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ON FINDING STABILIZING STATE FEEDBACK GAINS FOR A DISCRETE-TIME PERIODIC SYSTEM

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1. Introduction

Suppose it is required to find a control law of the form

$$u_k = -H_k x_k,\ H_{k+K} = H_k\ \forall k, \tag{1}$$

that stabilizes the linear discrete-time system

$$x_{k+1} = A_k x_k + B_k u_k, \tag{2}$$

where $A_k : \mathcal{Z} \to \mathcal{R}^{n\times n}$, $B_k : \mathcal{Z} \to \mathcal{R}^{n\times m}$ are known periodic matrices of integer period K, i.e., $A_{k+K} = A_k$, $B_{k+K} = B_k\ \forall k$. without having to transform A_k to a canonical form, and without regard to explicit closed-loop pole assignment. Such a situation arises, for instance, in iterative quasi-linearization methods for solving a discrete-time periodic Riccati equation [1]. There, to initiallize the algorithm, it is sufficient to find a control (1) that merely stabilizes (2) — exact values are not specified for the closed-loop characteristic multipliers. As Bittanti et. al. [1, §VIII-C] observe, the problem of choosing stabilizing (initial) gains could be solved by a pole-placement technique. Indeed, a technique of general validity for the assignment of closed-loop characteristic multipliers has been worked out recently by the authors [2], but such a procedure would be too elaborate for our purpose here, since the *precise* location of closed-loop poles is unimportant. Our present result is computationally cheaper too – it mainly involves the solution of a discrete periodic Lyapunov equation (DPLE), for which an efficient Schur technique exists [3].

2. Main result

Theorem 1 *Consider system (2), with the additional assumption that (A_k, B_k) is controllable, and that A_k is nonsingular. Then the periodic control law $u_k = -H_k x_k$,*

$$H_k = (I + B_k^T P_k^{-1} B_k)^{-1} B_k^T P_k^{-1} A_k \tag{3a}$$

$$= B_k^T (B_k B_k^T + P_k)^{-1} A_k, \tag{3b}$$

is stabilizing, where $P_k = P_k^T = P_{k+K} > 0$ solves

$$A_k P_{k+1} A_k^T - \alpha^2 P_k = 2\alpha^2 B_k B_k^T, \tag{4}$$

with α chosen such that

$$0 < \alpha^K < \min(1, \min|\lambda(\Psi_A^i)|). \tag{5}$$

Moreover, all the characteristic multipliers of $A_k - B_k H_k$ lie within the α-circle centered at the origin. ∎

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Proof: Omitted. See [4].

Remarks:

1. We have not assumed B_k has full column rank.

2. Theorem 1 specialized to the time-invariant case ($A_k \equiv A, B_k \equiv B$) essentially gives the result in [5]. However, there are some differences — for instance, their procedure does not guarantee that the closed-loop poles lie within the α-circle.

3. Theorem 1 considers only controllable and reversible sytems – this is not a restriction because more general systems can be handled quite easily. See section 2.1 for details.

Algorithm to implement theorem 1:

(i) Transform the monodromy matrix $\Psi_A = A_3 A_2 A_1$ to real Schur form (RSF). Store U_k, and compute $\tilde{A}_k \leftarrow U_{k+1}^T A_k U_k$, $\tilde{B}_k \leftarrow U_{k+1}^T B_k$. We are now working with the modified system

$$\tilde{x}_{k+1} = \tilde{A}_k \tilde{x}_k + \tilde{B}_k u_k. \tag{6}$$

(ii) Using knowledge of $\lambda(\Psi_A)$ from the RSF of Ψ_A, select α in (5).

(iii) Solve the following modified DPLE (for $\tilde{P}_k$):

$$\tilde{A}_k \tilde{P}_{k+1} \tilde{A}_k^T - \alpha^2 \tilde{P}_k = 2\alpha^2 \tilde{B}_k \tilde{B}_k^T. \tag{7}$$

By theorem 1, the stabilizing feedback for (6) is

$$\tilde{H}_k = \tilde{B}_k^T (\tilde{B}_k \tilde{B}_k^T + \tilde{P}_k)^{-1} \tilde{A}_k. \tag{8}$$

(iv) Solve $(\tilde{B}_k \tilde{B}_k^T + \tilde{P}_k) \cdot X = \tilde{B}_k$ for X.

