c) $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \left(u(t)-u^*(t)\right)^2=0$ a.s.

Proof: Straightforward combination of the techniques introduced in the proof of Theorem 2.1 together with the analysis in [6]. ∇∇∇

Remark 3.1: Theorem 3.1 uses a stochastic approximation type iteration with scalar gain $\mu/r(t-1)$. In practice, much more rapid convergence can be achieved with recursive least-squares type matrix gain sequences [12], [14]. The extension of the proofs given in [14] for least-squares algorithms with $\lambda \neq 0$ is straightforward.

Remark 3.2: Remarks (2.3) to (2.7) also apply *mutatis mutandis* to the stochastic case. For example stochastic adaptive controllers having integral action can be readily designed by weighting the input increment as in (2.30), (2.31), (2.46), and (2.47).

IV. CONCLUSIONS

This paper has established global convergence for adaptive one-step-ahead optimal controllers based on input matching. The key assumption is that the one-step-ahead optimal controller designed using the true system parameters leads to a stable closed-loop system. This is a very natural and obvious requirement for adaptive control using this approach. The results apply to a restricted class of linear systems including all stable nonminimum phase systems and some unstable-nonminimum phase systems. Previous results on the adaptive control of minimum phase system (see, for example, [1]–[7]) can be considered a special case of the current results when λ is taken to be zero. Thus, in practice it seems sensible to use the adaptive controllers described here rather than the usual minimum phase controllers since the parameter λ offers an extra degree of freedom in the design.

REFERENCES

[1] A. Feuer and A. S. Morse, "Adaptive control of single input, single output linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 557–570, Aug. 1978.
[2] B. Egardt, *Stability of Adaptive Controllers*. Berlin, Germany: Springer Verlag, 1979.
[3] G. C. Goodwin, P. J. Ramadge, and P. E. Caines, "Discrete time multivariable adaptive control," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 449–456, June 1980.
[4] A. S. Morse, "Global stability of parameter-adaptive control systems," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 433–439, June 1980.
[5] K. S. Narendra and Y-H. Lin, "Stable discrete adaptive control," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 456–461, June 1980.
[6] G. C. Goodwin, P. J. Ramadge, and P. E. Caines, "Discrete time stochastic adaptive control," *SIAM J. Contr. Optimiz.*, to be published.
[7] G. C. Goodwin and R. S. Long, "Generalization of results on multivariable adaptive control," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 1241–1245, Dec. 1980.
[8] W. R. E. Wouters, "Parameter adaptive regulatory control for stochastic SISO systems: Theory and an application," in *Proc. IFAC Symp. on Stochastic Control* (Budapest, Hungary), pp. 287–296, Sept. 1974.
[9] D. W. Clarke and J. P. Gawthrop, "Self-tuning controller," *Proc. Inst. Elec. Eng.*, vol. 122, no. 9, pp. 929–934, Sept. 1975.
[10] C. R. Johnson, Jr. and E. Tse, "Adaptive implementation of one-step-ahead optimal control via input matching," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 865–872, Oct. 1978.
[11] K. J. Astrom and B. Wittenmark, "On self-tuning regulators," *Automatica*, vol. 9, pp. 185–199, Mar. 1973.
[12] C. R. Johnson, Jr., "On single stage optimal control," in *Proc. 1978 IEEE Southeastcon* (Atlanta, GA), pp. 511–514, Apr. 1978.
[13] B. Wittenmark and P. K. Rao, "Comments on 'Single step versus multistep performance criteria for steady state SISO system,'" *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 140–141, Feb. 1979.
[14] K. S. Sin and G. C. Goodwin, "Stochastic adaptive control using modified least squares algorithms," to be published.
[15] R. J. Evans and R. Betz, "Discrete time adaptive control: deterministic results," Dep. Elect. Eng., Univ. Newcastle, NSW, Australia, Tech. Rep., Feb. 1980.
[16] C. R. Johnson, Jr., G. C. Goodwin, and K. S. Sin, "The global convergence of direct adaptive input matching control of some nonminimum phase plants," in *Proc. 18th Allerton Conf. Communication, Control, and Computing*, Monticello, IL, Oct. 1980.
[17] G. Kreisselmeier, "Adaptive control via adaptive observation and asymptotic feedback matrix synthesis," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 717–722, Aug. 1980.
[18] G. C. Goodwin and K. S. Sin, "Adaptive control of nonminimum phase systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 478–483, Apr. 1981.
[19] G. C. Goodwin and K. S. Sin, *Adaptive Filtering, Prediction and Control*, to be published.
[20] G. C. Goodwin, K. S. Sin, and K. K. Saluja, "Stochastic adaptive control and prediction: The general delay-colored noise case," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 946–950, Oct. 1980.

