Periodic Schur form and some matrix equations[‡]

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Abstract

We propose an elegant and conceptually simple method for computing the periodic solution of three classes of periodic matrix equations — Riccati, Lyapunov and Sylvester. Such equations arise naturally in several problems of linear system theory. Our approach is very attractive from a numerical point of view, since it is based on the periodic Schur form of matrix sequences $G_i, H_i,\ i = 0, \ldots, K-1$, which utilizes stable numerical techniques involving unitary (orthogonal in the real case) transformations only. Our approach readily extends to more general situations, such as when the equations are given in implicit or descriptor form.

1 Introduction

Systems and control theory has been a very active subject of research for a long time now. In recent years, there has been an increased cross-fertilization

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of ideas between this field and linear algebra — and this has played a crucial role in the development of *both* disciplines. One area of system theory where numerical linear algebra has had a direct impact is the development of stable, efficient and reliable algorithms. All too often in the past, 'textbook' methods to solve control problems have turned out to be numerically naive in the context of limited-precision arithmetic. This paper also focuses on the interplay between linear algebra and system theory, and proposes a powerful new tool for the numerical solution of discrete-time periodic Riccati, Lyapunov and Sylvester equations. These equations arise in a fundamental manner in all analysis and design problems involving periodic systems, especially those related to stability, optimal control and filtering, model reduction, observer design etc. Hence it is imperative that we seek better and more efficient ways of solving them in as general a setting as possible. The method described in this paper is a *theoretical* as well as an algorithmic tool, in that it extends the range of problems for which solutions exist. It gives a clear understanding of the periodic structure of these equations, and hence is more than a mere *extension* of existing procedures for solving the time-invariant versions of these equations.

The unifying thread to our solution techniques for the three equations is a Schur approach based on a recent implicit eigendecomposition algorithm [1, 2] which uses only unitary (orthogonal in the real case) transformations. It is well known that the use of unitary transformations promotes numerical stability in algorithms [3], thus our approach is numerically sound. Another notable feature of our method is that it easily generalizes to handle the case when the matrix equations are given in *descriptor* form. Descriptor systems, also referred to as generalized state-space systems, arise in diverse applications such as the study of large scale power systems, interconnected systems, robotics, econometrics, decentralized control and decision networks, population models, optimization problems etc. [4].

This paper is organized as follows. Section 1.1 introduces notation and some preliminary facts about linear periodic discrete-time systems. For easy reference, and to avoid undue repetition, we list below some oft-used acronyms:

DPLE: Discrete-time periodic Lyapunov equation
DALE: Discrete-time algebraic Lyapunov equation
DPRE: Discrete-time periodic Riccati equation
DARE: Discrete-time algebraic Riccati equation
SPPS: Symmetric periodic positive semi-definite
SDS: Stable deflating subspace
LQ: Linear quadratic

Section 1.2 briefly describes the periodic Schur decomposition, which lets us compute the Schur form of the product of several given matrices. This technique also gives a unitary basis (of Schur vectors) for any deflating subspace of the *periodic pencil* connected with these matrices. In Section 2, which deals with the DPRE, we motivate our solution procedure by considering a well known problem where the DPRE crops up naturally — the linear quadratic regulator problem. The discrete maximum principle leads to a periodic symplectic system of difference equations in the state and co-state vectors. We use the periodic Schur decomposition to compute the Riccati solution from the SDS of the (periodic) matrix pencil associated with this system. Sections 3 and 4 give the details of our treatment of the DPLE and the Sylvester equation respectively. For solving the DPLE $P_{k+1} = A_k P_k A_k^* + B_k B_k^*$, one procedure is to first upper-triangularize A_k via the periodic Schur decomposition, and then employ a back-substitution technique similar to the Hammarling method [5, 6] for a standard DALE. A similar procedure can be used for solving the periodic Sylvester equation. We also propose an alternative, deflating subspace approach for the Lyapunov and Sylvester equations. Finally, we end this paper with some concluding remarks in Section 5.

1.1 Discrete-time linear periodic systems

Throughout this paper, we denote the set of real numbers by $\boldsymbol{\mathcal{R}}$, and complex numbers by $\boldsymbol{\mathcal{C}}$. We write A^* for the conjugate-transpose of the matrix A, and $\Lambda(A)$ for the set of its eigenvalues. The system considered in this paper is

$$x_{k+1} = A_k x_k + B_k u_k, \tag{1}$$

where $x_k \in \boldsymbol{\mathcal{C}}^{\boldsymbol{n}}$ and $u_k \in \boldsymbol{\mathcal{C}}^{\boldsymbol{m}}$ are the state and the input of the system respectively. The matrices A_k and B_k are possibly complex, periodic with (integer) period $K > 1$, and have dimension $n \times n$ and $n \times m$ respectively. We allow A_k to be singular, thus the system might be non-reversible. The state transition matrix of (1) is given by

$$\Phi(t,\tau) := \begin{cases} I & t = \tau \\ A_{t-1}A_{t-2}\cdots A_{\tau+1}A_\tau & t > \tau \end{cases}$$
$$\Phi(t,\tau) \text{ undefined for } t < \tau.$$

It is easy to see that $\Phi(t+K, \tau+K) = \Phi(t,\tau)\ \forall t \geq \tau$ due to the periodicity of A_k. The matrix $\Psi_\tau := \Phi(\tau+K, \tau)$, $\tau = 0, 1, \ldots, K-1$, is known as the *monodromy matrix* of (1) at time τ. It is non-singular (resp. singular) for all τ if the system is reversible (non-reversible). The eigenvalues of Ψ_τ are known as the *characteristic multipliers* of (1). They are independent of τ. In other words, all Ψ_τ have the same spectrum. System (1), or equivalently Ψ_τ, is said to be *asymptotically stable* if all its characteristic multipliers lie inside the unit circle.