(v) Compute $\tilde{H}_k = X^T \tilde{A}_k$, the desired stabilizing periodic state feedback. (We can always change bases again : note that $H_k = \tilde{H}_k U_k^T = X^T \tilde{A}_k U_k^T$ in (1) would stabilize the original system (2).)

Numerical example: Consider system (2) with

$K = 3, n = 3, m = 2,$

$$A_3 = \begin{bmatrix} 0.7665 & 0.2749 & 0.4865 \\ 0.4777 & 0.3593 & 0.8977 \\ 0.2378 & 0.1665 & 0.9092 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.0606 & 0.5163 & 0.4940 \\ 0.9047 & 0.3190 & 0.2661 \\ 0.5045 & 0.9866 & 0.0907 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.9478 & 0.3841 & 0.5297 \\ 0.0737 & 0.2771 & 0.4644 \\ 0.5007 & 0.9138 & 0.9410 \end{bmatrix},\ B_3 = \begin{bmatrix} 0.0501 & 0.8278 \\ 0.7615 & 0.1254 \\ 0.7702 & 0.0159 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.6885 & 0.7362 \\ 0.8682 & 0.7254 \\ 0.6295 & 0.9995 \end{bmatrix},\ B_1 = \begin{bmatrix} 0.8886 & 0.3510 \\ 0.2332 & 0.5133 \\ 0.3063 & 0.5911 \end{bmatrix}.$$

This satisfies the conditions of theorem 1, and the characteristic multipliers are 2.9785 (unstable!), -0.0717 and

0.0165. We first use the periodic Schur decomposition algorithm [6] to obtain the modified system (6).

$$\tilde{A}_3 = \begin{matrix} 1.4427 & -0.5589 & -0.5639 \\ 0 & 0.2124 & -0.4018 \\ & 0 & 0.2357 \end{matrix}, \; \tilde{A}_2 = \begin{matrix} -1.1655 & -0.1518 & 0.8343 \\ 0 & -0.5861 & -0.1429 \\ & 0 & -0.5478 \end{matrix},$$

$$\tilde{A}_1 = \begin{matrix} -1.7714 & -0.0357 & 0.3271 \\ 0 & 0.5762 & -0.0323 \\ & 0 & -0.1276 \end{matrix}, \; \tilde{B}_3 = \begin{bmatrix} -0.9135 & -0.5568 \\ -0.5586 & 0.6212 \\ 0.1710 & -0.0728 \end{bmatrix},$$

$$\tilde{B}_2 = \begin{bmatrix} -1.2698 & -1.3888 \\ 0.0824 & 0.2595 \\ -0.0709 & 0.2668 \end{bmatrix}, \; \tilde{B}_1 = \begin{bmatrix} 0.8203 & 0.7911 \\ 0.4683 & -0.1987 \\ -0.2134 & -0.2660 \end{bmatrix}.$$

It is easy to see that Ψ_A is now upper-triangular too:

$$\tilde{A}_3\tilde{A}_2\tilde{A}_1 = \begin{bmatrix} 2.9785 & 0.1226 & -0.7568 \\ 0 & -0.0717 & -0.0202 \\ 0 & 0 & 0.0165 \end{bmatrix},$$

and that the characteristic multipliers have been found correctly. By (5), the maximum value α^K can take is 0.0165. So we take $\alpha = 0.25$. Solving (7) and (8), we get

$$\tilde{H}_3 = \begin{matrix} -0.8318 & 0.1949 & 0.6238 \\ -0.4744 & 0.3665 & -0.0073 \end{matrix}, \; \tilde{H}_2 = \begin{matrix} 0.4581 & 0.2489 & -0.2900 \\ 0.4292 & 0.0072 & -0.3836 \end{matrix},$$

$$\tilde{H}_1 = \begin{matrix} -0.6292 & 0.0237 & 0.1721 \\ -1.1986 & -0.2660 & 0.2532 \end{matrix}.$$