A System Theoretic Interpretation for GCD Extraction

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Abstract—**A new algorithm for the GCD extraction of a set of polynomial matrices is given. The approach is based on system theoretic notions of feedback.**

I. INTRODUCTION

A basic problem in linear system theory is that of finding the greatest common right divisor (or left divisor) of a set of polynomial matrices. Mathematically this can be reduced to the following. Given an $m \times r$ polynomial matrix

$$N(s)=N_0+N_1 s+\cdots+N_k s^k \tag{1.1}$$

find polynomial matrices $P(s), Q(s)$ such that

$$N(s)=P(s)Q(s) \tag{1.2}$$

and, for the case $N(s)$ has normal rank r, the matrix $P(s)$ has rank r for all s (i.e., has no multivariable zeros).

Two main classes of algorithms have been proposed for solving the problem. The more classical approaches are based on the Euclidean division algorithm as in [1] and [5], for example. More recently, several algorithms have been proposed based on real matrix operations on the coefficient matrices N_i [3], [4], [2]. In this paper, we present a new approach which while essentially being of the second type is novel in that it relies heavily on an isomorphism between polynomial matrices and a related state space model.

II. STATE-SPACE APPROACH

The basis of our approach is the conversion of $N(s)$ initially to a proper rational matrix via the mapping $s \to 1/\mu$. With this change of variables we obtain from (1.1)

$$\overline{N}(\mu)=N(1/\mu)=N_0+N_1 1/\mu+\cdots+N_k 1/\mu^k. \tag{2.1}$$

This proper rational matrix has the obvious *controllable* realization $W=\{A,B,C,D\}$ with

$$A=\begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}$$

$$C=\begin{bmatrix} N_k & N_{k-1} & \cdots & N_1 \end{bmatrix}, \quad D=[N_0]. \tag{2.2}$$

Notice that the A matrix of the realization of $\overline{N}(\mu)$ is nilpotent (all eigenvalues at $\mu=0$) and that this fact is sufficient to ensure that $\overline{N}$ is polynomial in $1/\mu$ (i.e., polynomial after the transformation $1/\mu=s$). We are interested first in the effect of a state feedback operation F on the system (2.2)

$$W \xrightarrow{F} W_F=\{A+BF, B, C+DF, D\}. \tag{2.3}$$

Lemma 2.1: For any F, $W_F=\{A+BF,B,C+DF,D\}$ has a transfer function $\overline{N}_F(\mu)$ that satisfies the relationship

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$$\bar{N}(\mu)=\bar{N}_F(\mu)\bar{Q}_F(\mu) \tag{2.4}$$

where $\bar{Q}_F$ is *polynomial* in $1/\mu$ and invertible.

Proof: The result of a feedback operation F on a system $W=\{A,B,C,D\}$ corresponding to $\bar{N}(\mu)$ can also be obtained by postmultiplying the matrix $\bar{N}(\mu)$ with $I+F(\mu I-A-BF)^{-1}B$. Inverting this matrix with the classical rule, we have

$$\bar{N}(\mu)=\bar{N}_F(\mu)\left(I-F(\mu I-A)^{-1}B\right)=\bar{N}_F(\mu)\bar{Q}_F(\mu).$$

The matrix $\bar{Q}_F$ is clearly invertible and is polynomial in $1/\mu$ since A is nilpotent.

The result shows that any state feedback operation on the system (2.2) 'corresponds' to a factorization of the type (2.4) where both $\bar{N}$ and $\bar{Q}_F$ are polynomial in $1/\mu$. The remaining problem is to *choose* F such that N_F is also polynomial in $1/\mu$ and has full rank for all μ.

As the following lemma shows, it is easy to ensure that $\bar{N}_F$ is polynomial.

Lemma 2.2: If the feedback F is such that the observable modes of W_F are all lying at $\mu=0$, then $\bar{N}_F$ in (2.4) is polynomial in $1/\mu$.

