AN ORTHOGONAL METHOD FOR THE CONTROLLABLE SUBSPACE OF A PERIODIC SYSTEM

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ABSTRACT

We describe a method for computing the controllable subspace of a linear periodic discrete time system. The method is based on the ordered periodic Schur form [1] of a matrix sequence A_i, $i = 0, \ldots, K-1$, and proceeds by reducing the state equation to a convenient form in which the controllable/uncontrollable states are clearly displayed. Its attractive features are simplicity, numerical accuracy and stability.

1. Introduction

A basic problem connected with linear systems is to compute the controllable subspace. Consider the following discrete-time system

$$x_{k+1} = A_k x_k + B_k u_k, \tag{1}$$

where $A_k \in \mathcal{C}^{n\times n}$, $B_k \in \mathcal{C}^{n\times m}$ are known periodic matrices of integer period K, i.e.,

$$A_{k+K} = A_k,\ B_{k+K} = B_k,\ \forall\, k \in \mathcal{Z};$$

and x_k, u_k are vectors of states and inputs respectively.

For the special case of *time-invariant* systems (period $K = 1$), this problem has been studied extensively, beginning with the work of Kalman – see for instance [2], [3], [4], [5]. We know that if the pair (A, B) has the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \tag{2}$$

then the modes of A_{22} are *uncontrollable* – they are neither affected by the input, nor by the modes of A_{11}.

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Since uncontrollable modes do not contribute to the input/output description of a dynamical system, we can *prune* the state vector to retain only controllable modes. This does not change the transfer function realized by the state equation

$$x_{k+1} = Ax_k + Bu_k. \tag{3}$$

In fact, this constitutes a valid procedure for finding a minimal realization of a given transfer function.

Let us stay a little bit longer with the time-invariant case, since it gives valuable insight into the periodic ($K > 1$) case. For a general pair (A, B) not having the transparent structure of (2), we would first find a (non-singular) state transformation to put (3) in a more convenient form, one in which A and B are as in (2). Such a state transformation always exists [6, p 130], and in fact, a numerically attractive procedure [5] uses only *unitary* transformations.

In this paper, we carry this basic idea through for periodic ($K > 1$) systems, and describe a general method to obtain the controllable subspace. The algorithm is based on the ordered periodic Schur form [1] of the matrix sequence A_i, $i = 0, \ldots, K-1$, and proceeds by transforming the state equation (1) to a convenient form in which the controllable/uncontrollable states are readily distinguished. This operation is carried out by orthogonal state transformations only, and the algorithm is numerically stable.

2. Linear periodic systems

2.1 Some preliminaries

Throughout this paper, we denote the set of integers and complex numbers by $\mathcal{Z}$ and $\mathcal{C}$ respectively. We write A^* for the conjugate-transpose of the matrix A, and A' for its transpose. We will denote the linear periodic discrete-time system under consideration by Σ. We assume that Σ is represented by equation (1).

The state-transition matrix for (1) is given by

$$\Phi(k,\ell) = \begin{cases} I & k=\ell \\ A_{k-1}A_{k-2}\cdots A_{\ell+1}A_\ell & k>\ell \end{cases}$$
$$\Phi(k,\ell) \text{ undefined for } k<\ell.$$

Using periodicity of A_k, it is easy to verify the following properties of $\Phi(k,\ell)$:

$$\begin{aligned} \Phi(\ell+kK,\ell) &= \Phi(\ell+K,\ell)^k \\ \Phi(\ell,\ell-kK) &= \Phi(\ell,\ell-K)^k \\ \Phi(k+K,\ell+K) &= \Phi(k,\ell)\,\forall k\geq\ell. \end{aligned}$$

The state-transition matrix over one period (starting at time i) is known as the *monodromy matrix* (at time i). We denote it by

$$\Psi_i := \Phi(i+K,i).$$

The monodromy matrix Ψ_i is non-singular if and only if A_k are, i.e., iff the system is reversible. The eigenvalues of Ψ_i are called the *characteristic multipliers* of the system (1). They are independent of i, since all Ψ_i have the same eigenvalues.

