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Linear Time-variable Systems: Stability of Reduced Models*

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Uniformly balanced realizations provide a method for approximating linear time-variable systems by lower order systems with guaranteed stability.

Key Words—Time-varying systems; system-order reduction; stability; linear systems.

Abstract—The notion of a 'uniformly balanced' realization for time-variable systems has been previously introduced. This representation is characterized by the fact that its controllability and observability grammians are equal and diagonal. Such a framework has many remarkable properties and leads to a setting where the subsystems can be taken as reduced model for time-variable systems. It turns out that once the stability of a subsystem is guaranteed, then the subsystem preserves many of the properties of the original system. In this paper, the stability of subsystems is fully explored.

1. INTRODUCTION

IN REALIZATION theory, the central concerns of most of the previous research has been the formal aspects of the problem, such as questions of existence and minimality of realizations obtained from a given impulse response matrix $H(t, \tau)$, and providing a description of the class of all possible realizations. It is clear, however, that not all realizations are equally useful for practical implementation or for answering various theoretical questions. Recently, Moore (1978, 1981) developed an essentially unique representation for linear time-invariant systems which he termed 'balanced'. This representation is characterized by the fact that its controllability and observability grammians are equal and diagonal. In such a realization one can always order the components of the state vector with respect to their influence on the system's input–output response. This framework leads to a very natural method for performing model reduction by deletion of states which are 'nearly uncontrollable' and 'nearly unobservable' and therefore 'nearly redundant'. For constant systems obtained in this way, reduced models are almost always stable if the original system is stable (Moore, 1978). Pernebo and Silverman (1979, 1982) showed that the stability of the reduced model is guaranteed if the diagonal elements of the controllability and observability grammians are distinct.

Shokoohi, Silverman and Van Dooren (1980, 1981, 1983) extended these balancing ideas to time-variable systems. They gave the necessary and sufficient conditions for a uniform realization (Silverman, 1968) to be uniformly balanced. This uniformly balanced framework led to the first systematic procedure for lower-order approximation of time-variable systems. In this paper we further study the properties of time-variable balanced systems, particularly, the asymptotic stability of reduced models. As in the time-invariant case, distinctness of the diagonal elements of the controllability and observability grammians is intimately connected with stability of subsystems but these connections are considerably more difficult to establish for time-variable systems. It is shown, however, if the diagonal elements are asymptotically distinct, and several technical conditions are satisfied, then asymptotic stability of system approximations can be guaranteed.

2. UNIFORMLY BALANCED ASYMPTOTICALLY STABLE REALIZATION

We start with a linear continuous-time system representation of the type

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t)$$

where the coefficient matrices $A(t)$, $B(t)$, $C(t)$ are continuous and bounded. Such a system will be

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denoted by the triplet $(A(\cdot), B(\cdot), C(\cdot))$. We first review several definitions.

Definition 1 (*Silverman and Anderson*, 1968)
A bounded realization (A, B, C) is said to be *uniformly completely controllable* if there exists a $\delta > 0$ such that

$$G_c(t-\delta, t) \geq \alpha_1(\delta)I > 0, \quad \forall t \in \mathscr{R}$$

where

$$G_c(t-\delta, t) = \int_{t-\delta}^{t} \Phi(t,\tau)B(\tau)B'(\tau)\Phi'(t,\tau)\,\mathrm{d}\tau. \quad \square$$

Definition 2
A bounded realization (A, B, C) is said to be *uniformly completely observable* if there exists a $\delta > 0$ such that

$$G_0(t, t+\delta) \geq \alpha_2(\delta)I > 0, \quad \forall t \in \mathscr{R}$$

where

$$G_0(t, t+\delta) = \int_{t}^{t+\delta} \Phi'(\tau,t)C'(\tau)C(\tau)\Phi(\tau,t)\,\mathrm{d}\tau. \quad \square$$

Definition 3 (*Silverman*, 1968)
A system representation (A, B, C) is said to be *uniform* if

(i) $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ are continuous and bounded
(ii) (A, B, C) is uniformly completely controllable and observable. □

Definition 4
A system representation (A, B, C) is said to be *uniformly balanced* if

(i) (A, B, C) is uniform
(ii) $G_c(t-\delta, t) = G_0(t, t+\delta) = \Sigma(t)$, where $\Sigma(t)$ is a diagonal matrix. □

If a uniform realization is asymptotically stable, then we can perform the balancing for $\delta = \infty$ (Shokoohi, Silverman and Van Dooren, 1983), and we obtain a uniformly balanced, asymptotically stable (u.b.a.s) realization (A, B, C) which satisfies the Liapunov equations

$$\dot{\Sigma}(t) = A(t)\Sigma(t) + \Sigma(t)A'(t) + B(t)B'(t) \quad \text{(1a)}$$
$$-\dot{\Sigma}(t) = A'(t)\Sigma(t) + \Sigma(t)A(t) + C'(t)C(t) \quad \text{(1b)}$$

where

$$G_c(-\infty, t) = G_0(t, \infty) = \Sigma(t) = \operatorname{diag}[\sigma_1(t), \sigma_2(t), \ldots, \sigma_n(t)] \geq \alpha I > 0, \quad \forall t. \quad (2)$$

Partitioning the matrices A, B, C as

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right], B = \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right], C = [C_1 \mid C_2].$$

Then *any* subsystem (A_{11}, B_1, C_1) also satisfies similar Liapunov equations (Shokoohi, Silverman and Van Dooren, 1983) with Σ_1 the corresponding submatrix of Σ

$$\dot{\Sigma}_1(t) = A_{11}(t)\Sigma_1(t) + \Sigma_1(t)A'_{11}(t) + B_1(t)B'_1(t) \quad \text{(3a)}$$
$$-\dot{\Sigma}_1(t) = A'_{11}(t)\Sigma_1(t) + \Sigma_1(t)A_{11}(t) + C'_1(t)C_1(t). \quad \text{(3b)}$$

It is justified (Shokoohi, Silverman and Van Dooren, 1983) that if Σ_1 contains the dominant (larger) singular values $\sigma_1(t)$, then the subsystem (A_{11}, B_1, C_1) corresponding to Σ_1 can be considered as a reasonable reduced model.

