POLE PLACEMENT VIA THE PERIODIC SCHUR DECOMPOSITION

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ABSTRACT

We present a new method for eigenvalue assignment in linear periodic discrete-time systems through the use of linear periodic state feedback. The proposed method uses reliable numerical techniques based on unitary transformations. In essence, it computes the Schur form of the open-loop monodromy matrix via a recent implicit eigen-decomposition algorithm, and shifts its eigenvalues sequentially. Given complete reachability of the open-loop system, we show that we can assign an arbitrary set of eigenvalues to the closed-loop monodromy matrix in this manner. Under the weaker assumption of complete controllability, this method can be used to place all eigenvalues at the origin, thus solving the so-called deadbeat control problem. The algorithm readily extends to more general situations, such as when the system equation is given in descriptor form.

1. Introduction

One of the most studied problems in modern control theory has been the modification of the dynamic response of a linear system through state feedback [1, 2]. Successful resolution of this problem for multivariable systems ranks as one of the cornerstones of the theory.

Within linear systems, an important subclass is that of *periodic* systems. Various processes in chemical, electrical and aerospace engineering can be modeled using linear periodic systems. An added incentive for studying such systems is that they represent the simplest case of general time-varying systems, consequently, their analysis is quite tractable. In fact, linear *time-invariant* theory serves as a guide to the study of linear periodic systems, and many classical concepts first developed for time-invariant systems have been extended and applied to the periodic case.

In this paper, we consider the eigenvalue assignment problem, henceforth referred to as the EAP, for linear periodic discrete-time systems. Though several authors have studied this problem [3, 4, 5], computational issues have not been adequately addressed so far. Here we propose a numerically sound procedure, based on a Schur approach, which uses only unitary transformations. It is well known that the use of unitary transformations promotes numerical stability in algorithms [6]. In essence, we compute an ordered Schur form of the open-loop monodromy matrix via an implicit eigen-decomposition algorithm [7], and shift its eigenvalues sequentially by linear periodic state feedback.

The paper is organized as follows. Section 2 introduces notation and some preliminary facts about linear periodic discrete-time systems. Section 3 states the EAP, and contrasts our solution procedure with previous ones. Section 4 describes our algorithm. Finally, after discussing possible extensions and an application in sections 5 and 6 respectively, we end this paper with some concluding remarks in section 7.

2. Linear periodic discrete-time systems

2.1 Some preliminaries

Throughout this paper, we denote the set of integers and complex numbers by $\mathcal{Z}$ and $\mathcal{C}$ respectively. We write A^* for the conjugate-transpose of the matrix A, A' for its transpose, and $\boldsymbol{\lambda}(A)$ for the set of its eigenvalues.

We will denote the linear periodic discrete-time system under consideration by Σ. Assume that Σ is represented by the following equation:

$$x_{k+1} = A_k x_k + B_k u_k, \tag{1}$$

where $A_k : \mathcal{Z} \to \mathcal{C}^{n\times n}$, $B_k : \mathcal{Z} \to \mathcal{C}^{n\times m}$ are known periodic matrices of integer period K, i.e., $A_{k+K} = A_k$, $B_{k+K} = B_k$; and x_k, u_k are vectors of states and inputs respectively. We allow A_k to be singular, thus the system might be non-reversible. The state-transition matrix of (1) is given by

$$\Phi(k,\ell) = \begin{cases} I & k = \ell \\ A_{k-1}A_{k-2}\cdots A_{\ell+1}A_\ell & k > \ell \end{cases}$$
$$\Phi(k,\ell) \text{ undefined for } k < \ell.$$

It is easy to see that $\Phi(k+K, \ell+K) = \Phi(k, \ell)$, for all $k \geq \ell$, due to the periodicity of A_k. The state-transition matrix over one period (starting at time i) is known as the *monodromy matrix* (at time i). We denote it by $\Psi_i := \Phi(i+K, i)$. It is non-singular if and only if the system is reversible. The eigenvalues of Ψ_i are called the *characteristic multipliers* of (1). They are independent of i, i.e., all Ψ_i have the same spectrum. System (1) is said to be *asymptotically stable* if all its characteristic multipliers lie inside the unit circle. When the system is unstable, we usually seek to *stabilize* it through feedback, i.e., move its characteristic multipliers to the interior of the unit circle.

