **AC-31**, pp. 831-838, 1986

[3] G. Golub and C. Van Loan, Matrix Computations, 2nd Ed., John Hopkins University Press, 1990.

[4] M. Morf and T. Kailath, Square-Root Algorithms for Least-Squares Estimation, *IEEE Trans. Aut. Contr.*, **AC-20**, pp. 487-497, 1975.

[5] X. Rao, Large scale stabilization with linear feedback, M.S. Thesis, Dept. Comp. Sc., Florida State University, 1999.

[6] X. Rao, K. Gallivan and P. Van Dooren, Stabilization of Large Scale Dynamical Systems, *Proceedings of the 14th Int. Symp. Math. Th. Netw. & Syst.*, MTNS2000 Perpignan, June 2000, 1999.

[7] Y. Saad, Projection and deflation methods for partial pole assignment in linear state feedback, *IEEE Trans. Aut. Contr.*, **AC-33**, pp. 290-297, 1988.

[8] M. Verhaegen and P. Van Dooren, Numerical aspects of different Kalman filter implementations, *IEEE Trans. Aut. Contr.*, **AC-31**, pp. 907-917, 1986.

[9] A. White and C. Sage, Optimal Systems Control, 2nd Ed., Prentice Hall, Englewood Cliffs, 1977.