We summarize the properties of subsystem (A_{11}, B_1, C_1) in the following lemma. Notice that every property holding for a subsystem (A_{11}, B_1, C_1) also holds for system (A, B, C) since it is a trivial subsystem of itself.

Lemma 1 (*Shokoohi, Silverman and Van Dooren*, 1983)
Let (A, B, C) be a u.b.a.s realization. Then any subsystem (A_{11}, B_1, C_1) has the following properties

(i) $\lambda_{\max}(A_{11}(t) + A'_{11}(t)) \leq 0, \quad \forall t$
(ii) $\mathscr{R}\lambda(A_{11}(t)) \leq 0, \quad \forall t$
(iii) $\|\Phi_{A_{11}}(t, t_0)\| \leq 1, \quad \forall t \geq t_0$
(iv) $\|x_1(t)\| \leq \|x_1(\tau)\|, \quad \forall t \geq \tau$

where $\mathscr{R}\lambda(A_{11})$ denotes the real part of eigenvalues of A_{11}.

Pernebo and Silverman (1979, 1982) showed that for balanced, a.s., time-invariant systems we have: $\|e^{At}\| < 1, \forall t > 0$, where $\|\cdot\|$ is the spectral norm. For time-varing systems, the following theorem holds.

Theorem 1
Let (A, B, C) be a u.b.a.s realization. If $A(t)$ is an analytic function of time, then $\|\Phi_A(t, t_0)\| < 1$, $\forall t > t_0$.

Proof
Suppose not. Then there exists a $t > t_0$ such that $\|\Phi_A(t, t_0)\| = 1$. We can always find $x_0 \triangleq x(t_0)$ such that $\|\Phi_A(t, t_0)x(t_0)\| = 1$ and $\|x(t_0)\| = 1$. Defining $x(t) = \Phi_A(t, t_0)x(t_0)$, we have $\|x(t)\| = \|x(t_0)\| = 1$. Since $\|x(t)\|$ is non-increasing (property (iv), Lemma 1), then $\|x(\tau)\| = 1$, $\forall \tau \in (t_0, t)$. But by analyticity, $\|x(\tau)\|^2 = x'(\tau)x(\tau) = 1$, $\forall \tau$, which contradicts asymptotic stability. □

Corollary 1

Let (A_{11}, B_1, C_1) be a subsystem of a u.b.a.s system (A, B, C). If $A_{11}(t)$ is analytic, then the solutions of $\dot{x}_1(t) = A_{11}(t)x_1(t)$ satisfy one of the following conditions:

(i) $\|x_1(t)\| = \text{constant}, \quad \forall t$
(ii) $\|x_1(t)\| < \|x_1(\tau)\|, \quad \forall t > \tau.$ □

The assumption of analyticity of $A(t)$ is necessary as shown by the following example.

Example 1

Define:

$$A(t) = \begin{cases} 0, & t \in [2k\pi, (2k+1)\pi) \\ -\dfrac{\sin^2(2t)}{2}, & t \in [(2k+1)\pi, (2k+2)\pi) \end{cases} \quad k \in \mathscr{Z}$$

$$B(t) = C(t) = \begin{cases} 0, & t \in [2k\pi, (2k+1)\pi) \\ \sin(2t), & t \in [(2k+1)\pi, (2k+2)\pi). \end{cases} \quad k \in \mathscr{Z}$$

Then (A, B, C) is uniformly balanced with $\Sigma(t) = 1$. The system is also asymptotically stable (see Corollary 2), but $A(t)$ is clearly not analytic. It is easily checked that $\Phi_A(t, 0) = 1$ for $t \in [0, \pi)$ and only starts to decrease after that. □

If the uniformly balanced framework is to be used for model reduction, then the stability of subsystems is of prime importance. The following definition and lemmas work to this end.

Definition 5

A real valued measurable function $f: [t_0, \infty) \to \mathscr{R}$ is said to be *integrable* or *summable* if $f \in \mathscr{L}_1[t_0, \infty)$, where

$$\mathscr{L}_1[t_0, \infty) = \{f: [t_0, \infty) \to \mathscr{R} \text{ such that } \int_{t_0}^{\infty} |f(t)|\, dt < \infty\}.$$

A matrix $F(t)$ is said to be integrable if $\|F(t)\| \in \mathscr{L}_1$. □

Lemma 2

If a matrix $F(t)$ is integrable, and $\|T(t)\| \le K$, $\forall t$, then $T(t)F(t)$ is integrable.

Lemma 3 (*Bellman*, 1970)

If a row and the corresponding column of a symmetric matrix G is deleted, then the ordered eigenvalues of λ_i of G and μ_j of the submatrix satisfy the interlacing property

$$\lambda_1 \le \mu_1 \le \lambda_2 \le \mu_2 \le \ldots \le \mu_{n-1} \le \lambda_n. \quad \square$$

Theorem 2 (*Shokoohi, Silverman and Van Dooren*, 1983)

Let (A, B, C) be a u.b.a.s realization. Then for *any* subsystem (A_{11}, B_1, C_1) the following are equivalent statements:

(i) $A_{11}(t)$ is a.s.
(ii) $(A_{11}(t), B_1(t))$ is uniformly controllable.
(iii) $(A_{11}(t), C_1(t))$ is uniformly observable.
(iv) $(A_{11}(t), B_1(t), C_1(t))$ is uniformly balanced with:

$$\Sigma_1(t) = \int_t^{\infty} \Phi'_{A_{11}}(\tau, t)C'_1(\tau)C_1(\tau)\Phi_{A_{11}}(\tau, t)\, d\tau$$
$$= \int_{-\infty}^{t} \Phi_{A_{11}}(t, \tau)B_1(\tau)B'_1(\tau)\Phi_{A_{11}}(t, \tau)\, d\tau. \quad \square$$

This theorem shows that any one of the conditions (i), (ii) and (iii) imply both asymptotic stability and uniform balancedness of the subsystem (A_{11}, B_1, C_1). The following theorem gives a sufficient condition which guarantees the asymptotic stability of *any* subsystem. In view of property (i) of Lemma 1, the properties of this theorem are 'generically' satisfied.