2.2 Time-invariant reformulation

In many instances, problems involving periodic systems can be tackled by recasting the periodic system as a time-invariant system. An advantage of this approach is that known results for time-invariant systems can then be immediately invoked. Often, this approach forms a first method of attack, because time-invariant theory is (currently) better understood than its periodic counterpart. The K-periodic system Σ described by (1) has K associated time-invariant representations. For $s = 0, 1, \ldots K-1$, these are

$$\theta_s(\ell+1) = \Psi_s \cdot \theta_s(\ell) + G_s \cdot v_s(\ell), \qquad (2)$$

where

$$\begin{aligned} \theta_s(\ell) &= x_{s+\ell K}, \\ \Psi_s &= \Phi(s+K, s), \\ G_s &= [\Phi(s+K, s+1)B_s \ \cdots \ B_{s+K-1}], \\ \text{and } v_s(\ell) &= \begin{bmatrix} u_{s+\ell K} \\ u_{s+\ell K+1} \\ \vdots \\ u_{s+(\ell+1)K-1} \end{bmatrix}. \end{aligned}$$

2.3 Reachability and controllability

The definition of reachability and controllability of Σ is standard, so we skip it here. The following lemma gives a simple criterion for these properties. It can be proved using the correspondence between systems (1) and (2).

Lemma 1 *System (1) is reachable (resp. controllable) at time s iff for each characteristic multiplier λ ($\lambda \neq 0$), rank $[\lambda I - \Psi_s \ \ G_s] = n$.* ∎

We now list some observations regarding reachability and lemma 1. Similar statements hold for controllability.

- Lemma 1 is the usual PBH test for the equivalent time-invariant system (2). Thus (1) is reachable at time s iff (2) is.
- If the matrix in lemma 1 loses rank for some eigenvalue λ of Ψ_s, then λ is an unreachable eigenvalue or 'mode' of Σ.
- Σ is *completely reachable* if lemma 1 holds for every $s, 0 \leq s \leq K-1$.

2.4 Coordinate transformations

It must be borne in mind that equation (1) is not the only (periodic) representation for Σ. We can let $x_k = T_k \tilde{x}_k$ in (1), where T_k is any non-singular periodic matrix, and arrive at the following alternative periodic realization

$$\begin{aligned} \tilde{x}_{k+1} &= \tilde{A}_k \tilde{x}_k + \tilde{B}_k u_k, \qquad (3) \\ \text{where } \tilde{A}_k &= T_{k+1}^{-1} A_k T_k, \\ \tilde{B}_k &= T_{k+1}^{-1} B_k. \end{aligned}$$

Such a transformation merely changes 'book-keeping'. It does not affect the characteristic multipliers, or structural properties like reachability or controllability. The pair $(\tilde{A}_k, \tilde{B}_k)$ in (3) is said to be *algebraically equivalent* to the pair (A_k, B_k) in (1).

It turns out that there exist some representations for Σ in which the EAP is very easy to solve. Given a particular realization (1) of Σ, a smart approach would be to first look for a coordinate transformation T_k which leads to a representation suitable for the EAP. For instance, in the time-invariant case ($K = 1$), we know that it helps to put (1) in controller canonical form if the desired characteristic equation is specified [2], or in Schur form[1] if the desired eigenvalues are given [8].

However, from a numerical point of view, just any T_k that accomplishes this task will not do, because it might be ill-conditioned with respect to inversion. For this reason, a favored class of transformations is that of unitary T_k. In this work, we consider unitary T_k which put the A_k in triangular form, while implicitly computing the Schur form of the monodromy matrices Ψ_i. The existence of such T_k is guaranteed by the following result:

Lemma 2 (Periodic Schur decomposition) *Given $n \times n$ matrices A_i, $i = 0, 1, \ldots, K-1$, there exist $n \times n$ unitary matrices T_i, $i = 0, 1, \ldots, K-1$, such that*

$$\begin{aligned} \tilde{A}_0 &= T_1^* A_0 T_0, \\ \tilde{A}_1 &= T_2^* A_1 T_1, \\ &\vdots \\ \tilde{A}_{K-2} &= T_{K-1}^* A_{K-2} T_{K-2}, \\ \textit{and } \tilde{A}_{K-1} &= T_0^* A_{K-1} T_{K-1} \end{aligned}$$

[1] More precisely, the system matrix is reduced to Schur form.

is each upper-triangular. Moreover, T_k can be chosen so that the diagonal elements (eigenvalues) of the products $(\tilde{A}_{i+K-1}\cdots\tilde{A}_{i+1}\tilde{A}_i)$ appear in any desired order.