Proof: Clearly, if $W_F=\{A+BF,B,C+DF,D\}$ has all its observable modes at $\mu=0$ there exists an equivalent *minimal* realization $\tilde{W}_F=\{\tilde{A},\tilde{B},\tilde{C},D\}$ for $\bar{N}_F(\mu)$ with $\tilde{A}$ nilpotent, which means $\bar{N}_F(\mu)$ is polynomial in $1/\mu$. □

Notice that the system (2.2) is controllable. Therefore, all modes are freely assignable by state feedback and it is always possible to construct a feedback satisfying the previous lemma. To each such feedback there corresponds a factorization

$$\bar{N}(\mu)=\bar{N}_F(\mu)\cdot\bar{Q}_F(\mu)$$

or also, using the fact that

$$\bar{N}(1/s)=N(s),\quad \bar{N}_F(1/s)=N_F(s)\quad\text{and}\quad \bar{Q}_F(1/s)=Q_F(s),$$

$$N(s)=N_F(s)\cdot Q_F(s) \tag{2.5}$$

where all three matrices are polynomial.

The remaining problem is to choose F so that $N_F(s)$ also has no multivariable zeros. To do this we first need to recall the connection between multivariable zeros and system observability under feedback.

Let $W=\{A,B,C,D\}$ represent an arbitrary linear system [not necessarily the one specified by (2.2)] and as before let $W_F=\{A+BF,B,C+DF,D\}$. Also, let

$$\mathcal{O}^i=\begin{bmatrix}C\\CA\\\vdots\\CA^{i-1}\end{bmatrix}$$

denote the i-observability matrix of W and $\mathcal{O}^i_F$ the corresponding matrix for W_F.

Definition 2.3 [6]: A system W is said to be *strongly observable* iff W_F is observable for all F (i.e., $\mathcal{O}^n_F$ has full column rank for all F, where n is the order of the system W). □

An important property for our purposes is the following.

Lemma 2.4 [6]: If a system is strongly observable, then its transfer function matrix has no multivariable zeros. □

If a system is *not* strongly observable, then a feedback law exists which makes it as unobservable as possible in the following sense (let $\mathfrak{N}$ denote "null space of").

Lemma 2.5 [6]: For any system $W=\{A,B,C,D\}$, there exists a F_0 such that $\mathfrak{N}(\mathcal{O}^n_F)\subset\mathfrak{N}(\mathcal{O}^n_{F_0})$ for all F. For any such F_0, W_{F_0} is termed *maximally unobservable*. □

We shall see in the next section that in our context, the problem of GCD extraction reduces essentially to that of finding an appropriate F_0.

III. GCD Extraction

Based on the relationships developed in the previous section we now show that the GCD extraction problem is equivalent, under mild conditions, to that of finding a specific type of feedback for the system (2.2). The restriction, which will be removed later, is that $N(s)$ has no zero at $s=0$ (i.e., N_0 in (1.1) has full column rank r).

Theorem 3.1: Let rank $N(s)=r$ at $s=0$ and let $W=\{A,B,C,D\}$ be the realization of $N(1/\mu)$ as defined by (2.2). If F_0 is such that W_{F_0} is maximally unobservable and has all its observable eigenvalues at $\mu=0$, then

$$N(s)=N_{F_0}(s)Q_{F_0}(s)$$

where

$$Q_{F_0}(s)=I-F_0(1/sI-A)^{-1}B$$

and

$$N_{F_0}(s)=(C+DF_0)(1/sI-A-BF_0)^{-1}B+D$$

are polynomial in s and $N_{F_0}(s)$ has no multivariable zeros.

Proof: Since W_{F_0} is maximally unobservable, $\bar{N}_{F_0}(\mu)$ has no multivariable zeros or, equivalently, has full rank for all (finite) values of μ. By Lemma 2.2, since the observable modes of W_{F_0} are at $\mu=0$, $\bar{N}_{F_0}(\mu)$ is polynomial in $1/\mu$. Hence, $N_{F_0}(s)=\bar{N}_{F_0}(1/s)$ is polynomial in s and has full rank for all s save possibly at $s=0$ ($\mu=\infty$). But $N_{F_0}(0)=N(0)$ has full rank by assumption so that $N_{F_0}(s)$ has full rank for all s. The form of $Q_{F_0}(s)$ and the fact that it is polynomial have already been established in Lemma 2.1.