2.2 Time-invariant reformulation

In many instances, problems involving periodic systems can be tackled by viewing the periodic system as a time-invariant system. An advantage of this approach is that known results for time-invariant systems can then be immediately invoked. The K-periodic system Σ given by (1) has K associated time-invariant representations. These are, for $s=0,1,\ldots K-1$:

$$\theta_s(\ell+1) = \Psi_s\cdot\theta_s(\ell) + G_s\cdot v_s(\ell), \qquad (4)$$

where

$$\begin{aligned} \theta_s(\ell) &= x_{s+\ell K}, \\ \Psi_s &= \Phi(s+K,s), \\ G_s &= [\Phi(s+K,s+1)B_s \ \cdots \ B_{s+K-1}], \\ \text{and } v_s(\ell) &= \begin{bmatrix} u_{s+\ell K} \\ u_{s+\ell K+1} \\ \vdots \\ u_{s+(\ell+1)K-1} \end{bmatrix}. \end{aligned}$$

2.3 Coordinate transformations

It must be borne in mind that equation (1) is not the only (periodic) representation for Σ. We can let $x_k = T_k\tilde{x}_k$ in (1), where T_k is any non-singular periodic matrix, and arrive at the following alternative periodic realization

$$\begin{aligned} \tilde{x}_{k+1} &= \tilde{A}_k\tilde{x}_k + \tilde{B}_k u_k, \qquad (5) \\ \text{where} \quad \tilde{A}_k &= T_{k+1}^{-1}A_kT_k, \\ \tilde{B}_k &= T_{k+1}^{-1}B_k. \end{aligned}$$

Such a transformation of the state-space merely alters 'book-keeping'. It does not affect the characteristic multipliers, or structural properties like reachability or controllability. The pair $(\tilde{A}_k,\tilde{B}_k)$ in (5) is said to be *algebraically equivalent* to the pair (A_k,B_k) in (1).

Much simplicity can be gained by representing Σ in appropriate state coordinates. For instance, it turns out that just as in the time-invariant case ($K=1$), there exist some representations in which the controllable/uncontrollable states can be read off by inspection [7]. Given a particular representation (1), it is clearly worthwhile, for our purpose of finding the controllable subspace, to look for a coordinate transformation T_k which will put the state equation in such a nice form.

However, from a numerical point of view, just any T_k that accomplishes this task will not do, because it might be ill-conditioned with respect to inversion. For this reason, a favored class of transformations is that of unitary T_k. In this work, we consider unitary T_k which put the A_k in triangular form, while implicitly computing the Schur form of the monodromy matrices Ψ_i. The existence of such T_k is guaranteed by the following theorem:

Theorem 1 (Periodic Schur decomposition) *Given $n\times n$ matrices A_i, $i=0,1,\ldots,K-1$, there exist $n\times n$ unitary matrices T_i, $i=0,1,\ldots,K-1$, such that*

$$\begin{aligned} \tilde{A}_0 &= T_1^*A_0T_0, \\ \tilde{A}_1 &= T_2^*A_1T_1, \\ &\vdots \\ \tilde{A}_{K-2} &= T_{K-1}^*A_{K-2}T_{K-2}, \\ \text{and } \tilde{A}_{K-1} &= T_0^*A_{K-1}T_{K-1} \end{aligned}$$

is each upper-triangular. Moreover, T_k can be chosen so that the diagonal elements (eigenvalues) of the products $(\tilde{A}_{i+K-1}\cdots\tilde{A}_{i+1}\tilde{A}_i)$ appear in any desired order.

Proof: See [1]. A constructive proof, as well as a numerical algorithm on the lines of the classical QR algorithm, is described therein. ∎

Note that a (unitary) similarity transformation with T_i puts the monodromy matrix Ψ_i in Schur form:

$$T_i^*\Psi_iT_i = T_i^*A_{i+K-1}\cdots A_iT_i = \tilde{A}_{i+K-1}\cdots\tilde{A}_i := \tilde{\Psi}_i.$$

In other words, the periodic Schur decomposition really computes the Schur form of Ψ_i. However, it does so *implicitly,* without ever forming the matrix products! The algorithm described in [1] works directly on the A_i matrices, and reduces them to upper-triangular form. This results in lesser computation, and greater accuracy.

