DISCRETE-TIME PERIODIC SYSTEMS: A FLOQUET APPROACH

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Designation: Short paper

Key Words: Floquet theory, Schur Form

Abstract. *We apply a Floquet-like theory to linear discrete-time periodic systems, and present an algorithm to compute the state-transformation matrices needed to give a time-invariant representation. The algorithm uses the periodic-Schur decomposition of a matrix sequence* $A_i,\ i = 0, \ldots, K-1$.

1 Introduction

In the literature, two approaches are commonly used for analyzing discrete-time periodic systems. In one, the modified z-transform technique is used [2]. In the other, the periodic sytem is mapped isomorphically to a LTI system, either by state-sampling [3], or by state-grouping [4]. In this paper, we present an alternative, and perhaps more direct method, along the lines of classical Floquet theory for continuous-time periodic systems [5, 6] .

2 Floquet theory

2.1 Continuous-time case

Floquet theory for continuous-time linear periodic systems can be summarized in the following result.

Theorem 1. *Given the linear periodic system*

$$\frac{dx}{dt} = A(t)x,\ A(t+T) = A(t) \tag{1}$$

there exists a nonsingular matrix $P(t)$, periodic of period T with $P(0) = I$, such that the change of variables $x = P(t)y$ transforms the system into a linear system with constant coefficients.

The proof proceeds by finding a (constant) matrix R such that the state-transition matrix over one period[1] $\Phi(T, 0) = e^{RT}$, and defining $P(t) = \Phi(t, 0)e^{-Rt}$. It is easy to verify that $P(0) = I$. We have (see [5] for details)

$$\begin{aligned} P(t+T) &= \Phi(t+T, 0)e^{-R(t+T)} \\ &= \Phi(t, 0)\Phi(T, 0)e^{-RT}e^{-Rt} \\ &= \Phi(t, 0)e^{-Rt} = P(t). \end{aligned}$$

Thus $P(t)$ is periodic with period T. Performing the change of variables $x = P(t)y$ in (1), we see that the solution passing through x_0 at $t = 0$ is given by

$$\begin{aligned} x(t; x_0) &= \Phi(t, 0)x_0 \\ &= P(t)e^{Rt}x_0 \\ &= P(t)y(t; x_0). \end{aligned}$$

[1] the so-called *monodromy* matrix

Thus $y(t; x_0) = e^{Rt}x_0$, and is the solution of the linear system with constant coefficients $dy/dt = Ry$. Note that the periodicity of the original system has been absorbed into the transformation $P(t)$. Indeed, we have, in obvious notation,

$$\begin{aligned} x(t; t_0, x_0) &= P(t)y(t; t_0, x_0) = P(t)e^{R(t-t_0)}y_0 \\ &= P(t)e^{R(t-t_0)}P^{-1}(t_0)x_0. \end{aligned}$$

¿From the foregoing, we deduce that the whole behavior of the solutions of a linear system with periodic coefficients such as (1) depends upon the eigenvalues of matrix R. These eigenvalues are of the form $\frac{1}{T}\ln\lambda_j$, where λ_j are the eigenvalues of the (monodromy) matrix $\Phi(T, 0)$.

2.2 Discrete-time case

Here, we consider the system

$$x_{k+1} = A_k x_k,\ A_{k+K} = A_k. \tag{2}$$

As in the continuous-time case, we expect to gain insight into the structure of the solutions of a linear system with periodic coefficients by transformation into an equivalent time-invariant system. Accordingly, we would like to find non-singular T_k, periodic of period K, such that the change of variables

$$x_k = T_k \hat{x}_k$$

transforms (2) into a linear system with constant coefficients

$$\hat{x}_{k+1} = A\hat{x}_k.$$

In other words, we seek matrices $T_k = T_{k+K}$ satisfying

$$T_{k+1}^{-1}A_kT_k = A,\ \text{for all k}, \tag{3}$$

where A is a constant matrix.

The question now arises as to the conditions under which we can solve for A and T_k. Equation (3) gives

$$\begin{aligned} T_K^{-1}A_{K-1}T_{K-1}\cdots T_1^{-1}A_0T_0 &= T_0^{-1}(A_{K-1}\cdots A_0)T_0 \\ &= A^K \end{aligned}$$

which means A^K is similar to the monodromy matrix

$$\Psi := \Phi(K, 0) = A_{K-1}\cdots A_0.$$

Thus, to compute A, we *definitely* need the eigenvalues of the monodromy matrix [7]. The K-th root A exists whenever A^K is non-singular (or when its zero eigenvalue has a full set of eigenvectors). Refer to [7] for details about the theory of K-th roots of a given matrix.