The (finite-window) reachability grammian matrix of $(A(\cdot), B(\cdot))$ is given by

$$W_r(t,\tau) := \sum_{j=\tau}^{t-1} \Phi(t, j+1) B_j B_j^* \Phi^*(t, j+1), \; t > \tau. \tag{2}$$

Various system properties such as controllability, observability, stabilizability, detectability can be defined just as for the time-invariant case [7, 8]. A stabilizability criterion is that the system (1) or the periodic pair $(A(\cdot), B(\cdot))$ is stabilizable at time τ iff the pair $(\Psi_\tau, \; W_r(\tau + K, \tau))$ is stabilizable [8].

1.2 Periodic Schur decomposition

Consider the set of (implicit) difference equations

$$G_k z_{k+1} = H_k z_k, \quad k = 0, 1, \ldots \tag{3}$$

with periodic coefficients $H_k = H_{k+K}$, $G_k = G_{k+K}$. For period $K = 1$, one has the constant coefficient case $H_k \equiv H$, $G_k \equiv G$, and it is well known that the generalized eigenvalues of the pair (H, G) yield important information about the system (3). When $K > 1$, we first assume, for simplicity, that all G_k are invertible. Letting $S_k := G_k^{-1} H_k$, (3) becomes

$$z_{k+1} = G_k^{-1} H_k \cdot z_k = S_k z_k, \quad k = 0, 1, \ldots \tag{4}$$

which is an explicit system of difference equations in z_k, again with periodic coefficients $S_k = S_{k+K}$. Now, the characteristic multipliers of (4) are crucial in understanding the system behavior — these are of course the eigenvalues of the monodromy matrices

$$S^{(k)} = S_{k+K-1} \cdots S_{k+1} S_k, \; k = 0, 1, \ldots, K-1. \tag{5}$$

Note that all $S^{(k)}$ have the same eigenvalues.

In order to fully describe the behavior of (4), we require the eigenvalues and eigenvectors of all the *cyclically shifted* matrix products $S^{(k)}$. While computing these, we must avoid, if possible, explicit formation of $S^{(k)}$ themselves. An implicit decomposition of these matrices to achieve this is described in [1], [2]. Its existence can be described by the following theorem, whose proof is given in [1]:

Theorem 1. **(Periodic Schur decomposition)** *Let matrices H_i, G_i, $i = 0, \ldots, K-1$ be all $n \times n$ and complex. Then there exist unitary matrices Q_i, Z_i, $i = 0, ..., K-1$, such that*

$$\begin{array}{ll} \hat{G}_0 = Z_0^* \cdot G_0 \cdot Q_1 & \hat{H}_0 = Z_0^* \cdot H_0 \cdot Q_0 \\ \hat{G}_1 = Z_1^* \cdot G_1 \cdot Q_2 & \hat{H}_1 = Z_1^* \cdot H_1 \cdot Q_1 \\ \hat{G}_2 = Z_2^* \cdot G_2 \cdot Q_3 & \hat{H}_2 = Z_2^* \cdot H_2 \cdot Q_2 \\ \vdots & \vdots \\ \hat{G}_{K-1} = Z_{K-1}^* \cdot G_{K-1} \cdot Q_0 & \hat{H}_{K-1} = Z_{K-1}^* \cdot H_{K-1} \cdot Q_{K-1}, \end{array} \tag{6}$$

where all matrices $\hat{G}_i$, $\hat{H}_i$ are upper-triangular. Moreover, Q_i and Z_i can be chosen so that the eigenvalues of $S^{(k)}$ appear in any desired order. ∎

Corollary. An easy observation is that in the case of a single sequence, say H_i, $i = 0, \ldots, K-1$, Q_i alone suffice to put Ψ_H (implicitly) in Schur form. Just take $G_i \equiv I$ in theorem 1, then $Q_{i \bmod K} = Z_{i-1}$, $i = 1, 2, \ldots, K$, will work. ∎

Clearly, if the matrices G_i are invertible, then each Q_i puts the matrix $S^{(i)}$ in Schur form, i.e., $Q_i^* S^{(i)} Q_i$ is upper triangular. However, the periodic Schur decomposition result is more general, and works even for singular G_i, H_i — it solves the *periodic eigenvalue problem* associated with (3). It is an extension of the standard QZ-algorithm [9] to K matrices, $K > 1$. The QZ-algorithm for a matrix pair (G, H) avoids inverses while computing the Schur form of $G^{-1}H$, the periodic Schur decomposition does likewise while computing the Schur form of $S^{(k)}$. Other features common to the two methods are simultaneous reduction, implicit shifts, reordering of diagonal elements, and of course, numerical stability.

We remark here that, modulo some minor modifications, the periodic Schur decomposition has a real matrix version also, in which one of the matrices in (6) has a quasi-triangular form.

2 Discrete-time periodic Riccati equation

As an example of the occurrence of the DPRE in the analysis of periodic systems, we consider the periodic LQ control problem. We shall see that this problem is a natural setting for introducing our approach to the DPRE. It is known that the optimal periodic solution to the LQ problem is a linear state-feedback control, and that the feedback gains can be obtained by solving an appropriate DPRE. Using variational theory to study the LQ problem, we arrive at a set of homogeneous difference equations with properties intimately connected to those of the above DPRE. We show that we can actually *solve* the DPRE (and hence the LQ problem) by using a geometric approach to study this set of difference equations.

Linear quadratic optimal control problem:

The LQ problem is to find a control function which minimizes a given quadratic cost functional. It can be stated as follows:

$$\left.\begin{array}{l}\text{Find } u_k \text{ to minimize}\\ J = \dfrac{1}{2}\displaystyle\sum_{k=0}^{\infty}(x_k^* Q_k x_k + u_k^* R_k u_k), \quad Q_k = Q_k^* \geq 0,\ R_k = R_k^* > 0,\\ \text{subject to } x_{k+1} = A_k x_k + B_k u_k,\end{array}\right\} \tag{7}$$

where the matrices $A_k, Q_k \in \mathcal{C}^{n\times n}$, $R_k \in \mathcal{C}^{m\times m}$, and $B_k \in \mathcal{C}^{n\times m}$ are periodic of period K. ∎

Under the assumption that the system in (7) is stabilizable and detectable, the optimal periodic control $u_{opt}(k)$ is unique and stabilizing [10]. It is given in feedback form by

$$u_{opt}(k) = -(R_k + B_k^* P_{k+1} B_k)^{-1} B_k^* P_{k+1} A_k x_k, \tag{8}$$

where P_k is the unique SPPS stabilizing solution of

$$P_k = A_k^* P_{k+1} A_k - A_k^* P_{k+1} B_k (R_k + B_k^* P_{k+1} B_k)^{-1} B_k^* P_{k+1} A_k + Q_k. \tag{9}$$

Equation (9) is the DPRE underlying the LQ problem (7).