In the remainder of this section we shall be concerned with finding an algorithm to construct a F_0 satisfying the requirements of this theorem. Conceptually, this is no problem. We know [6] how to characterize all feedback laws making a system maximally unobservable and within this class pole placement could be performed to place the remaining observable modes at zero. However, this approach is not very efficient algorithmically. A major contribution of this paper is an algorithm which is considerably simpler than the "obvious" solution. The key to this algorithm is the construction of a special right inverse for the observability matrix of an arbitrary maximally unobservable system W_{F_0}, as described below.

Let L be a matrix whose rows form a basis for the row space of $\mathcal{O}^n_{F_0}$ and let L^* be a right inverse for L such that L^*L is upper triangular (this can always be obtained as shown in the next section). Further, define

$$F_*=-D^+C(I-L^*L)$$

where D^+ is any left inverse for D (which exists since $D=N_0$ has full column rank). The major characterization is then given by the following.

Theorem 3.2: Let W satisfy the conditions of Theorem 3.1 and let L^*, F_* be as defined above. Then

i) W_{F_*} is maximally unobservable;

ii) $N_{F_*}(s)=CL^*(1/sI-LAL^*)^{-1}LB+D$ is polynomial in s and has no multivariable zeros.

Several preliminary results will be required to prove this theorem. The first is a characterization of the class of all feedback laws which make a system maximally unobservable.

Lemma 3.3 [6]: If D has full column rank then W_F is maximally unobservable iff $F=-D^+C+HL$ for some H.

Observe that F_* certainly has the above form with $H=D^+CL^*$. The following lemma shows the role of L^*.

Lemma 3.4: Let A have the form (2.2) then LAL^* is nilpotent.

Proof: LAL^* is nilpotent iff $(LAL^*)^d=0$ for some finite d. But $(LAL^*)^d=L(AL^*L)^{d-1}AL^*$. Since A and L^*L are both upper triangular and since A has all zeros on its diagonal, AL^*L is nilpotent. Therefore, there clearly is a d such that $(AL^*L)^{d-1}=0$ and thus also $(LAL^*)^d=0$.

We note that an arbitrary right inverse L^- will not yield the same result as Lemma 3.4. To prove Theorem 3.2 we now observe that by an obvious coordinate transformation

$$\tilde{W}_{F_*}=\left\{L\left(A+BF_*\right)L^*,LB,\left(C+DF_*\right)L^*,D\right\}$$

is equivalent to W_{F_*} and is strongly observable. Note, however, that $(I-L^+L)L^+=0$ for *any* right inverse of L. Hence,

$$\tilde{W}_{F_*}=\{LAL^*,LB,CL^*,D\}$$

and by Lemma 3.4 LAL^* has all of its eigenvalues at $\mu=0$. The conclusion of Theorem 3.2 then follows immediately.

IV. ALGORITHM

Two remaining details that have to be discussed are how to ensure that N_0 in (1.1) is left invertible (see Theorem 3.1) and how to construct L^* (see Theorem 3.2). We will be able to give the general algorithm after clarifying these two points. A full column rank r for N_0 can be obtained in either of two ways.

1) Transform $s \to s'+\alpha$, then $N(s'+\alpha)=N'(s')$ has constant term $N_0'=N(\alpha)$ which has full rank for almost any α [namely, all α that are not a zero of $N(s)$]. By choosing $|\alpha|<1$ this transformation does not give any numerical problem.

2) Apply the structure algorithm to make the System 2.2 "invertible" (i.e., to make D invertible see [8]). The operations performed on the system 2.2 can as well be performed directly on the polynomial matrix $N(s)$ [3], [9]. This algorithm is based on the following idea. If $N(s)$ has constant term N_0 with rank $\rho<r$ then construct a unitary transformation U such that $N_0U=[\underbrace{\tilde{N}_0}_{\rho}|0]$. Then the $(r-\rho)$ rightmost columns of $N(s)U$ can be divided by s resulting in a new polynomial matrix $N'(s)$. This is repeated until $N'(s)$ has constant term N_0' with rank r. It is easy to check that

$$N(s)=N'(s)Q_0(s) \tag{4.1}$$

where $Q_0(s)$ is a polynomial matrix containing the zeros of $N(s)$ at $s=0$. The algorithm is stable since it uses only unitary operations and terminates after a finite number of steps [8].