2.2.1 'Naive' algorithm for T_k:

We have the following algorithm for finding A and T_k:

1. Set $T_0 = I$. Solve for A in
$$A^K = \Psi \tag{4}$$
using the method outlined in [7]. Note that the K-th root, when it exists, is not unique.

2. Solve for T_k, $k = 1, \ldots, K-1$, using the recurrence
$$T_{k+1}A = A_k T_k. \tag{5}$$

One easily checks that the above steps yield $T_K A^K = \Psi T_0$, and therefore, $T_K = I$. Periodicity of T_k is thus satisfied. In fact, we imposed this to obtain equation (4).

Remark:

1. This procedure always works when Ψ is non-singular, because then the K-th root A exists, is non-singular, and so (5) has a unique solution. The case of singular Ψ is more complicated, equation (5) may not have a solution *even if there exists* a K-th root A of Ψ. This is currently under investigation.

2. Unfortunately, as with all straight-forward procedures, the above algorithm may suffer from severe numerical difficulties. Firstly, it involves the computation of the K-th root of Ψ, which is a dense matrix. Secondly, the solution described in [7] requires not only the eigenvalues of Ψ, but also its Jordan form; and there is no reliable computer program to compute Jordan forms for repeated eigenvalues [8]. Thirdly, equation (5) has no structure, and so its solution is computationally intensive.

Some simplification would result if equation (5) has some structure, e.g., if A_k and A are in upper-triangular form. (We could then solve for T_k also to be upper-triangular). For A to be upper-triangular, it would be helpful if $A^K = \Psi$ were also so. We are naturally led to wonder whether there is a numerically sound procedure which triangularises A_k, *and* puts the matrix product Ψ in Schur Form (SF). It turns out that the periodic Schur decomposition lemma (see [1] for the proof of existence and algorithm) gives precisely such a method.

Lemma 1. *[Periodic Schur decomposition] Given $n \times n$ matrices A_i, $i = 0, 1, \ldots, K-1$, there exist $n \times n$ unitary matrices Q_i, $i = 0, 1, \ldots, K-1$ such that*

$$\begin{aligned}
\tilde{A}_0 &= Q_1^H A_0 Q_0 \\
\tilde{A}_1 &= Q_2^H A_1 Q_1 \\
&\vdots \\
\tilde{A}_{K-2} &= Q_{K-1}^H A_{K-2} Q_{K-2} \\
\tilde{A}_{K-1} &= Q_0^H A_{K-1} Q_{K-1}
\end{aligned}$$

are upper-triangular. ∎

To recapitulate, we take recourse to the periodic Schur decomposition for the following reason(s):

- To find the K-th root of Ψ, we would have to find its eigenvalues anyway. The periodic Schur decomposition gives us these, *in addition to* upper-triangularising each A_k. The algorithm in [1], while working only on the A_k matrices, *implicitly* puts Ψ in SF .

- Finding the K-th root of an upper-triangular matrix is relatively simple [9].

- The K-th root A is not unique, though its eigenvalues are known (K-th roots of the eigenvalues of Ψ). So why not find A also in SF? This will simplify solution of (5).

- 'Pre-processing' with the Q_k is numerically reliable since only unitary transformations are used.

2.2.2 'Good' algorithm for T_k:

We will exhibit T_k of (3) in the form

$$T_k = Q_k \tilde{T}_k, \; Q_k \text{ unitary}, \; \tilde{T}_k \text{ upper-triangular}. \tag{6}$$

Note that this is the QR-factorization of T_k, which is a useful representation. In (6), Q_k, $\tilde{T}_k$ are periodic matrices with period K.

Finding (6) involves the following steps:

- Compute unitary Q_k to put the A_k matrices in upper-triangular form, using the algorithm described in [1].