2.1 Previous methods for DPRE solution

Earlier procedures for solving (9) have used either quasi-linearization or time-invariant reformulation. See [11] for a description and comparison of these two techniques. In the quasi-linearization method, the SPPS solution of the DPRE is obtained as the limit of periodic solutions of a sequence of DPLEs. This procedure is the periodic analog of the well known Newton-type algorithm for solving a DARE [12]. It follows the general philosophy of tackling a non-linear (quadratic) problem through a sequence of linear problems.

In the second method, namely time-invariant reformulation, the underlying periodic system is viewed as a time-invariant system [13]. A periodic generator [10] of the DPRE is obtained as a solution of an 'equivalent' DARE. While this connection with a time-invariant equation provides interesting insight into the periodic case, a *direct* procedure which exploits the fundamental periodic structure of the problem is clearly of value. We provide such a direct method in this paper using the periodic Schur form. Similar ideas are also described in [1, 14, 24].

2.2 Proposed Schur method for the DPRE

The discrete maximum principle applied to the LQ control problem (7) gives us the following Hamiltonian difference equations in the state x_k and co-state γ_k:

$$\begin{bmatrix} I & B_k R_k^{-1} B_k^* \\ 0 & A_k^* \end{bmatrix} \begin{bmatrix} x_{k+1} \\ \gamma_{k+1} \end{bmatrix} = \begin{bmatrix} A_k & 0 \\ -Q_k & I \end{bmatrix} \begin{bmatrix} x_k \\ \gamma_k \end{bmatrix}. \tag{10}$$

Equation (10) has the same form as (3), with the following correspondences

$$z_k \doteq \begin{bmatrix} x_k \\ \gamma_k \end{bmatrix}, G_k \doteq \begin{bmatrix} I & B_k R_k^{-1} B_k^* \\ 0 & A_k^* \end{bmatrix}, F_k \doteq \begin{bmatrix} A_k & 0 \\ -Q_k & I \end{bmatrix}. \tag{11}$$

If A_k is invertible, so is G_k. Then (10) takes the form of (4), with the additional feature that $S_k \doteq G_k^{-1} H_k$ is now symplectic :

$$\begin{aligned} S_k &= \begin{bmatrix} I & B_k R_k^{-1} B_k^* \\ 0 & A_k^* \end{bmatrix}^{-1} \begin{bmatrix} A_k & 0 \\ -Q_k & I \end{bmatrix} \\ &= \begin{bmatrix} A_k + B_k R_k^{-1} B_k^* A_k^{*-1} Q_k & -B_k R_k^{-1} B_k^* A_k^{*-1} \\ -A_k^{*-1} Q_k & A_k^{*-1} \end{bmatrix}. \end{aligned} \tag{12}$$

The periodicity of S_k implies that the system (10) *sampled* over K steps becomes

$$z_{k+K} = S^{(k)} z_k, \tag{13}$$

with $S^{(k)}$ defined as in (5). Now assuming that $\gamma_k = P_k x_k$, where P_k is SPPS of period K, we get

$$\begin{bmatrix} I \\ P_k \end{bmatrix} x_{k+K} = S^{(k)} \begin{bmatrix} I \\ P_k \end{bmatrix} x_k, \tag{14}$$

from which it can be shown that $\operatorname{Im} \begin{bmatrix} I \\ P_k \end{bmatrix}$ is an *invariant subspace* of $S^{(k)}$. Indeed, for each k, a DARE can be associated with the symplectic matrix $S^{(k)}$:

$$[\, -P_k \;\; I \,]\, S^{(k)} \begin{bmatrix} I \\ P_k \end{bmatrix} = 0, \tag{15}$$

and it is not hard to see that P_k in (15) is a periodic generator [8] for the DPRE (9). Invariant subspace methods to solve the DARE (15) are well known. In essence, these find the stabilizing solution of the DARE by computing a basis for the stable invariant subspace of the associated symplectic matrix. In order to use such a procedure to solve (15), we have to find a basis for the stable invariant subspace of $S^{(k)}$. In principle, one can do this by reducing $S^{(k)}$ to Jordan form, and taking the stable eigenvectors as the required basis. However, such a computation is fraught with numerical difficulties and is best avoided. A more appropriate technique is to use a Schur reduction instead. Schur methods to solve Riccati equations have been popularized by Laub [15, 16] and Van Dooren [17]. For the DPRE, it goes as follows. Find a unitary matrix T_k to put $S^{(k)}$ in Schur form with a particular ordering:

$$\begin{bmatrix} T_{11k} & T_{12k} \\ T_{21k} & T_{22k} \end{bmatrix}^* S^{(k)} \begin{bmatrix} T_{11k} & T_{12k} \\ T_{21k} & T_{22k} \end{bmatrix} = \begin{bmatrix} S_{11k} & S_{12k} \\ 0 & S_{22k} \end{bmatrix}, \tag{16}$$

where the partitioning conforms to that of S_k in equation (12), and S_{11k} (resp. S_{22k}) is an upper-triangular matrix with stable (unstable) eigenvalues. Standard assumptions [16, 8] on the problem guarantee that $S^{(k)}$ has no eigenvalue on the unit circle, and precisely n eigenvalues in the open unit disk. Also, T_{11k} is then invertible. The SPPS stabilizing solution of the DPRE can be written as

$$P_k = T_{21k} T_{11k}^{-1}. \tag{17}$$

As a slight modification, we could use the idea in [18] to find P_k from just one of T_{21k}, T_{11k}^{-1} — this also guarantees positive semidefinite-ness of P_k.