Theorem 3

Let (A, B, C) be a u.b.a.s realization. If

$$\lambda_{\max}\left(\frac{A(t) + A'(t)}{2}\right) \notin \mathscr{L}_1[t_0, \infty)$$

then *every* subsystem (A_{11}, B_1, C_1) is u.b.a.s.

Proof

Since $\lambda_{\max}\left(\frac{A(t) + A'(t)}{2}\right) \notin \mathscr{L}_1[t_0, \infty)$ and is always non-positive, we have

$$\lim_{t \to \infty} \int_{t_0}^{t} \lambda_{\max}\left(\frac{A(\tau) + A'(\tau)}{2}\right) d\tau = -\infty.$$

The matrix $(A_{11}(\tau) + A'_{11}(\tau))$ is a submatrix of $(A(\tau) + A'(\tau))$, so by Lemma 3: $\lambda_{\max}(A_{11}(\tau) + A'_{11}(\tau)) \le \lambda_{\max}(A(\tau) + A'(\tau))$, $\forall \tau$. This implies

$$\lim_{t \to \infty} \int_{t_0}^{t} \lambda_{\max}\left(\frac{A_{11}(\tau) + A'_{11}(\tau)}{2}\right) d\tau = -\infty.$$

Using Wazewski's inequality (6), we have $\lim_{t \to \infty} \|\Phi A_{11}(t, t_0)\| = 0$. This implies a.s. and by Theorem 2, the u.b. of the subsystem. □

Corollary 2

Let (A, B, C) be a u.b.a.s realization with $A(t)$ periodic. If there exists a t_1 such that

$\lambda_{\max}(A(t_1) + A'(t_1)/2) < 0$, then *every* subsystem (A_{11}, B_1, C_1) is u.b.a.s.

Proof

Since there exists a t_1 such that $\lambda_{\max}(A(t_1) + A'(t_1)/2) < 0$, then by continuity of $\lambda_{\max}(t)(A(t)$ is assumed to be continuous) there exists an $\varepsilon > 0$ such that

$$\lambda_{\max}\left(\frac{A(t) + A'(t)}{2}\right) < 0, \quad \forall t \in (t_1 - \varepsilon, t_1 + \varepsilon).$$

The periodicity of $\lambda_{\max}$ then implies that

$$\lambda_{\max}\left(\frac{A + A'}{2}\right) \notin \mathscr{L}_1[t_0, \infty).$$

The result then follows from Theorem 3.

Corollary 3

Let the time-invariant realization (A, B, C) be balanced and a.s. If $\lambda_{\max}(A + A')/2 < 0$, then *every* subsystem (A_{11}, B_1, C_1) is balanced and a.s. □

The importance of Theorem 3 is due to the simplicity of the condition, i.e., no additional constraints are required for the elements of $\Sigma(t)$. Corollary 2 implies that if $A(t)$ is periodic, then the stability and balancedness of the subsystem (A_{11}, B_1, C_1) hold generically, and Corollary 3 gives a new condition for the stability of the subsystems of time-invariant systems. However, we emphasize that the condition on $\lambda_{\max}(t)$ is only sufficient and not necessary, as can be verified by the following example.

Example 2

Let (A, B, C) be given by

$$\left(\begin{bmatrix} -1 & 0 \\ -2\sqrt{2} & -2 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}, [\sqrt{2}, 2]\right)$$

then it is a.s. and u.b. with $\Sigma = I_{2\times 2}$. We note that $\lambda_{\max}(A + A') = 0 \in \mathscr{L}_1$, yet every subsystem is u.b.a.s. □

3. STABILITY OF REDUCED MODEL AND SINGULAR VALUES

For time-invariant systems there exists a direct relationship between distinctness of singular values of Σ and guaranteed asymptotic stability of the subsystems (Pernebo and Silverman, 1982). In this section, we extend this connection to the time-variable case by providing conditions under which the hypothesis of Theorem 3 and Corollary 2 holds. The following lemma is designed for periodic systems.

Lemma 4

Let (A, B, C) be a u.b.a.s realization. If there exists a point t_1 where $\Sigma(t_1)$ has distinct singular values and $\mathscr{R}\lambda(A(t_1)) < 0$, then $\lambda_{\max}(A(t_1) + A'(t_1)) < 0$.