Proof: See [7]. A constructive proof, as well as a numerical algorithm on the lines of the classical QR algorithm, is described therein. ∎

Note that a (unitary) similarity transformation with T_i puts the monodromy matrix Ψ_i in Schur form:

$$T_i^*\Psi_i T_i = T_i^* A_{i+K-1}\cdots A_i T_i = \tilde{A}_{i+K-1}\cdots\tilde{A}_i := \tilde{\Psi}_i.$$

In other words, the periodic Schur decomposition really computes the Schur form of Ψ_i. However, it does so *implicitly,* without ever forming the matrix products! The algorithm described in [7] works directly on the A_i matrices, and reduces them to upper-triangular form. This results in lesser computation, and greater accuracy.

We mention here that, with minor modifications, the periodic Schur decomposition has a real-matrix version also [7].

3. Pole placement in periodic systems

Consider the system Σ described by (1). If we apply linear state-variable feedback of the form

$$u_k = F_k x_k + v_k,\ F_{k+K} = F_k, \tag{4}$$

where v_k is the new external input, we obtain the *closed-loop* system

$$x_{k+1} = (A_k + B_k F_k)x_k + B_k v_k, \tag{5}$$

which is again K-periodic. We denote the closed-loop transition matrix by $\hat{\Phi}(k,i), k \geq i$, and the corresponding monodromy matrix by $\hat{\Psi}_i$. That is,

$$\hat{\Psi}_i := \hat{\Phi}(i+K,i). \tag{6}$$

It is well known that the eigenvalues of $\hat{\Psi}_i$, which are the closed-loop characteristic multipliers, can be arbitrarily chosen by state feedback if and only if Σ is completely reachable [3].

Let $\Gamma \subset \mathcal{C}$ be an arbitrary set of n complex numbers representing the desired eigenvalues for $\hat{\Psi}_i$. Then the problem considered in this paper can be stated as follows:

Periodic eigenvalue assignment problem: *Let (A_k, B_k) be a completely reachable periodic pair. Find periodic $m \times n$ matrices F_k such that*

$$\boldsymbol{\lambda}(\hat{\Psi}_i) = \Gamma.$$

When (1) is a minimal representation of Σ, the poles and characteristic multipliers are the same, hence the above problem is also known as the *pole placement* problem. ∎

3.1 Previous algorithms for periodic EAP

In the past, two approaches have been used to solve the periodic eigenvalue assignment problem. The first approach [4] transforms the EAP for Σ into an EAP for the associated time-invariant system (2). The desired periodic matrices F_k are then found from the feedback matrix of one of the K systems in (2). Such indirect methods can be cumbersome, and it is worthwhile searching for algorithms which directly exploit the periodicity of the problem.

The second approach [5] is based on computing the Jordan form of Ψ_i, the open-loop monodromy matrix. It is a periodic extension of the well-known Simon-Mitter recursive pole placement algorithm for the time-invariant case [1]. An attractive feature of this approach is that it is recursive — hence, the fewer the eigenvalues to be shifted, the lesser the computation. Unfortunately, the computations required are rather involved, since it is necessary to update the Jordan bases at each stage. Moreover, this algorithm is best suited only for the case of distinct eigenvalues, since reliable computation of the Jordan form for repeated eigenvalues is a very delicate numerical problem [9].

3.2 Proposed Schur approach

We seek to ameliorate, via a Schur approach, the above-mentioned difficulties associated with existing solution procedures for the periodic EAP. Our algorithm is recursive, it saves on computations by shifting only the 'bad' eigenvalues. It tries to minimize the norms of the feedback matrices to be used at each stage, leading to an acceptable suboptimal solution. It is a numerically sound approach, since it is based on the periodic Schur decomposition technique described in lemma 2, which uses only unitary transformations. It combines the good features of, and offers improvement over, all previous methods.