For the construction of L^* notice that a row transformation of L does not affect the problem. Indeed, if $LL^*=I$ and L^*L is upper triangular, then this is also true for $L_1=RL$ and $L_1^*=L^*R^{-1}$. And since one can always construct a (unitary) R such that L_1 is in echelon form, we may assume, without loss of generality, that L is already in this form. Solving L^* from $LL^*=I$ by back substitution and filling in $l_{ij}^*=0$ whenever the element is not uniquely determined we obtain a "dual" echelon form for L^*. The product L^*L is then upper triangular. This is illustrated in the example (nonzero elements are marked by an $\times$)

$$L=\begin{bmatrix} \times & \times & \times & \times & \times & \times \\ & \times & \times & \times & \times & \times \\ & & & \times & \times & \times \\ & & & & & \times \end{bmatrix},$$

$$L^*=\begin{bmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & & 0 & 0 \\ & & \times & \times \\ & & & 0 \\ & & & \times \end{bmatrix},$$

$$L^*L=\begin{bmatrix} \times & \times & \times & \times & \times & \times \\ & \times & \times & \times & \times & \times \\ & & 0 & 0 & 0 & 0 \\ & & & \times & \times & \times \\ & & & & 0 & 0 \\ & & & & & \times \end{bmatrix}.$$

The complete algorithm now goes as follows.

i) Extract the multivariable zeros at $s=0$ from $N(s)$ by the factorization $N(s)=N'(s)Q_0(s)$ (see 4.1), so that N_0' has full rank. The matrix $Q_0(s)$ will be part of the GCD. We only have to remove the remaining zeros from $N'(s)$. Transform $VN'(s)=N''(s)$ such that $VN_0'=\begin{bmatrix}\bar{D}\\0\end{bmatrix}$ where V is unitary and $\bar{D}$ invertible. This left transformation does not affect the GCD since the latter one is a *right* factor.

ii) Let the system $W=\{A,B,C,D\}$ for $N''(1/\mu)$ be partitioned as

$$\begin{array}{cc} \begin{bmatrix} 0 & I & \cdots & 0 \\ 0 & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \\ \begin{bmatrix} \bar{C}_k & \cdots & \bar{C}_1 \\ \tilde{C}_k & \cdots & \tilde{C}_1 \end{bmatrix} & \begin{bmatrix} \bar{D} \\ 0 \end{bmatrix} \end{array} \Big\} \, r. \tag{4.2}$$

Then construct $F_0=-D^+C \triangleq [\bar{A}_k \ \cdots \ \bar{A}_1]$ where $\bar{A}_i=-\bar{D}^{-1}\bar{C}_i$ to make the system W_{F_0} maximally unobservable.

iii) Construct a basis L (in echelon form) for $\mathcal{O}_{F_0}^n$. Since

$$A_{F_0}=A+BF_0=\begin{bmatrix} 0 & I & \cdots & 0 \\ \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \\ \bar{A}_k & & \cdots & \bar{A}_1 \end{bmatrix}$$

$$C_{F_0}=C+DF_0=\begin{bmatrix} 0 & \cdots & 0 \\ \tilde{C}_k & \cdots & \tilde{C}_1 \end{bmatrix} \tag{4.3}$$

are both sparse matrices an economic scheme can be developed to compute $C_{F_0}A_{F_0}^i$. Moreover, if for some β we notice that rank $\mathcal{O}_{F_0}^{\beta}=$ rank $\mathcal{O}_{F_0}^{\beta-1}$ we know we have a basis for $\mathcal{O}_{F_0}^n$ even if $\beta<n$ (see [6]). While recursively building up a basis in echelon form, this rank test can easily be performed.

iv) Construct L^* and $F_*=F_0(I-L^*L)$.

The GCD is now immediately given in a polynomial form because of the special shape of A and B in its realization. If $F_* \triangleq [-Q_k \ \cdots \ -Q_1]$, then

$$Q_{F_*}(s)=I+Q_1s+\cdots+Q_ks^k$$

and the overall GCD is $Q(s)=Q_{F_*}(s)\cdot Q_0(s)$.

Comments:

1) If we are also interested in the 'remainder' of the GCD extraction, we have to compute the matrices $CL^*(LAL^*)^iLB$.

2) The time consuming step in this algorithm is the construction of L in echelon form. This can be done using unitary transformations. The number of computations is of the order of $[\beta(m-r)]^2rk$.

An example may clarify the procedure.