We mention here that, with minor modifications, the periodic Schur decomposition [1] has a real-matrix version also, where all A_k are reduced to upper triangular form, except for one which is made quasi upper triangular, viz., with possibly 2×2 diagonal blocks. Moreover, T_k can now be chosen so that the 1×1 and 2×2 blocks of the product $\tilde{A}_{i+K-1} \cdots \tilde{A}_{i+1}\tilde{A}_i$ appear in any desired order.

2.4 Standard controllable form

The definition of controllability of Σ is standard, so we skip it here. It is known that the dimension of Σ's controllable subspace is constant (or time-invariant) [8]. Thus, Σ is controllable if and only if it is controllable at an arbitrary time instant, say $k = 0$. The following lemma gives a simple criterion for controllability.

Lemma 1 (Controllability)
Σ is controllable at time s iff for each characteristic multiplier $\lambda \neq 0$, rank $[\lambda I - \Psi_s \;\; G_s] = n$.

Proof: Follows from the equivalence of systems (1) and (4). See also theorem 4 of [8]. ∎

We now list some observations regarding controllability and lemma 1.

- Lemma 1 is the usual PBH test for the equivalent time-invariant system (4). Thus (1) is controllable iff (4) is.
- If the matrix in lemma 1 is not of full rank for some eigenvalue $\lambda \neq 0$ of Ψ_s, then that λ is an uncontrollable eigenvalue or 'mode' of Σ.
- Σ is *completely controllable* if lemma 1 holds for any $s, 0 \leq s \leq K-1$.
- The set of states controllable at time k, denoted by $\mathcal{X}_c(k)$, is a subspace. This subspace is invariant under $A(\cdot)$, in the sense that

$$x \in \mathcal{X}_c(k) \Rightarrow A_k x \in \mathcal{X}_c(k+1).$$

Using this, it can be shown that there exist state-transformations which put A_k, B_k in the form

$$\tilde{A}_k = \begin{bmatrix} \tilde{A}_{11}(k) & \tilde{A}_{12}(k) \\ 0 & \tilde{A}_{22}(k) \end{bmatrix}, \; \tilde{B}_k = \begin{bmatrix} \tilde{B}_1(k) \\ 0 \end{bmatrix}, \quad (6)$$

which is the 'natural' representation (alluded to earlier) from the controllability point of view. When A_k, B_k are transformed to (6), (Ψ_s, G_s) take the form

$$\tilde{\Psi}_s = \begin{bmatrix} \tilde{\phi}_{11} & \tilde{\phi}_{12} \\ 0 & \tilde{\phi}_{22} \end{bmatrix}, \; \tilde{G}_s = \begin{bmatrix} \tilde{\gamma}_1 \\ 0 \end{bmatrix}. \quad (7)$$

Moreover, in (6), $\tilde{A}_{22}(k)$ is non-singular for all k, a result which strengthens the intuition that zero eigenvalues are *trivially* controllable (the corresponding state can always be driven to zero). This also explains why only non-zero eigenvalues need to be considered in lemma 1.

3. Algorithm description

In this section, we describe the main result of this paper, namely an algorithm to compute the controllable subspace of a linear periodic discrete-time system. We treat the complex and real matrix cases side by side.

The following lemma is key.

Lemma 2 *If (A, B) is controllable, with*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

then (A_{22}, B_2) is controllable. Also, in particular, if A_{22} is a 1×1 (resp. 2×2) diagonal block (of the Schur form) of A corresponding to a real eigenvalue (a pair of complex conjugate eigenvalues), then (A_{22}, B_2) is controllable iff $B_2 \neq 0$. ∎

In order to introduce the main idea of our algorithm, we first restrict attention to the simpler $K = 1$ case (time-invariant system). A fuller treatment of the time-invariant case can be found in Varga [5].