The key idea of our proposed method is that we can use the periodic Schur decomposition of the sequence S_k, $k = 0, 1, \ldots, K-1$, to find the above ordered Schur form of $S^{(k)}$. What is more, we can do this *without* forming the matrix products $S^{(k)}$. This is crucial in situations where one or more G_k is singular, for then the corresponding S_k [cf. (10)-(12)] does not exist, and none of the $S^{(k)}$ can be formed! This happens, for instance, when A_k is singular [cf. (11)]. However, our procedure still works because we use only the implicit form (10) of the Hamiltonian equations.

To summarize, we use a deflating subspace approach to solve the DPRE (9). Methods based on deflating subspaces have many advantages, and have been used successfully in the solution of the DARE [17, 16]. Our algorithm brings all those advantages to the periodic case.

Numerical example:

As an illustration of our Schur approach, suppose we take, in (7), $R_k \equiv I$, $Q_k \equiv I$, and

$$
\begin{aligned}
&K = 3, n = 3, m = 2, \\
&A_2 = \begin{bmatrix} 0.5586 & -0.4254 & 0.4685 \\ -1.0659 & -0.3666 & -0.4905 \\ 0.6874 & 0.0786 & -0.1981 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.7362 & 0.8886 \\ 0.7254 & 0.2332 \\ 0.9995 & 0.3063 \end{bmatrix}, \\
&A_1 = \begin{bmatrix} 0.0919 & 0.5419 & -1.5145 \\ 0.2432 & -0.4114 & 0.7030 \\ -0.4407 & 0.1707 & 0.1933 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.3510 & 0.8460 \\ 0.5133 & 0.4121 \\ 0.5911 & 0.8415 \end{bmatrix}, \\
&A_0 = \begin{bmatrix} -0.1376 & -0.0124 & 0.1057 \\ 0.1127 & -0.1821 & 0.0378 \\ -0.0179 & 0.2828 & -0.2265 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0.2693 & 0.4679 \\ 0.4154 & 0.2872 \\ 0.5373 & 0.1783 \end{bmatrix}.
\end{aligned}
\tag{18}
$$

where we have rounded off all data to four decimal places. The monodromy matrix

$$
\Psi_0 = \Phi(3,0) = A_2 A_1 A_0 = \begin{bmatrix} 0.1173 & -0.3965 & 0.2326 \\ -0.0841 & 0.4494 & -0.3020 \\ 0.0295 & -0.3475 & 0.2615 \end{bmatrix}
$$

has eigenvalues 0, 0.0739 and 0.7543, and *infinite* condition number. The zero characteristic multiplier is due to A_2 which is singular — because of this, the symplectic matrix S_k defined in (12) does not exist. Nevertheless, we can use the periodic Schur decomposition algorithm to compute a unitary basis for the SDS of (G_k, H_k) defined in (11). This gives us the SPPS stabilizing solution of the DPRE (9) as

$$
P_2 = \begin{bmatrix} 3.8442 & 0.5588 & 0.8751 \\ 0.5588 & 1.2582 & 0.0421 \\ 0.8751 & 0.0421 & 1.5015 \end{bmatrix}, P_1 = \begin{bmatrix} 1.3340 & -0.0973 & -0.2283 \\ -0.0973 & 1.5624 & -1.2967 \\ -0.2283 & -1.2967 & 4.6357 \end{bmatrix}, P_0 = \begin{bmatrix} 1.0495 & -0.0756 & 0.0214 \\ -0.0756 & 1.4094 & -0.2699 \\ 0.0214 & -0.2699 & 1.2011 \end{bmatrix},
$$

The relative error, measured by

$$
\| A_k^* P_{k+1} A_k - A_k^* P_{k+1} B_k (R_k + B_k^* P_{k+1} B_k)^{-1} B_k^* P_{k+1} A_k + Q_k - P_k \| \;/\; \| P_k \|,
$$

is 5.1408×10^{-16}, 5.6533×10^{-16} and 1.0674×10^{-15} for $k = 0$, 1 and 2 respectively, which is comparable to the machine precision ($\epsilon = 2.2204 \times 10^{-16}$).

2.3 Extensions

The Schur procedure we just described for the solution of the DPRE is very general. It can be easily adapted to deal with a broad range of situations, as sketched below :

- Consider the case where the constraint is a generalized state-equation

$$E_k x_{k+1} = A_k x_k + B_k u_k. \tag{19}$$

We assume that E_k is non-singular, with formation of ${E_k}^{-1} A_k$ being undesirable. The Hamiltonian difference equations (10) now become

$$\begin{bmatrix} E_k & B_k R_k^{-1} B_k^* \\ 0 & A_k^* \end{bmatrix} \begin{bmatrix} x_{k+1} \\ \gamma_{k+1} \end{bmatrix} = \begin{bmatrix} A_k & 0 \\ -Q_k & E_k^* \end{bmatrix} \begin{bmatrix} x_k \\ \gamma_k \end{bmatrix}, \tag{20}$$

which can be handled using essentially the same procedure as before.

- Cross-weighting (between the state and control vectors) can be incorporated just as easily in the performance index in (7).

- Often, one wishes to avoid forming R_k^{-1}. This might be because R_k, even though non-singular, is ill-conditioned with respect to inversion. We can handle this case by staying with the original Hamiltonian difference equations (where u_k is given implicitly),

$$\begin{bmatrix} I & 0 & 0 \\ 0 & A_k^* & 0 \\ 0 & -B_k^* & 0 \end{bmatrix} \begin{bmatrix} x_{k+1} \\ \gamma_{k+1} \\ u_{k+1} \end{bmatrix} = \begin{bmatrix} A_k & 0 & B_k \\ -Q_k & I & 0 \\ 0 & 0 & R_k \end{bmatrix} \begin{bmatrix} x_k \\ \gamma_k \\ u_k \end{bmatrix}. \tag{21}$$

and using an orthogonal reduction procedure on (21) to construct a system equivalent to (10):

$$\tilde{G}_k \begin{bmatrix} x_{k+1} \\ \gamma_{k+1} \end{bmatrix} = \tilde{H}_k \begin{bmatrix} x_k \\ \gamma_k \end{bmatrix}. \tag{22}$$

The rest is as before. We note that this extension follows the approach described in [17].