Proof

We prove the results by contradiction. All the matrices are evaluated at t_1. Let $\lambda_{\max}(A + A') = 0$, and let the columns of V span the nullspace of $A + A'$

$$(A + A')V = 0. \tag{4}$$

Adding (1a) and (1b) we obtain

$$(A + A')\Sigma + \Sigma(A + A') = -(BB' + C'C). \tag{5}$$

Pre- and postmultiplying above equation with V' and V respectively, we have

$$B'V = 0, \; CV = 0. \tag{6}$$

Postmultiplying (5) by V, we then obtain

$$(A + A')\Sigma V = 0. \tag{7}$$

From (4) and (7) it follows that the columns of ΣV are in the right nullspace of $(A + A')$. Therefore there exists a matrix $\bar{\Sigma}$ such that

$$\Sigma V = \Sigma \tag{8}$$

where w.l.o.g. it is possible to choose V such that $\bar{\Sigma}$ is diagonal. In such a coordinate system, the diagonal entries of $\bar{\Sigma}$ are then a subset of the diagonal entries of Σ. Since Σ is simple (no repeated eigenvalues), then V is spanned by a set of unit vectors. From (1a) and (1b) we have

$$2\dot{\Sigma} = (A - A')\Sigma + \Sigma(A' - A) + BB' - C'C. \tag{9}$$

On the diagonal we thus have

$$2\dot{\Sigma} = \operatorname{diag}[BB' - C'C]. \tag{10}$$

Since V is spanned by unit vectors and from (7), $BB'V = 0$, $C'CV = 0$, then we have

$$\dot{\Sigma}V = 0. \tag{11}$$

Postmultiplying (1b) by V, and using (6) and (11), we obtain

$$A'\Sigma V + \Sigma A V = 0.$$

But since $(A + A')V = 0$ and $\Sigma V = V\Sigma$, we have $\Sigma A V = A V \Sigma$, $\Sigma V = V\Sigma$. Writing the above equations columnwise, we obtain

$$\Sigma A\mathscr{V}_i = \sigma_i A\mathscr{V}_i, \Sigma\mathscr{V}_i = \sigma_i\mathscr{V}_i, \quad i = 1,\ldots,r. \quad (12)$$

Since Σ has distinct eigenvalues, then

$$A\mathscr{V}_i = \lambda_i\mathscr{V}_i. \quad (13)$$

Pre- and postmultiplying (1b) with $\mathscr{V}_i^*$ and $\mathscr{V}_i$ respectively, we obtain

$$2\mathscr{R}(\lambda_i(A))\mathscr{V}_i^*\Sigma\mathscr{V}_i = 0.$$

Since $\mathscr{R}\lambda_i(A) \neq 0$, this implies that Σ is singular and thus contradicts the uniformity of (A, B, C).

The following example shows that $\mathscr{R}\lambda(A(t_1)) < 0$ is a necessary condition for $\lambda_{max}(A(t_1) + A'(t_1)) < 0$.

Example 3

Let $a_{11}(t) =$

$$\begin{cases} 0, & t \in [2k\pi, (2k+1)\pi) \\ -\dfrac{\sin^2(2t)}{2}, & t \in [(2k+1)\pi, (2k+2)\pi) \end{cases} \quad k \in \mathscr{Z}$$

$$a_{22}(t) = a_{11}(t+\pi), b_{11}^2(t) = c_{11}^2(t) = -2a_{11}(t),$$
$$b_{22}^2(t) = c_{22}^2(t) = -a_{22}(t).$$

Then

$$\left(\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix}, \begin{bmatrix} c_{11} & 0 \\ 0 & c_{22} \end{bmatrix}\right)$$

is a u.b.a.s. realization (see Example 1) with

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

having distinct singular values. Since $\mathscr{R}\lambda(A) \nless 0$, then we cannot conclude $\lambda_{max}(A + A') < 0$, in fact,

$$\lambda_{max}(A + A') = \lambda_{max}\begin{bmatrix} 2a_{11} & 0 \\ 0 & 2a_{22} \end{bmatrix} = 0, \quad \forall t.$$

Yet both subsystems

$$\left(a_{11}, [b_{11}, 0], \begin{bmatrix} c_{11} \\ 0 \end{bmatrix}\right) \text{ and } \left(a_{22}, [0, b_{22}], \begin{bmatrix} 0 \\ c_{22} \end{bmatrix}\right)$$

are u.b.a.s. □

Theorem 4

Assume (A, B, C) is u.b.a.s. realization with $A(t)$ periodic. If $\exists t_1 \ni \Sigma(t_1)$ has distinct diagonal elements and $\mathscr{R}\lambda(A(t_1)) < 0$, then *every* subsystem (A_{11}, B_1, C_1) is u.b.a.s.

Proof

According to Lemma 4, $\lambda_{max}(A(t_1) + A'(t_1)) < 0$. Corollary 2 then yeilds the required result. □

Corollary 4

Let the time-invariant realization (A, B, C) be balanced and a.s. If Σ has distinct diagonal elements, then *every* subsystem (A_{11}, B_1, C_1) is balanced and a.s.

Proof

Since A is a.s, then $\mathscr{R}\lambda(A) < 0$ and the result follows from Theorem 4. □

Example 3 shows that the conditions of Theorem 4 are sufficient but not necessary to guarantee the stability of the subsystem. For time-invariant systems, Corollaries 3 and 4 give two different sufficient conditions for stability of the subsystems. The following theorem shows that in many cases, these two conditions are equivalent, which is another remarkable property of balanced realizations.

Theorem, 5

Let the time-invariant realization (A, B, C) be single input *or* single output, a.s, and balanced. Then

$$\lambda_{max}\left(\frac{A + A'}{2}\right) < 0 \leftrightarrow \Sigma \text{ has distinct diagonal elements.} \quad (14)$$

Proof

($\leftarrow$) It follows from Lemma 4. ($\rightarrow$) It suffices to show that if Σ has equal diagonal elements, then $\lambda_{max}(A + A'/2) = 0$. Suppose (A, B, C) is single input and w.l.o.g. Suppose that the first two diagonal elements of Σ are equal. The subsystem (A_{11}, B_1) corresponding to the first two diagonal elements satisfies the following Liapunov equation

$$\sigma_1 I(A_{11} + A'_{11}) = -B_1 B'_1. \quad (15)$$

Since $B_1 B'_1$ is 2×2 but has rank 1 at the most, then

$$\sigma_1^2 \det(A_{11} + A'_{11}) = \det(B_1 B'_1) = 0. \quad (16)$$

This implies that $(A_{11} + A'_{11})$ is singular. Using Lemmas 1 and 3 we have

$$0 = \lambda_{max}(A_{11} + A'_{11}) \leq \lambda_{max}(A + A') \leq 0 \rightarrow \lambda_{max}(A + A') = 0 \quad (16b)$$

which proves the result.