3.1 Algorithm for time-invariant systems

Lemma 2 motivates a simple procedure for identifying uncontrollable states of an uncontrollable pair (A, B). The algorithm is the following. Start by putting A in Schur form with any particular ordering of the diagonal blocks. If the last 1×1 (resp. 2×2 in the real case) diagonal block corresponds to an uncontrollable eigenvalue (an uncontrollable complex conjugate pair of eigenvalues), then, by lemma 2, the last row (last two rows) of the transformed B matrix *vanishes* (with some minor assumptions about repeated eigenvalues). If this happens, leave that block at the bottom, and reorder the remaining blocks to check for another uncontrollable mode. Continuing in this fashion, accumulate all uncontrollable eigenvalues in $\tilde{A}_{22}$, where $\tilde{A}$ and $\tilde{B}$ are partitioned as in lemma 2. $\tilde{B}_2$ is now zero, and $(\tilde{A}_{11}, \tilde{B}_1)$ represents the controllable part of (A, B).

3.2 Algorithm for periodic systems

In view of lemma 1, it is clear how the foregoing algorithm for time-invariant systems (described in section 3.1) applies in the periodic case. In effect, we run Algorithm 3.1 on the pair (Ψ_s, G_s). But we do so *implicitly*, working only with the A_k's and B_k's.

3.2.1 Computational steps in brief:

1. Use theorem 1 to find a coordinate transformation which puts Ψ_s in Schur form (denoted by $\tilde{\Psi}_s$). Choose the Schur form in which only zero eigenvalues occur in the first diagonal block of $\tilde{\Psi}_s$, where $\tilde{\Psi}_s$, G_s are partitioned as

$$\tilde{\Psi}_s = \begin{bmatrix} \tilde{\psi}_{11} & \tilde{\psi}_{12} \\ 0 & \tilde{\psi}_{22} \end{bmatrix}, \; G_s = \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix} \quad (8)$$

   This is done because the zero eigenvalue is known to be controllable.

2. Apply Algorithm 3.1 to the subsystem $(\tilde{\psi}_{22}, \tilde{C}_2)$ in (8).

   It is helpful to note that the last row of G_s vanishes iff the last row of *each* B_k vanishes.

3. Continue reordering the diagonal blocks of Ψ_s, till $(\tilde{A}_k, \tilde{B}_k)$ is exhibited in the form shown in (6). When this happens, Ψ_s, G_s will be as shown in (7).

4. Accumulate the transformations thus far to find T_k, the desired coordinate-transformation.

The case of repeated eigenvalues which are not all controllable (or uncontrollable) presents some technical difficulties, just as in the time-invariant case [5]. This is left out here for the sake of simplicity.

4. Extensions

- *Descriptor systems*:
  Generalized state-space (or descriptor) systems

$$E_k x_{k+1} = A_k x_k + B_k u_k$$

  can be handled just as easily, using the generalized periodic Schur decomposition of the two sequences $A_i, E_i, \; i = 0, \ldots, K-1$. See [1] for details about this decomposition.

- *Real matrix case*:
  As shown in section 3, all this goes through for real case too. The transformations are orthogonal then.

5. Applications

- *Pole-placement*:
  Using a similar approach, the pole-placement problem for periodic systems can be solved. Theorem 1 can be used to choose the state-feedback matrices to assign particular eigenvalues of Ψ_s. The interested reader is referred to [9] for further details.

- *Dead-beat control*:
  It is possible to do dead-beat control of periodic systems [9].

- *Canonical decomposition*:
  Since controllability and reconstructibility are duals of each other, the method outlined in this summary is clearly extendable to finding the *full* Kalman canonical decomposition for linear periodic discrete-time systems.

6. Conclusion

A computational procedure has been proposed for finding the controllable subspace of linear periodic discrete-time systems. It uses only unitary state transformations, and is numerically stable. It readily extends to more general situations, such as when the system equation is given in descriptor form.

7. References

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