- Combinations of the foregoing can be tackled.

- Finally, everything we have described so far in Section 2 applies in the real case as well.

3 Discrete-time periodic Lyapunov equation

Suppose we wish to find the steady-state periodic solution of the difference equation

$$P_{k+1} = A_k P_k A_k^* + B_k B_k^*, \tag{23}$$

where $A_k \in \mathcal{C}^{n\times n}$, $B_k \in \mathcal{C}^{n\times m}$ are known periodic matrices with integer period $K > 1$, viz., $A_{k+K} = A_k$, $B_{k+K} = B_k$, $\forall k$.

Equation (23) is called the discrete-time periodic Lyapunov equation (DPLE) and is of interest in linear periodic system theory, especially in the areas of optimal control/prediction, and stability. To give just one example of its use, it is known [7] that determining the SPPS solutions of the DPLE corresponds to finding the cyclostationary processes compatible with the discrete-time periodic stochastic system

$$x_{k+1} = A_k x_k + B_k w_k, \tag{24}$$

where w_k is Gaussian white noise with zero mean and identity covariance matrix: $w_k \sim \mathrm{N}(0, I_m)$, w_k and x_k are jointly independent, and the initial condition (at time k_0) is Gaussian with zero mean: $x_{k_0} \sim \mathrm{N}(0, P_{k_0})$. Note that the state covariance matrix $P_k = E[x_k x_k^*]$ satisfies (23). Thus, if we initialize system (24) with P_{k_0} equal to one of the (steady-state) solutions[1] of (23), the resulting process x_k is cyclostationary (of period K).

For another interpretation of the DPLE (23), note that its steady-state solution gives the (infinite-window) reachability grammian, which is defined as

$$W_k := W_r(k, -\infty) = \sum_{j=-\infty}^{k-1} \Phi(k, j+1) B_j B_j^* \Phi^*(k, j+1). \tag{25}$$

3.1 Alternative formulations for the DPLE

Equation (23) is an algebraic equation to be solved for K unknowns $P_0, \ldots, P_{K-1}$. It can be rewritten as the discrete-time algebraic Lyapunov equation (DALE)

$$\mathcal{P} = \mathcal{A}\mathcal{P}\mathcal{A}^* + \mathcal{B}\mathcal{B}^*, \tag{26}$$

[1] also called *periodic generators*

where

$$\mathcal{P} := \begin{bmatrix} P_1 & 0 & \cdots & \cdots & 0 \\ 0 & P_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & P_{K-1} & 0 \\ 0 & 0 & \cdots & 0 & P_0 \end{bmatrix}, \tag{27}$$

and

$$\mathcal{A} := \begin{bmatrix} 0 & 0 & \cdots & 0 & A_0 \\ A_1 & 0 & 0 & \cdots & 0 \\ \vdots & A_2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & A_{K-1} & 0 \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} B_0 & 0 & \cdots & \cdots & 0 \\ 0 & B_1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & B_{K-2} & 0 \\ 0 & 0 & \cdots & 0 & B_{K-1} \end{bmatrix}. \tag{28}$$

To solve (26) for $P_0, \ldots, P_{K-1}$, we cannot blindly use well known DALE techniques (Hammarling [5, 6]), since those methods would ignore the structure of the matrices and solve for a dense $\mathcal{P}$. However, it is interesting to note that the eigenvalues of $\mathcal{A}$ are just the K-th roots of those of $\Psi_0 = A_{K-1}A_{K-2}\cdots A_1A_0$ (or any other monodromy matrix Ψ_τ), so we can justifiably hope to simplify $\mathcal{A}$ using the periodic Schur decomposition. In fact, the following will work — first use theorem 1 (corollary) to upper-triangularize A_k, then do a *perfect shuffle* (permute rows and columns) of equation (26) to go to

$$\hat{\mathcal{P}} = \hat{\mathcal{A}}\hat{\mathcal{P}}\hat{\mathcal{A}}^* + \hat{\mathcal{B}}\hat{\mathcal{B}}^*,$$

where $\hat{\mathcal{A}}$ is block upper-triangular with cyclic diagonal blocks, and $\hat{\mathcal{P}}$ is "dense" with diagonal submatrices (the (i,j)-th block-element of $\hat{\mathcal{P}}$ is an $n \times n$ diagonal matrix with the (i,j)-th elements of $P_0, \ldots, P_{K-1}$ on its diagonal). After these preliminaries, the Hammarling method [5, 6] can be used to solve for $\hat{\mathcal{P}}$, and hence for P_k, $k = 0, \ldots, K-1$. The (i,j)-th elements of all P_k are found simultaneously — this would be important in a parallel implementation. However, we shall not deal any further with this viewpoint in this paper.

Alternatively, to find the desired P_τ, $0 \le \tau \le K-1$, all we have to do is solve

$$P_\tau = \Phi(\tau + K, \tau)P_\tau\Phi^*(\tau + K, \tau) + W_r(\tau + K, \tau) \tag{29}$$

for each value of τ, where as already noted in Section 1.1, $\Phi(\tau + K, \tau)$ is the monodromy matrix at τ (also denoted by Ψ_τ), and $W_r(\tau + K, \tau)$ is the finite window reachability grammian over one period starting at τ. Note that (29) is a DALE. This method is elaborated further in Section 3.2.

Previous methods for DPLE solution:

Equation (23) has been studied extensively, and conditions for the existence and uniqueness of solutions are now well known [19]. Surprisingly however, little attention has been paid to the efficient numerical computation of these solutions. For instance, DPLEs arise in the iterative solution by linearization of the DPRE [8, 11], and the authors mention [8, Section VIII-A] that at each iteration, the bottleneck consists of solving a DPLE similar to (23). Whilst tackling (23) through (29), they suggest multiplying A_k and B_k to form $\Phi(\cdot)$ and $W_r(\cdot)$, and then solving (29) via standard methods [5, 6]. Now, formation of matrix products should be avoided as far as possible, both to reduce the number of flops and for numerical accuracy. Moreover, *explicit* computation of (29) followed by application of standard DALE techniques amounts to ignoring the underlying structure, viz. periodicity, of the original problem — which is to solve (23). And it is a cardinal rule of numerical analysis that *structure should be exploited whenever possible.*[2]

In what follows, we describe two approaches for finding the SPPS solution of (23). In the first procedure, which is described in Section 3.2, we reformulate (23) as (29), but this reformulation is only *implicit*. The distinguishing feature of our method is that while solving (29), the matrix products Ψ_τ and $W_r(\cdot,\cdot)$ are never actually computed — instead we implicitly triangularize Ψ_τ using unitary transformations (via the periodic Schur decomposition). The second method, described in Section 3.3, treats (23) as a special case of the DPRE (9), and solves it using ideas explained in Section 2.2.