Example:

$$N(s)=\begin{bmatrix} s-1 & s-1 \\ -2s^2-4s & 2s+4 \\ s^2+3s-1 & -3 \\ -3s & s^2+2 \end{bmatrix}$$

$$=\begin{bmatrix} -1 & -1 \\ 0 & 4 \\ -1 & -3 \\ 0 & 2 \end{bmatrix}+\begin{bmatrix} 1 & 1 \\ -4 & 2 \\ 3 & 0 \\ -3 & 0 \end{bmatrix}s+\begin{bmatrix} 0 & 0 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}s^2.$$

i)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix}N(s)=N'(s)=\begin{bmatrix} -1 & -1 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} 1 & 1 \\ -4 & 2 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}s+\begin{bmatrix} 0 & 0 \\ -2 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}s^2.$$

ii)

$$W=\begin{array}{cc} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ -2 & 0 & -4 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix} & \begin{bmatrix} -1 & -1 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad F_0=\left[\begin{array}{cc|cc} -\frac{1}{2} & 0 & 0 & 1\frac{1}{2} \\ \frac{1}{2} & 0 & 1 & -\frac{1}{2} \end{array}\right].$$

iii)

$$A_{F_0}=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{2} & 0 & 0 & 1\frac{1}{2} \\ \frac{1}{2} & 0 & 1 & -\frac{1}{2} \end{bmatrix} \quad C_{F_0}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix}.$$

$$L=[1 \quad 1 \quad -1 \quad -1] \qquad (\text{with } \beta=2)$$

iv)

$$L^*=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\quad W_{F_*}=\begin{bmatrix}0\\0\\-2\\0\\1\end{bmatrix}\begin{bmatrix}-1&-1\\-1&-1\\0&4\\0&0\\0&0\end{bmatrix},\quad F_*=\begin{bmatrix}0&\frac{1}{2}&-\frac{1}{2}&1\\0&-\frac{1}{2}&1\frac{1}{2}&0\end{bmatrix}.$$

Result:

$$Q_{F_*}=\begin{bmatrix}\frac{1}{2}s+1&-\frac{1}{2}s^2-s\\-1\frac{1}{2}s&\frac{1}{2}s^2+1\end{bmatrix}\quad N_{F_*}=\begin{bmatrix}-1&-1\\2s&2s+4\\0&0\\-1&-1\end{bmatrix}.$$

Note that for illustrative simplicity we only use fractional numbers and elementary transformations in this example. For the implementation of this method in a program, the use of unitary transformation is recommended where possible (namely, for the construction of $\bar{D}$ and of the basis in echelon form of L). This will indeed improve the numerical stability of the algorithm. Unfortunately, one still has to invert $\bar{D}$ and L and these steps are poorly conditioned when $\bar{D}$ and L have a large condition number [13]. Only part of this is due to the possible bad conditioning of the GCD problem itself. The conditioning of $\bar{D}$ is indeed independent of the conditioning of the GCD problem since $\bar{D}$ can be modified by a "shift" α as described in Section IV and since the inversion of $\bar{D}$, e.g., does not appear in other GCD extraction methods (see [3], [5], [10], [11], [14]). On the other hand, it is exactly the inversion of $\bar{D}$ that allows us to perform the rank test required in the GCD problem on a smaller matrix than, e.g., other methods [3], [4], [11], [14]. One should note here that the so-called "fast orthogonalization method" used in [14] is in fact a type of Schmidt orthogonalization procedure without reorthogonalization and is therefore unstable [13]; reorthogonalization would again yield a speed which is comparable to the other methods mentioned above. Finally, we want to mention that some of the algorithms for the minimal design problem can be used to solve the GCD problem as well (see [11], [12], [14] for further references), which makes the task of comparing methods for GCD extraction rather lengthy. The main advantages of the method presented here are its simplicity and its ease for implementation. It should also be noted that the obtained factors Q_{F*} and N_{F*} have degree in s lower or equal to the corresponding degree of N (namely k). This is not always the case for other methods [1], [3], [5], [10], [11], [14].

V. Conclusion

After the initial submission of this paper, we received a report of Emre's [10] which makes an alternate connection between common divisors and (A, B)-invariant subspaces (closely related to the null space of L for an appropriate C). However, detailed algorithms of the type given here are not provided. Further, connections between the two approaches should be explored, however, and a full numerical comparisons between the various GCD extraction methods proposed in the literature should be performed.