We note that $\lambda_{max}(A + A')/2 < 0$ does not in general imply that Σ has distinct diagonal elements as shown in the following example.

Example 4

The system (A, B, C) with

$$A = \begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} \tag{17}$$

is balanced and a.s with $\lambda_{\max}(A + A')/2 = -2 < 0$. However, the diagonal elements of Σ are not distinct, in fact, $\Sigma = I_{2\times 2}$.

In the sequel we investigate the case where the diagonal elements of $\Sigma(t)$ are distinct for large t, and relate it to the stability of the subsystems. However, such a connection is considerably more difficult to establish compared to the time invariant or periodic case. The following definition is useful.

Definition 6

Two real valued functions $f_1(t)$ and $f_2(t)$ are said to be *asymptotically distinct*, if there exists a $t_0 \in \mathscr{R}$ and $\varepsilon > 0$ such that

$$|f_1(t) - f_2(t)| \geq \varepsilon, \forall t \in [t_0, \infty).$$

Let us divide the spectrum of $(A + A')$ into two sets $S_1 = \{\lambda_i(t)\}_{i=1}^{r}$ and $S_2 = \{\lambda_j(t)\}_{j=r+1}^{n}$, where $\lambda_i(t) \in \mathscr{L}_1$ for $i = 1, \ldots, r$ and $\lambda_j(t) \notin \mathscr{L}_1$ for $j = r+1, \ldots, n$. Notice that eigenvalues $\lambda_i(t)$ can be chosen continuous since $(A(t) + A'(t))$ is continuous (Kato, 1976). Let $\lambda_{\min}(S_1)$ be the pointwise minimum of the set S_1, and $\lambda_{\max}(S_2)$ be the pointwise maximum of the set S_2. We now make the following assumption.

Assumption 1

Assume that $\lambda_{\max}(S_2)$ and $\min(\lambda_{\min}(S_1), 0)$ are asymptotically distinct from each other.

Let us define orthonormal bases $V^{(i)}(t)$ and $V^{(n)}(t)$ for the invariant subspaces corresponding to the groups of eigenvalues, then we have the following eigenvalue decomposition:

$$\left(\frac{A(t) + A'(t)}{2}\right)(V^{(i)}(t) \mid V^{(n)}(t))$$

$$= (V^{(i)}(t) \mid V^{(n)}(t)) \left(\begin{array}{c|c} \Lambda^{(i)}(t) & 0 \\ \hline 0 & \Lambda^{(n)}(t) \end{array}\right) \tag{18}$$

where $\Lambda^{(i)}(t) = \operatorname{diag}[\lambda_1(t), \ldots, \lambda_r(t)]$ and $\Lambda^{(n)}(t) = \operatorname{diag}[\lambda_{r+1}(t), \ldots, \lambda_n(t)]$. Because of Assumption 1, it follows from Kato (1976) that projectors

$$P^{(i)}(t) = V^{(i)}(t)V^{(i)\prime}(t),\ P^{(n)}(t) = V^{(n)}(t)V^{(n)\prime}(t)$$

can be chosen continuous in $[t_0, \infty)$. By the Theorem of Dolezal (1964), it also follows that $V^{(i)}(t)$ and $V^{(n)}(t)$ can be chosen continuous in t, since these are bases of the range of $P^{(i)}(t)$ and $P^{(n)}(t)$ respectively, and these projectors have constant rank.

We now have the following four technical lemmas which are stated in the framework of having a u.b.a.s realization (A, B, C).

Lemma 5

If Assumption 1 holds, then

$$[(A(t) + A'(t))\Sigma(t)V^{(i)}(t)] \in \mathscr{L}_1$$

implies that there exists a diagonal continuous matrix $\bar{\Sigma}(t) > \alpha I > 0$ such that

$$[\Sigma(t)\bar{V}^{(i)}(t) - \bar{V}^{(i)}(t)\bar{\Sigma}(t)] \in \mathscr{L}_1$$

where $\bar{V}^{(i)}(t) = V^{(i)}(t)P(t)$ with $P(t)$ a unitary matrix.

Proof

We can always find continuous matrices $K(t)$ and $L(t)$ such that

$$\Sigma(t)V^{(i)}(t) = V^{(i)}(t)K(t) + V^{(n)}(t)L(t) \tag{19}$$

where $K(t) \geq \alpha I > 0$. Postmultiplying the above equation with $(A + A')/2$, we have $(A + A')/2\ \Sigma V^{(i)} = V^{(i)}\Lambda^{(i)}K + V^{(n)}\Lambda^{(n)}L$ which implies

$$V^{(n)}\Lambda^{(n)}L \in \mathscr{L}_1. \tag{20}$$

By asumption 1, $(\Lambda^{(n)})^{-1}$ is uniformly bounded. Premultiplying (20) with $(\Lambda^{(n)})^{-1}V^{(n)\prime}$ and using Lemma 2, gives $L(t) \in \mathscr{L}_1$ and (19) implies

$$[\Sigma(t)V^{(i)}(t) - V^{(i)}(t)K(t)] \in \mathscr{L}_1.$$

Using the eigenvalue decomposition $K(t) = P(t)\bar{\Sigma}(t)P'(t)$, we have $[\Sigma(t)\bar{V}^{(i)}(t) - \bar{V}^{(i)}(t)\bar{\Sigma}(t)] \in \mathscr{L}_1$ where $\bar{V}^{(i)}(t) = V^{(i)}(t)P(t)$ and $\bar{\Sigma}$ is continuous. □

Note that $\bar{\Sigma}$ in the above lemma is continuous and bounded, we now further assume that $\bar{\Sigma}(t)$ behaves as nicely as $\Sigma(t)$. Since $\dot{\Sigma}(t)$ is continuous and bounded (Shokoohi, Silverman and Van Dooren, 1983), we assume the same for $\bar{\Sigma}(t)$.