3.2 Proposed method 1 – periodic Hammarling

This is basically a straightforward extension of Hammarling's procedure [6] to the periodic case. It is known that for a stabilizable periodic pair $(A(\cdot), B(\cdot))$, the DPLE (23) has a unique SPPS solution iff $A(\cdot)$ is asymptotically stable [8]. The periodic generators P_τ, $0 \le \tau \le K-1$ of this solution must satisfy the DALE (29). To verify this for P_0, consider equation (23) for $k = 0, 1, \ldots, K$:

$$
\begin{aligned}
P_1 &= A_0 P_0 A_0^* + B_0 B_0^* \\
P_2 &= A_1 P_1 A_1^* + B_1 B_1^* \\
&= A_1 A_0 P_0 A_0^* A_1^* + A_1 B_0 B_0^* A_1^* + B_1 B_1^* \\
&\;\;\vdots
\end{aligned}
$$

[2] quoted from the book *Matrix Computations* by Golub and Van Loan.

$$P_K = \Phi(K,0)P_0\Phi^*(K,0) + \sum_{j=0}^{K-1}\Phi(K,j+1)B_jB_j^*\Phi^*(K,j+1), \tag{30}$$

which is just (29) since $P_K = P_0$, and the summation on the right is $W_r(K,0)$. Equation (30) can therefore be rewritten as the DALE

$$P_0 = \Psi_0 P_0 \Psi_0^* + \sum_{j=0}^{K-1}\Phi(K,j+1)B_jB_j^*\Phi^*(K,j+1), \tag{31}$$

which must be solved for P_0. We can apply to (31) a procedure similar to Hammarling [5], and obtain P_0 as

$$P_0 = V_0V_0^*, \tag{32}$$

where V_0 is upper-triangular. Then, using the recursion (23), the other matrices P_τ, $0 < \tau \leq K-1$, can be found in such a decomposition $V_\tau V_\tau^*$ as well, where V_τ is upper-triangular.

Finding P_0:

To solve (31), it would clearly help to simplify the coefficient matrices using stable transformations, as explained in [6, 5]. To this end, we appeal to the periodic Schur decomposition (corollary of theorem 1). We saw that Q_i puts Ψ_i in Schur form, viz.,

$$Q_i^*\Psi_iQ_i = \hat{A}_{i+K-1}\cdots\hat{A}_{i+1}\hat{A}_i =: \tilde{\Psi}_i.$$

In particular, (unitary) similarity transformation with Q_0 puts Ψ_0 in upper triangular form. Thus pre and post multiplication, by Q_0^* and Q_0 respectively, would reduce (31) to the following equivalent DALE

$$\tilde{P}_0 = \tilde{\Psi}_0\tilde{P}_0\tilde{\Psi}_0^* + \sum_{j=0}^{K-1}(Q_0^*\Phi(K,j+1)B_j)(Q_0^*\Phi(K,j+1)B_j)^*, \tag{33}$$

where $\tilde{P}_0 = Q_0^*P_0Q_0$, and $\tilde{\Psi}_0 = Q_0^*\Psi_0Q_0 = \hat{A}_{K-1}\hat{A}_{K-2}\cdots\hat{A}_1\hat{A}_0$. This transformation is carried out implicitly, meaning that Ψ_0 and the Φ's are not actually computed. Instead, the periodic Schur algorithm [1] works on A_k directly to give Q_0 and $\hat{A}_k$.

The transformed equation (33) has a particularly simple form. This makes the computation of $\tilde{P}_0$ very easy. First of all, as mentioned earlier, $\tilde{\Psi}_0$ is an upper-triangular matrix. What is more, each term of the summation on the right hand side can be put in the form GG^*, where G is upper-triangular. This becomes clear when we examine the individual terms. For $0 \leq j \leq K-2$, we have

$$\begin{aligned} Q_0^*\Phi(K,j+1)B_j &= Q_0^* A_{K-1}A_{K-2}\cdots A_{j+2}A_{j+1}B_j \\ &= \hat{A}_{K-1}\hat{A}_{K-2}\cdots \hat{A}_{j+2}\hat{A}_{j+1}(Q_{j+1}^* B_j) \end{aligned} \tag{34}$$

upon inserting $Q_iQ_i^*$, $j < i < K$, at appropriate places. Further, we can replace $(Q_{j+1}^*B_j)$ by the upper-triangular matrix of its RQ-factorization.

3.3 Proposed method 2 — deflating subspace approach

It is well known that every Lyapunov equation is also a Riccati equation. This simple observation leads us to yet another method, which we believe is novel, for solving the DPLE (23). And that is to treat it as a DPRE, and use Riccati techniques such as those described in Section 2.2 for its solution. If we make the substitutions

$$A_k \leftarrow {A_k}^*, \; B_k \leftarrow 0, \; Q_k \leftarrow B_k {B_k}^*$$

in the DPRE (9), it reduces to the DPLE

$$P_k = A_k P_{k+1} A_k^* + B_k {B_k}^*,$$

and hence the latter equation can be handled by considering the *periodic eigenvalue problem:*

$$\begin{bmatrix} I & 0 \\ 0 & A_k \end{bmatrix} z_{k+1} = \begin{bmatrix} A_k^* & 0 \\ -B_k{B_k}^* & I \end{bmatrix} z_k. \tag{35}$$

Notice that for the time-invariant continuous-time case, this idea boils down to eliminating an off-diagonal block in a block-triangular matrix (see Section 4).