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References

[1] C. C. McDuffee, *The Theory of Matrices*. New York: Chelsea, 1946.
[2] E. Emre and L. Silverman, "New criteria and system theoretic interpretations for relatively prime polynomial matrices," in *Proc. IEEE Conf. Decision Contr.*, pp. 1244–1249, 1976; also in *IEEE Trans. Automat. Contr.*, vol. AC-22, pp. 239–242, 1977.
[3] S. Kung, T. Kailath, and M. Morf, "A generalized resultant matrix for polynomial matrices," in *Proc. IEEE Conf. Decision Contr.*, pp. 892–895, 1976.
[4] B. D. O. Anderson and E. I. Jury, "Generalized Bezoutian and Sylvester matrices in multivariable linear control," in *Proc. IEEE Conf. Decision Contr.*, pp. 901–906, 1976.
[5] W. A. Wolovich, *Linear Multivariable Systems*. New York: Springer-Verlag, 1974.
[6] L. Silverman, *Discrete Riccati Equations: Alternative Algorithms, Asymptotic Properties and System Theory Interpretations, in Control and Dynamic Systems*. New York: Academic, 1976.
[7] B. C. Moore and L. Silverman, "A time domain characterization of the invariant factors of a system transfer function," in *Proc. JACC*, 1974.
[8] L. Silverman, "Inversion of multivariable linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-14, pp. 270–276, 1969.
[9] P. Van Dooren, P. Dewilde, and J. Vandewalle, "On the determination of the Smith–McMillan form of a rational matrix from its Laurent expansion," *IEEE Trans. Circuits Syst.*, vol. CAS-26, pp. 180–189, Mar. 1979.
[10] E. Emre, "Nonsingular factors of polynomial matrices and (A, B)-invariant subspaces," Memo COSOR 78-12, Dep. Math., Eindhoven Univ. Technol., May 1978; also in *SIAM J. Contr.*, vol. 18, pp. 288–296, May 1980.
[11] R. Bitmead, S. Kung, B. Anderson, and T. Kailath, "Greatest common divisor via generalized Sylvester and Bezout matrices," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 1043–1407, Dec. 1978.
[12] G. Forney, "Minimal bases of rational vector spaces with applications to multivariable linear systems," *SIAM J. Contr.*, vol. 13, pp. 493–520, May 1975.
[13] J. Wilkinson, *The Algebraic Eigenvalue Problem*. London: Oxford Univ. Press, 1965.
[14] S. Kung, T. Kailath, and M. Morf, "Fast and stable algorithms for minimal design problem" in *Proc. 4th IFAC Symp.*, Pergamon, London, 1979.

Minimal-Time Minimal-Order Deadbeat Regulator with Internal Stability

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Abstract—**This paper is concerned with the output deadbeat (finite settling time) regulation with internal stability for linear discrete-time multivariable systems. The two cases are treated separately; the one is the regulation by state feedback and the other is the regulation through function observers. For both cases, the basic solvability conditions, the minimal settling time and the characterizations of the minimal-time regulators are derived. Throughout the paper, the well-posedness condition plays a fundamental role. A deterministic version of the so-called separation theorem in LQG problem is derived, namely, the minimal-time state-feedback regulator combined with the minimal-time minimal-order function observer yields the overall minimal-time minimal-order regulator by output feedback.**

I. Introduction

Since the early stage of the development of the sampled-data control theory, the deadbeat regulation (the finite settling time control) has been regarded as a simple and effective control policy [1]. In the framework of modern state-space approach, a number of contributions have been made on the analysis and synthesis of multivariable deadbeat regulators [2]–[5]. These works were mainly concerned with the *state* deadbeat regulation, the deadbeat regulation for the entire state vector. It is only recently that the *output* deadbeat regulation began to be investigated under the influence of Wonham–Morse geometric approach [6]. Leden [7] derived a solvability condition for the problem of output deadbeat regulator showing that the minimal-time deadbeat regulator was constructed through the cancellation of all the transmission zeros. He did not give a clear answer for the stability problem induced by the cancellation of unstable transmission zeros. Akashi and Imai [8] extended this result to the case of output feedback with some stability considerations. They derived an elegant geometric characterization of the minimal settling time. However, their formulation of stability excludes all the systems with nondecaying external signals (steps, ramps, sinusoids, etc.). They also considered the deadbeat regulation from the viewpoint of disturbance decoupling [9].

In this paper, we consider the problem of output deadbeat regulation with internal stability in its full generality. The results of [7], [8] are extended in the following lines: "stability" in [8] is replaced by "internal

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