Assumption 2

$\|\dot{\bar{\Sigma}}(t)\| \leq M,\ \forall t \in [t_0, \infty)$.

Lemma 6

If Assumption 2 holds and the diagonal elements of $\Sigma(t)$ are uniformly distinct, then $[\Sigma(t)\bar{V}^{(i)}(t) - \bar{V}^{(i)}(t)\bar{\Sigma}(t)] \in \mathscr{L}_1$ implies that the basis

$$\bar{V}^{(i)}(t) = \begin{bmatrix} \bar{V}_1^{(i)}(t) \\ \hline \bar{V}_2^{(i)}(t) \end{bmatrix}$$

is continuous, and w.l.o.g we have:

(i) non-diagonal elements of $r \times r$ matrix $\bar{V}_1^{(i)}(t)$ are integrable.
(ii) diagonal elements of $\bar{V}_1^{(i)}(t)$ have magnitudes close to 1 for large t.
(iii) $\bar{V}_2^{(i)}(t) \in \mathscr{L}_1$.

Proof

Defining $\bar{V}^{(i)}(t) \triangleq [V_1 \vdots V_2 \vdots \ldots \vdots V_r]$, and taking the first column of $[\Sigma(t)\bar{V}^{(i)}(t) - \bar{V}^{(i)}(t)\bar{\Sigma}(t)]$, we have

$$[\Sigma(t)V_1(t) - \bar{\sigma}_1(t)V_1(t)] \in \mathscr{L}_1 \tag{21}$$

where $\bar{\sigma}(t)$ is the first diagonal element of $\bar{\Sigma}(t)$. Rewriting (21) we obtain

$$\begin{bmatrix} (\sigma_1(t) - \bar{\sigma}_1(t))\mathscr{V}_{11}(t) \\ (\sigma_2(t) - \bar{\sigma}_1(t))\mathscr{V}_{21}(t) \\ \vdots \\ (\sigma_n(t) - \bar{\sigma}_1(t))\mathscr{V}_{n1}(t) \end{bmatrix} \in \mathscr{L}_1. \tag{22}$$

We claim that at most one of the functions $(\sigma_i(t) - \bar{\sigma}_1(t))$ $i = 1,\ldots,n$ is integrable since, if we have $(\sigma_i(t) - \bar{\sigma}_1(t)) \in \mathscr{L}_1$ and $(\sigma_j(t) - \bar{\sigma}_1(t)) \in \mathscr{L}_1$ for $i \neq j$, then the difference of these two functions must also be integrable, that is

$$(\sigma_i(t) - \sigma_j(t)) = [(\sigma_i(t) - \bar{\sigma}_1(t)) - (\sigma_j(t) - \bar{\sigma}_1(t))] \in \mathscr{L}_1.$$

However, this is a contradiction to assumption of asymptotically distinct singular values. Assumption 2 implies that $\bar{\sigma}_1(t)$ cannot jump from $\sigma_i(t)$ to $\sigma_j(t)$ nor can it go continuously from $\sigma_i(t)$ to $\sigma_j(t)$ with infinitely large slope. Therefore integrability in (22) implies that $\bar{\sigma}_1(t)$ becomes essentially close to one of the diagonal elements of $\Sigma(t)$, for large t. Suppose w.l.o.g that $(\sigma_1(t) - \bar{\sigma}_1(t))$ is the only possible integrable function, then $(\sigma_i(t) - \bar{\sigma}_1(t) \notin \mathscr{L}_1$, $i = 2,\ldots,n$ and by asymptotic distinctness of singular values, they become essentially bounded away from zero for large t. Therefore (22) implies that $\mathscr{V}_{i1}(t) \in \mathscr{L}_1$ for $i = 2,\ldots,n$ and since

$$\mathscr{V}_{11}^2(t) + \sum_{i=2}^{n} \mathscr{V}_{i1}^2(t) = 1,$$

then $(\mathscr{V}_{11}^2(t) - 1) \in \mathscr{L}_1$, and for t sufficiently large, $\mathscr{V}_{i1}(t)$, $i = 2,\ldots,n$ becomes close to zero while $|\mathscr{V}_{11}(t)|$ becomes close to 1. Similar proof follows for other eigenvectors $V_2,\ldots,V_r$, noting that the structure of $\bar{V}^{(i)}(t)$ prevents two diagonal elements of $\bar{\Sigma}(t)$ from being the same. In fact, by asymptotic distinctness of singular values of $\Sigma(t)$, the diagonal elements of $\bar{\Sigma}(t)$ also are distinct for large t. This implies (Kato, 1976) that decomposition $K(t) = P(t)\bar{\Sigma}(t)P'(t)$ in Lemma 5 is in fact a continuous decomposition, therefore, $\bar{V}^{(i)}(t) = V^{(i)}(t)$ $P(t)$ is also continuous. Finally, we note that a constant permutation will always yield the proper ordering for $\bar{V}^{(i)}(t)$ as described in the lemma. □

Lemma 7

Suppose $B'(t)\bar{V}^{(i)}(t) \in \mathscr{L}_1$ and $C(t)\bar{V}^{(i)}(t) \in \mathscr{L}_1$ where $\bar{V}^{(i)}(t)$ is defined as in Lemma 6. Then $\dot{\Sigma}(t)\bar{V}^{(i)}(t) \in \mathscr{L}_1$.