For simplicity, we explain this approach for a time-invariant Lyapunov equation (DALE) only. The basic idea carries over with minor modifications to the periodic case, to illustrate which we shall give a numerical example later. Accordingly, let us now consider

$$P = APA^* + BB^* \tag{36}$$

with A already in Schur form. Under the assumption that A is stable, (36) has a unique symmetric solution $P > 0$. As pointed out above, we can view (36) as an algebraic Riccati equation and invoke standard Riccati techniques. In particular, we can construct its solution from a basis for the SDS of the generalized eigenvalue problem

$$\lambda G z = H z, \quad \text{with} \quad G = \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}, \quad H = \begin{bmatrix} A^* & 0 \\ -BB^* & I \end{bmatrix}. \tag{37}$$

If A is invertible, $S := G^{-1}H$ is symplectic,

$$S := \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}^{-1} \begin{bmatrix} A^* & 0 \\ -BB^* & I \end{bmatrix} = \begin{bmatrix} A^* & 0 \\ -A^{-1}BB^* & A^{-1} \end{bmatrix},$$

and we know its eigenvalues — just those of A^* and A^{-1}. Hence we could compute S and find a basis for its stable eigenspace to determine the desired solution P. However, for numerical reasons, it is preferable to work directly with the pencil $\lambda G - H$ and compute its SDS.

Stable deflating subspace:

As explained in [16, 17], we must find matrices Q, Z to upper triangularize G and H :

$$QGZ = \overline{\diagdown}\big|, \qquad QHZ = \overline{\diagdown}\big|, \tag{38}$$

with the stable generalized eigenvalues on top. Since A is in Schur form, G is already upper-triangular; so we need to upper-triangularize H only (while maintaining upper-triangularity of G). Once Q and Z have been found, with $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$, the solution to (36) is given by $P = Z_{21}Z_{11}^{-1}$.

Unitary Q and Z satisfying (38) can be computed using the QZ-algorithm [9]. Alternatively, *non-unitary* transformations of the type $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ -AX & I \end{bmatrix}$, each of which form a group, can be used to construct Q and Z in (38) to have the form

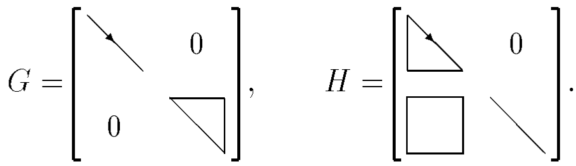
$$Q = \begin{bmatrix} I & 0 \\ -AP & I \end{bmatrix}, \quad Z = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix}, \tag{39}$$

where the sub-block P directly gives the solution to (36). To see this, observe that the matrices Q and Z shown in (39) satisfy

$$
\begin{aligned}
QGZ &= \begin{bmatrix} I & 0 \\ -AP & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} = G \\
QHZ &= \begin{bmatrix} I & 0 \\ -AP & I \end{bmatrix} \begin{bmatrix} A^* & 0 \\ -BB^* & I \end{bmatrix} \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} = \begin{bmatrix} A^* & 0 \\ -APA^* - BB^* + P & I \end{bmatrix}.
\end{aligned} \tag{40}
$$

Hence P solves (36) provided QHZ is upper-triangular in (40).

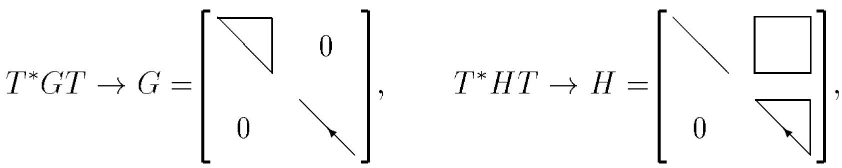
A different, and perhaps simpler, procedure for finding Q and Z in (38) is to perform permutation on G and H followed by reordering of diagonal elements. As before, there exist unitary and non-unitary variants to accomplish the reordering step. Briefly, the procedure is as follows. Since A is already in Schur form, G and H (defined in (37)) look like



Put G and H in upper-triangular form by block permutation (at two levels) using

$$
T = \begin{bmatrix} 0 & \tilde{I} \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & \begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix} \\ \begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix} & 0 \end{bmatrix}, \tag{41}
$$

where $\tilde{I}$ is the *reversal matrix* with 1s on its secondary diagonal. This permutation essentially involves no work, and gives



where $\tilde{I}$ had the effect of reversing the order of rows and columns as indicated by the arrows. We have almost reached our goal now, except that the stable generalized eigenvalues are at the bottom! Next we take these to the top by *reordering*. This reordering can be accomplished by either unitary transformations or non-unitary. In the case of unitary transformations, at each step we can exchange adjacent diagonal elements only. We refer to [20] for details on reordering diagonal elements.

It is important to notice that in contrast to a 'regular' Riccati problem where one must find the (ordered) Schur form of a $2n \times 2n$ matrix, here in the special case of Lyapunov, one needs to do only an $n \times n$ Schur decomposition of the A matrix (followed by eigenvalue reordering). Thus the computational complexity of this Lyapunov solver is comparable to other direct solution methods — we do *not* increase complexity by going via Riccati.

Numerical example:

As an illustration of our proposed Schur method for the DPLE, we present a numerical example. Consider the system of (18) again, with characteristic multipliers 0, 0.0739 and 0.7543. Before beginning to solve the Lyapunov equation (23), we perform a state-space transformation (theorem 1, corollary) to put A_k in upper-triangular form. The updated A_k and B_k are

$$K = 3, n = 3, m = 2,$$

$$A_2 = \begin{bmatrix} -1.1698 & 0.1174 & 0.8326 \\ 0 & 0.5839 & -0.1585 \\ 0 & 0 & -0.5479 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.4503 & 0.5171 \\ -0.1303 & 0.4241 \\ 1.3592 & 0.7003 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -1.7604 & 0.2725 & -0.2578 \\ 0 & 0.5789 & 0.0663 \\ 0.0000 & 0.0000 & -0.0000 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.0550 & -0.5335 \\ 0.1998 & 0.3540 \\ 0.8325 & 1.0880 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 0.3663 & -0.1154 & -0.1157 \\ 0 & 0.2186 & 0.0110 \\ 0 & 0 & 0.0186 \end{bmatrix}, \quad B_0 = \begin{bmatrix} -0.2328 & -0.0157 \\ 0.1593 & -0.1887 \\ -0.6740 & -0.5453 \end{bmatrix},$$