4. Algorithm description

In section 2.4, we stated that we can gain much simplicity vis-a-vis the EAP by representing Σ in appropriate state coordinates. We now demonstrate that the periodic Schur decomposition described in lemma 2 gives one such convenient representation.

Starting with the state equation (1), perform a transformation $x_k = T_k\tilde{x}_k$, with T_k given by lemma 2. As already noted, this puts Ψ_i in Schur form $\tilde{\Psi}_i$, but does not alter the system structural properties. Also, in the new state equation (3), $\tilde{A}_k$ are upper-triangular.

4.1 Basic step – placing one pole

In what follows, we describe how to shift one eigenvalue of $\tilde{\Psi}_i$. Let the n-th diagonal element (also eigenvalue) of $\tilde{\Psi}_i$ be λ_n^{old}. We compute feedback matrices $\tilde{F}_k$ so that the n-th eigenvalue of

$$S \doteq (\tilde{A}_K + \tilde{B}_K \tilde{F}_K) \cdots (\tilde{A}_1 + \tilde{B}_1 \tilde{F}_1) \tag{7}$$

has a desired value λ_n^{new}. Since $\tilde{A}_k$ are triangular, we take $\tilde{F}_k$ to have zero entries in all but the last column. This ensures that all $(\tilde{A}_i + \tilde{B}_i \tilde{F}_i)$ in (7), and hence S, remain upper-triangular. Let f_k denote the n-th column of $\tilde{F}_k$. It now remains to determine f_k.

Suboptimal solution — only 1 f_k is nonzero: We show that we can accomplish the basic step in section 4.1 through feedback at only one u_k. In other words, we change only one $\tilde{A}_k$.

Since Σ is completely reachable, from lemma 1, *atleast* one $\tilde{B}_k$ must have a nonzero bottom row. Suppose that the last row of $\tilde{B}_j$, denoted by b_j', is nonzero. Let $\tilde{A}_j$ be partitioned as

$$\tilde{A}_j = \begin{bmatrix} ? & a_j \\ 0 & \alpha_j \end{bmatrix}, \tag{8}$$

where α_j is the (n,n) element of $\tilde{A}_j$. Then it is a simple matter to choose f_j so that

$$\alpha_j + b_j' f_j = \frac{\lambda_n^{new}}{\lambda_n^{old}} \alpha_j := \hat{\alpha}_j \tag{9}$$

All $f_k, k \neq j$, are taken to be zero.

At this point, we have placed one eigenvalue. Note that equation (9) has infinitely many solutions – we mention two choices here:

1. f_j = *minimum norm* solution of (9). This reduces the feedback gains of the overall solution.
2. f_j = the solution of (9) which minimizes $||\hat{a}_j||$, where $\hat{a}_j$ is the $(1,2)$-block of $(\tilde{A}_j + \tilde{B}_j \tilde{F}_j)$ partitioned as in (8). This choice gives better robustness of the closed-loop poles, by keeping S close to a normal matrix.

4.2 Placing other poles after reordering

Through the procedure described in section 4.1, we moved one 'bad' eigenvalue of $\tilde{\Psi}_i$ to a new location, while leaving the others untouched. Now, by means of interchange operations using only unitary transformations, we bring the eigenvalue which is to be relocated next, to the bottom of S. This reordering of eigenvalues is a classical Schur idea, and has been incorporated into the periodic Schur algorithm [7]. The individual matrices of the product in (7) are maintained triangular during this reordering process.

Once we have a new 'bad' eigenvalue at the bottom, we shift it by starting the basic step 4.1 all over again. We continue this process till all the 'bad' eigenvalues (of Ψ_i) have been shifted. By keeping track of the various transformations applied, we accumulate the feedback matrices found at each step, to compute the final answer to the EAP.

This concludes the description of our algorithm for the periodic EAP.