Proof

Subtracting (1b) from (1a) we get $2\dot{\Sigma}(t) = \operatorname{diag}\{BB' - C'C\}$ where $\operatorname{diag}\{M\}$ contains only the diagonal elements of M. By assumption we have $(BB' - C'C)\bar{V}^{(i)} \in \mathscr{L}_1$. Using this, coupled with the special structure of $\bar{V}^{(i)}(t)$ as in Lemma 6, we have $2\dot{\Sigma}\bar{V}^{(i)} = [\operatorname{diag}(BB' - C'C)]\bar{V}^{(i)} \in \mathscr{L}_1$.

Lemma 8

Suppose $[\Sigma\bar{V}^{(i)} - \bar{V}^{(i)}\bar{\Sigma}] \in \mathscr{L}_1$ and $[\Sigma A\bar{V}^{(i)} - A\bar{V}^{(i)}\bar{\Sigma}] \in \mathscr{L}_1$, then there exists a diagonal matrix $\bar{\Lambda}(t)$ such that $[A\bar{V}^{(i)} - \bar{V}^{(i)}\bar{\Lambda}] \in \mathscr{L}_1$.

Proof

Working columnwise, we want to show that if $(\Sigma - \bar{\sigma}_1 I)V_1 \in \mathscr{L}_1$ and $(\Sigma - \bar{\sigma}_1 I)AV_1 \in \mathscr{L}_1$, then $[AV_1 - \bar{\lambda}_1 V_1] \in \mathscr{L}_1$. We have

$$AV_1 - \bar{\lambda}_1 V_1 = \sum_{i=2}^{n} \bar{\lambda}_i V_i \tag{23}$$

and we wish to show the r.h.s of (23) is integrable. Premultiplying (23) by $(\Sigma(t) - \bar{\sigma}_1 I)$ we obtain

$$(\Sigma(t) - \bar{\sigma}_1 I)(AV_1 - \bar{\lambda}_1 V_1) = \sum_{i=2}^{n} \lambda_i(\Sigma - \bar{\sigma}_1 I)V_i$$

$$\Rightarrow \sum_{i=2}^{n} \bar{\lambda}_i(\Sigma - \bar{\sigma}_1 I)V_i \in \mathscr{L}_1. \tag{24}$$

Now using the distinctness of singular values and results of Lemma 6 we can easily deduce

$$\sum_{i=2}^{n} \bar{\lambda}_i V_i \in \mathscr{L}_1. \qquad \square$$

We now arrive at the main result of the section, Theorem 6. Remarkable in this theorem is that we do *not* need the existence of limits of $A(t)$, $B(t)$, $C(t)$, $\Sigma(t)$ and $\dot{\Sigma}(t)$, as is the case for many asymptotic results in time-varying systems. On the other hand, we do need the following technical assumption, in addition to Assumptions 1 and 2. This assumption ensures the integrability of a matrix whenever it is squarely integrable.

Assumption 3

$\begin{bmatrix} B' \\ \hline C \end{bmatrix} \bar{V}^{(i)} \in \mathscr{L}_1$ whenever

$$\bar{V}^{(i)'}[B \vdots C']\begin{bmatrix} B' \\ \hline C \end{bmatrix}\bar{V}^{(i)} \in \mathscr{L}_1.$$

Theorem 6

Let (A, B, C) be u.b.a.s realization, and suppose Assumptions 1, 2, and 3 hold. Then the asymptotic distinctness of the elements of $\Sigma(t)$ implies the asymptotic distinctness of $\lambda_{\max}(A + A')/2$ from zero, i.e. $S_1 = \phi$.

Proof

Suppose not. Then $S_1 \neq \phi$, and by Assumption 1 there exists a continuous $V^{(i)}$ such that

$$(A + A')V^{(i)} \in \mathscr{L}_1$$

which also implies

$$(A + A')\bar{V}^{(i)} \in \mathscr{L}_1 \tag{25}$$

where $\bar{V}^{(i)} = V^{(i)}P$ with P a unitary matrix. Pre- and postmultiplying $(1(a) + 1(b))$ with $\bar{V}^{(i)'}$ and $\bar{V}^{(i)}$ respectively, we have

$$\bar{V}^{(i)'}(A + A')\Sigma\bar{V}^{(i)} + \bar{V}^{(i)'}\Sigma(A + A')\bar{V}^{(i)} = -\bar{V}^{(i)'}(BB' + C'C)\bar{V}^{(i)}.$$

From (25) it follows that: $\bar{V}^{(i)'}(BB' + C'C)\bar{V}^{(i)} \in \mathscr{L}_1$, and by Assumption 3, we have

$$B'\bar{V}^{(i)} \in \mathscr{L}_1, C\bar{V}^{(i)} \in \mathscr{L}_1. \tag{26}$$

Postmultiplying 1(a) + 1(b) by $\bar{V}^{(i)}$, using (25) and (26) we obtain

$$(A + A')\Sigma\bar{V}^{(i)} \in \mathscr{L}_1. \tag{27}$$

Lemma 5 implies that there exists a continuous diagonal matrix $\bar{\Sigma}(H) \geq \alpha I > 0$ such that

$$[\Sigma\bar{V}^{(i)} - \bar{V}^{(i)}\bar{\Sigma}] \in \mathscr{L}_1 \tag{28}$$

where $\bar{V}^{(i)}$ is defined as in Lemma 6. Lemma 7 implies that $\dot{\Sigma}\bar{V}^{(i)} \in \mathscr{L}_1$. Postmultiplying (1b) by $\bar{V}^{(i)}$ we get

$$[\Sigma A\bar{V}^{(i)} + A'\Sigma\bar{V}^{(i)}] \in \mathscr{L}_1. \tag{29}$$

Adding and subtracting $A\Sigma\bar{V}^{(i)}$ to (29) and using (28) we get

$$[\Sigma A\bar{V}^{(i)} - A\bar{V}^{(i)}\bar{\Sigma}] \in \mathscr{L}_1. \tag{30}$$