- Find upper-triangular matrices $\tilde{T}_k$ such that
$$\hat{A}_k = \tilde{T}_{k+1}^{-1} \tilde{A}_k \tilde{T}_k \tag{7}$$
is a constant matrix A for all k. This step involves first finding A. Note that (7) gives
$$\hat{A}_{K-1} \cdots \hat{A}_0 = \tilde{T}_K^{-1} \tilde{A}_{K-1} \tilde{T}_{K-1} \cdots \tilde{T}_1^{-1} \tilde{A}_0 \tilde{T}_0$$
which is the same as
$$A^K = \tilde{T}_0^{-1} (\tilde{A}_{K-1} \cdots \tilde{A}_0) \tilde{T}_0. \tag{8}$$
Thus, if we take $\tilde{T}_0 = I$ in (8), A can be computed as the K-th root of $(\tilde{A}_{K-1} \cdots \tilde{A}_0)$. We outline a procedure for computing A in the appendix. $\tilde{T}_k$ is then found by the recurrence
$$\tilde{T}_{k+1} A = \tilde{A}_k \tilde{T}_k, \;\; k = 1, \ldots, K-1$$
which is a triangular system of equations.

For the case of singular Ψ, statements similar to those made in remark 1 hold.

3 Conclusion

A Floquet approach is presented for linear discrete-time periodic systems. This could be useful in the analysis of such systems. A numerical algorithm is described to compute the state-transformation matrices which put the periodic system in time-invariant form.

4 Acknowledgement

The suggestion to look at this problem was made by A. Laub and J. Hench of the Univ. of California at Santa Barbara. Their internal report [10] introduces the main idea behind this paper.

5 Appendix: K-th root of a triangular matrix

Problem Statement:

Let S be a given $n \times n$ upper-triangular matrix. Assume that the zero eigenvalue of S, if it exists, is diagonalizable. Find an $n \times n$ upper-triangular matrix A satisfying

$$A^K = S. \tag{9}$$

Solution:

We will solve for A one column at a time, starting with the first and proceeding towards the right. When S is singular, we assume that the Jordan blocks (each of dimension 1) corresponding to the zero eigenvalue are grouped together at the top. In this case, for some $r < n$, the leading principal sub-matrix of order r is zero in S, and we choose that sub-matrix to be zero in A also. Hence, for singular S, we solve for columns of A beginning with the $(r+1)$-th column.

When solving for the j-th column, it is helpful to partition the leading principal sub-matrices of order j of A and S as follows

$$A_j = \begin{bmatrix} A_{j-1} & a_j \\ 0 & \alpha_{jj} \end{bmatrix}, \; S_j = \begin{bmatrix} S_{j-1} & s_j \\ 0 & \sigma_{jj} \end{bmatrix}. \tag{10}$$

where a_j, s_j are $(j-1)$ column-vectors, α_{jj}, σ_{jj} are scalars, and A_{j-1}, S_{j-1} are upper-triangular matrices of size $(j-1)$. Of course, A_{j-1} and S_{j-1} are the leading principal sub-matrices of order $(j-1)$ of A and S respectively.

¿From equation (9), we see that

$$\begin{aligned} A_{j-1}^K &= S_{j-1} \\ \alpha_{jj}^K &= \sigma_{jj} \end{aligned} \tag{11}$$

and

$$(A_{j-1}^{K-1} + A_{j-1}^{K-2}\alpha_{jj} + \cdots + \alpha_{jj}^{K-1} I)a_j = s_j. \tag{12}$$

Since A_{j-1} is known when solving for the j-th column of A, and α_{jj} is given by (11), we can solve for a_j from equation (12). The coefficient matrix of (12) is upper-triangular, with the i-th diagonal entry given by

$$\begin{aligned} d_i &= \sum_{\ell=0}^{\ell=K-1} \alpha_{ii}^{\ell} \alpha_{jj}^{K-1-\ell} \\ &= \frac{(\alpha_{ii}^K - \alpha_{jj}^K)}{(\alpha_{ii} - \alpha_{jj})}. \end{aligned}$$

We see that $d_i = 0 \Leftrightarrow \alpha_{ii}^K = \alpha_{jj}^K$ *and* $\alpha_{ii} \neq \alpha_{jj}$. Thus, (12) can be solved uniquely for a_j provided S has distinct eigenvalues, or if we take $\alpha_{ii} = \alpha_{jj}$ whenever $\sigma_{ii} = \sigma_{jj}$. Similar conditions are encountered in the $K = 2$ case [9].

When the zero eigenvalue of S has Jordan blocks of size greater than one, certain inequalities on the sizes of Jordan blocks have to be satisfied for the K-th root to exist [7].

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