It can be verified that the closed-loop monodromy matrix

$$(\tilde{A}_3 - \tilde{B}_3\tilde{H}_3)(\tilde{A}_2 - \tilde{B}_2\tilde{H}_2)(\tilde{A}_1 - \tilde{B}_1\tilde{H}_1) = \begin{bmatrix} 0.0120 & 0.0990 & -0.0111 \\ -0.0159 & -0.0487 & 0.0049 \\ 0.0343 & 0.0158 & 0.0001 \end{bmatrix}$$

has eigenvalues $-0.0182 \pm 0.0308i$ (abs. value 0.0358) and -4.69×10^{-5}, which all have magnitude less than α. Thus the stabilizing feedback given by Theorem 1 does indeed place the characteristic multipliers within the α-circle.

2.1 General case

The '(A_k, B_k) only stabilizable' case is simple, we apply theorem 1 to *just* those controllable modes which need to be moved, viz., which are outside the α-circle. More precisely, we first use the periodic Schur algorithm [6,7] to put Ψ_A in Schur form *as well as* standard controllable form:

$$\Psi_A = \begin{bmatrix} \psi_{11} & \psi_{12} \\ 0 & \psi_{22} \end{bmatrix}, \tag{9}$$

where ψ_{22} is non-singular and defines the uncontrollable modes. Furthermore, within ψ_{11}, we order the eigenvalues so that those within the α-circle are on top. These need not be touched, and *include* the zero eigenvalue. Then A_k, B_k can be partitioned as

$$A_k = \left[\begin{array}{cc|c} A_g(k) & \times & \\ 0 & A_b(k) & A_{12}(k) \\ \hline \multicolumn{2}{c|}{0} & A_{\bar{c}}(k) \end{array}\right], \; B_k = \left[\begin{array}{c} B_g(k) \\ B_b(k) \\ \hline 0 \end{array}\right], \tag{10}$$

where the pair

$$\begin{bmatrix} A_g(k) & \times \\ 0 & A_b(k) \end{bmatrix} \begin{matrix} \}n_c - n_1 \\ \}n_1 \end{matrix}, \quad \begin{bmatrix} B_g(k) \\ B_b(k) \end{bmatrix} \Big\} n_c$$

is controllable; and the characteristic multipliers of the $n_1 \times n_1$ matrix $A_b(k)$ are 'bad', or outside the α-circle. Note that since all subsystems of a controllable system are themselves controllable, the periodic pair $[A_b(k), B_b(k)]$ is also controllable. Now just apply theorem 1 to the (controllable) pair $[A_b(k), B_b(k)]$, and find $n_1 \times m$ (periodic) matrices $H_b(k)$ such that $A_b(k) - B_b(k) \cdot H_b(k)$ has its characteristic multipliers inside the α-circle. Since all $A_b(k)$ are non-singular, we can always choose α so small that $\alpha^{-1}A_b(k)$ is stable (in a discrete periodic sense). Then the desired state feedback matrix for the overall system is given by $H_k = \left[\, 0 \;\middle|\; H_b(k) \;\middle|\; 0 \,\right]$.

Remarks:

1. Note that we cannot shift the eigenvalues of the uncontrollable part ψ_{22}. However, since $[A_k, B_k]$ is stabilizable, these uncontrollable modes are stable. In general, these may not be within the α-circle. So unless we assume that $\lambda(\psi_{22})$ are within the α-circle, we cannot achieve our goal of putting *all* characteristic multipliers inside the α-circle.

2. We only need to work with a sub-system of size n_1 — this leads to savings in computation, especially if n_1 is much smaller than n.

3. Concluding remarks

We have presented an elegant algorithm for stabilizing a linear periodic discrete-time system using periodic state feedback. The algorithm is very simple, and mainly involves the solution of a periodic Lyapunov equation. It gives a number $0 < \alpha < 1$ such that all closed-loop poles have magnitude less than α. Moreover, it works only with that sub-system whose poles need to be shifted, and is cheaper than an explicit pole placement routine [4]. Thus it is attractive for stabilization problems where the exact location of closed-loop poles is unimportant.

4. References

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