From Lemma 8, therefore, there exists a diagonal matrix $\bar{\Lambda}(t)$ such that

$$[A\bar{V}^{(i)} - \bar{V}^{(i)}\bar{\Lambda}] \in \mathscr{L}_1. \tag{31}$$

Partitioning A in (31) in accordance with $\bar{V}^{(i)}$ in Lemma 6, we have

$$\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \bar{V}_1^{(i)} \\ \bar{V}_2^{(i)} \end{bmatrix} - \begin{bmatrix} \bar{V}_1^{(i)} \\ \bar{V}_2^{(i)} \end{bmatrix}\bar{\Lambda}\right) \in \mathscr{L}_1$$

which implies that $A_{21}(t) \in \mathscr{L}_1$. Similarly partitioning equation (25) we get $A_{12}(t) \in \mathscr{L}_1$ and $[A_{11}(t) + A'_{11}(t)] \in \mathscr{L}_1$. Using Wazewski's inequality (Zadeh and Desoer, 1963), we then have

$$0 < \alpha = \exp\int_{t_0}^{t} \lambda_{\min}\left(\frac{A_{11} + A'_{11}}{2}\right) d\tau \leq \|\Phi_{A_{11}}(t, t_0)\|$$

which implies

$$\|\Phi_{A_{11}}(t, t_0)\| \geq \alpha > 0, \ \forall t \geq 0. \tag{32}$$

On the other hand, considering the original system (A, B, C), and subdividing its transition matrix $\dot{\Phi}_A(t, t_0) = A(t)\Phi_A(t, t_0)$, $\Phi_A(t_0, t_0) = I$ accordingly, we get

$$\begin{bmatrix} \dot{\phi}_{11}(t, t_0) & \dot{\phi}_{12}(t, t_0) \\ \dot{\phi}_{21}(t, t_0) & \dot{\phi}_{22}(t, t_0) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \phi_{11}(t, t_0) & \phi_{12}(t, t_0) \\ \phi_{21}(t, t_0) & \phi_{22}(t, t_0) \end{bmatrix} \tag{33}$$

with

$$\begin{bmatrix} \phi_{11}(t_0, t_0) & \phi_{12}(t_0, t_0) \\ \phi_{21}(t_0, t_0) & \phi_{22}(t_0, t_0) \end{bmatrix} = \begin{bmatrix} I_{r \times r} & 0 \\ 0 & I \end{bmatrix}.$$

By a.s since $\|\Phi_A(t, t_0)\| \to 0$ as $t \to \infty$, then

$$\|\phi_{11}(t, t_0)\| \to 0 \text{ as } t \to \infty. \tag{34}$$

Using (33) with its boundary condition, together with the variation of constant formula we have

$$\phi_{21}(t, t_0) = \int_{t_0}^{t} \Phi_{A_{22}}(t, \tau)A_{21}(\tau)\phi_{11}(\tau, t_0)\, d\tau$$

$$\phi_{12}(t, t_0) = \int_{t_0}^{t} \Phi_{A_{11}}(t, \tau)A_{12}(\tau)\phi_{22}(\tau, t_0)\, d\tau$$

$$\phi_{11}(t, t_0) = \Phi_{A_{11}}(t, t_0) + \int_{t_0}^{t} \Phi_{A_{11}}(t, \tau)A_{12}(\tau)\phi_{21}(\tau, t_0)\, d\tau.$$

Taking the norm of the above equations, we get

$$\|\phi_{21}(t,t_0)\| \leq \int_{t_0}^{t} \|A_{21}(\tau)\| \, \mathrm{d}\tau \tag{35}$$

$$\|\phi_{12}(t,t_0)\| \leq \int_{t_0}^{t} \|A_{12}(\tau)\| \, \mathrm{d}\tau \tag{36}$$

$$\|\phi_{11}(t,t_0) - \Phi_{A_{11}}(t,t_0)\| \leq \int_{t_0}^{t} \|A_{12}(\tau)\| \, \mathrm{d}\tau. \tag{37}$$

Considering that $A_{12} \in \mathscr{L}_1$ and $A_{21} \in \mathscr{L}_1$, by taking t_0 sufficiently large, we can make the right-hand sides of (35)–(37) arbitrarily small, but this contradicts (32) and (34). □

Theorem 7

Let (A, B, C) be a u.b.a.s realization, and suppose Assumptions 1, 2, 3 hold. Then the asymptotic distinctness of the elements of $\Sigma(t)$ implies that *every* subsystem (A_{11}, B_1, C_1) is u.b.a.s.

Proof

Combining the results of Theorem 6 and Theorem 3, we have the conclusion of Theorem 7. □

We note that while Assumption 1 is fundamental in the proof of Lemma 5, Assumptions 2 and 3 are only technical assumptions which are satisfied in many cases. They are designed to exclude possible pathological behavior.

4. CONCLUSIONS

This paper is the continuation of the authors previous work (Shokoohi, Silverman and Van Dooren, 1980, 1981, 1983) where a 'balanced' realization for time-variable systems was introduced. Balanced realizations lead to a natural technique for model reduction, using the concepts of controllability and observability. In this paper we have studied stability of the resulting approximation. Several sufficient conditions to guarantee the stability of subsystems of time-variable, periodic and time-invariant systems are given. The idea of taking the subsystem, in the balanced framework, as a reduced model is very appealing and attractive. However, much work needs to be done to further understand the balanced realization, its variations and implications in linear system theory. In particular, more research is desirable to justify that reduced model is in fact a good one. At this point, however, balanced realizations yield a new perspective in view of linear systems and model reduction.

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