5. Extensions

- The periodic EAP for descriptor systems

$$E_k x_{k+1} = A_k x_k + B_k u_k \tag{10}$$

can be handled just as easily. Assume that E_k is non-singular, with formation of ${E_k}^{-1} A_k$ being undesirable. For such systems, periodic state feedback $u_k = F_k x_k + v_k$, $F_{k+K} = F_k$, results in the closed loop system

$$E_k x_{k+1} = (A_k + B_k F_k) x_k + B_k v_k, \tag{11}$$

the underlying characteristic multipliers of which are the eigenvalues of the matrix

$$S_F \doteq E_K^{-1} (A_K + B_K F_K) \cdots E_1^{-1} (A_1 + B_1 F_1).$$

The problem is to choose F_k so that S_F has desired eigenvalues.

We use the generalized periodic Schur decomposition [7] to triangularize the two sequences A_i, E_i, $i = 0, \ldots, K-1$, while *implicitly* computing the Schur form of the monodromy matrix of (10). We can then choose F_k to have, as before, nonzero elements only in the last column. This will preserve the triangular form of the matrices $A_k + B_k F_k$. The rest of the algorithm is the same as for the case $E_k = I$.

- The algorithm described in this paper can be modified to solve the following two special cases:

  1. For a given stability margin, compute stabilizing feedback matrices F_k with small norms.
  2. Conversely, for given constraints on the norms of F_k, ensure the fastest dynamics of the closed-loop system, i.e., minimize the spectral radius of $\hat{\Psi}_i$.

- Under the weaker assumption of complete controllability of Σ, the procedure presented in this paper can be used to achieve state deadbeat control, by setting $\Gamma = \{0\}$. When all eigenvalues of $\hat{\Psi}_i$ are equal to 0, starting with any initial value x_0 at t_0, the state of (1) goes to the origin in at most $\mu_c K$ steps, where μ_c is the maximal controllability index of Σ:

$$\mu_c := \max_{0 \leq i \leq K-1} \mu_{ci}.$$

See [10] for the definition of the controllability indices μ_{ci}, and for a derivation of this result. Note that this is not the tightest possible bound, we *might* be able to drive any state to the origin in fewer steps.

- We considered Σ to be completely reachable in this paper, which is a necessary and sufficient condition for arbitrary eigenvalue assignment [3]. In a recent paper [5], it is shown that for nonreachable systems, it is still possible to assign freely a *core spectrum* of the monodromy matrix of the reachable part. We can use our algorithm to do this in a numerically sound manner. This issue is currently being investigated.
- With minor modifications, all the results in this paper hold for the real-matrix case too. When A_k, B_k are real, one likes to compute the feedback matrices F_k to be real also, while avoiding complex arithmetic altogether. In that case, we would require Γ to be a symmetric set[2], and use the real-matrix version of lemma 2 to place one or two (complex-conjugate) poles in each iteration. Of course, placing a pair of complex-conjugate poles corresponds to shifting a 2×2 diagonal block of the matrix product. A version of the periodic Schur decomposition algorithm which uses only real arithmetic is described in [7].

6. Application

6.1 Finding the controllability subspace

In this paper, the pole placement (resp. deadbeat control) problem for periodic systems was solved under the assumption of complete reachability (controllability). While it is true that the algorithm outlined here breaks down when the system is not completely controllable, *precisely this* condition can be used to construct the controllable subspace of Σ. Refer to [11] for details.

7. Conclusion

A computational procedure has been proposed for pole placement in linear periodic discrete-time systems by means of linear periodic state feedback. This is useful in many problems, e.g. stabilization of unstable plants.

The algorithm performs a suboptimal minimization of the norms of the feedback matrices used for pole-shifting. It is more reliable, from a numerical point of view, than existing methods for pole placement. This is because it uses only unitary state transformations. The algorithm is recursive, it reduces the original problem to a sequence of subproblems, in each of which 1 eigenvalue is shifted (possibly 2 in the real case). An advantage of this recursive nature is that only 'bad' eigenvalues are shifted, resulting in computational savings — the fewer the eigenvalues to be shifted, the lesser the computational effort.

[2] We use the word *symmetric* to mean $\lambda \in \Gamma \Rightarrow \lambda^* \in \Gamma$.

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9. References

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