where we have only shown four digits of accuracy. The new monodromy matrix is, just for the record,

$$\Psi_0 = \Phi(3,0) = A_2 A_1 A_0 = \begin{bmatrix} 0.7543 & -0.2924 & -0.2354 \\ -0.0000 & 0.0739 & 0.0044 \\ -0.0000 & 0.0000 & 0.0000 \end{bmatrix},$$

showing that the characteristic multipliers have been found correctly. Next, after the reordering trick explained above, we use the periodic Schur decomposition algorithm to compute a unitary basis for the SDS of the periodic pencil defined in (35). Finally, we obtain

$$P_2 = \begin{bmatrix} 5.0254 & -0.1872 & -0.6263 \\ -0.1872 & 0.1923 & 0.5515 \\ -0.6263 & 0.5515 & 1.8769 \end{bmatrix}, P_1 = \begin{bmatrix} 1.4551 & -0.0315 & 0.1568 \\ -0.0315 & 0.0718 & -0.0034 \\ 0.1568 & -0.0034 & 0.7526 \end{bmatrix}, P_0 = \begin{bmatrix} 10.0295 & 0.1957 & -0.3187 \\ 0.1957 & 0.2075 & 0.1064 \\ -0.3187 & 0.1064 & 2.9013 \end{bmatrix},$$

which satisfy (23) with relative errors 1.8494×10^{-16}, 1.6047×10^{-16} and 3.6080×10^{-16} respectively, the relative error in computing P_k being measured by

$$\|A_{k-1}P_{k-1}A_{k-1}^* + B_{k-1}B_{k-1}^* - P_k\| \;/\; \|P_k\|.$$

The relative errors are of the order of machine epsilon, $\epsilon = 2.2204 \times 10^{-16}$.

3.4 Extensions

• The methods extend with minor modifications to the implicit Lyapunov equation

$$E_i P_{i+1} E_i^* = A_i P_i A_i^* + B_i B_i^*, \quad \text{where } A_i = A_{i+K},\ B_i = B_{i+K},\ E_i = E_{i+K},$$

by using the periodic Schur form of the sequence $A_i, E_i,\ i = 0, 1, \ldots, K-1$.

• Everything we have described so far in Section 3 applies in the real case as well. In that case, we would have 2×2 blocks on diagonal corresponding to complex conjugate eigenvalues.

4 Sylvester equation

When considering periodic observers, say

$$z_{k+1} = F_k z_k + G_k u_k + H_k y_k, \tag{42}$$

with $F_k = F_{k+K}$, $G_k = G_{k+K}$, $H_k = H_{k+K}$, to a periodic system

$$x_{k+1} = A_k x_k + B_k u_k \tag{43}$$
$$y_k = C_k x_k + D_k u_k, \tag{44}$$

$A_k = A_{k+K}$, $B_k = B_{k+K}$, $C_k = C_{k+K}$, $D_k = D_{k+K}$, we encounter the following periodic Sylvester equation

$$T_{k+1} A_k - F_k T_k = H_k C_k, \tag{45}$$

where the monodromy matrix $\Psi_F = F_K \cdots F_1$ has to be chosen stable, and T_k has to be periodic and invertible. If we take $T_k = I$, this problem becomes one of periodic pole placement, and we can use our Schur idea, as described in [21]. Alternatively, we can first choose F_k to satisfy the stability constraints (take e.g.

F_k constant and stable), and then solve for T_k from (45). In this case, recursively applying (45) leads to

$$T_k \Psi_A - \Psi_F T_k = \Omega_k, \tag{46}$$

where Ω_k is a summation of matrices involving A_k, F_k, C_k and H_k. (The monodromy matrix Ψ_F is independent of k when F_k is chosen constant.) A straightforward approach is to use a periodic analog (involving the periodic Schur form) of the Bartels-Stewart algorithm [23] to solve (46); and then recurse on (45) assuming invertibility of A_k. However, instead of involving ourselves with complicated expressions like Ω_k which arise above, we can *directly* use the periodic Schur idea as follows. Equation (45) is equivalent to

$$\begin{bmatrix} I & 0 \\ T_{k+1} & I \end{bmatrix}^{-1} \begin{bmatrix} A_k & 0 \\ H_k C_k & F_k \end{bmatrix} \begin{bmatrix} I & 0 \\ T_k & I \end{bmatrix} = \begin{bmatrix} A_k & 0 \\ 0 & F_k \end{bmatrix}. \tag{47}$$

Denoting $S_k = \begin{bmatrix} A_k & 0 \\ H_k C_k & F_k \end{bmatrix}$, we can define $S^{(k)}$ in the usual manner (5). Since $S^{(k)} = \begin{bmatrix} \Psi_A & 0 \\ \mathrm{X} & \Psi_F \end{bmatrix}$, its spectrum is clearly that of Ψ_A and of Ψ_F. Also (47) implies

$$\begin{bmatrix} I & 0 \\ T_k & I \end{bmatrix}^{-1} S^{(k)} \begin{bmatrix} I & 0 \\ T_k & I \end{bmatrix} = \begin{bmatrix} \Psi_A & 0 \\ 0 & \Psi_F \end{bmatrix}, \tag{48}$$

so $\mathrm{Im}\begin{bmatrix} I \\ T_k \end{bmatrix}$ is an invariant subspace of $S^{(k)}$ with spectrum Ψ_A. This invariant subspace is unique if $\Lambda(\Psi_A) \cap \Lambda(\Psi_F) = \phi$. So the periodic Schur algorithm with reordering to put $\Lambda(\Psi_A)$ on top of the product $S^{(k)}$ will give bases for the spaces $\mathrm{Im}\begin{bmatrix} I \\ T_k \end{bmatrix}$ from the first n columns of the transformation matrices Q_k:

$$\begin{bmatrix} I \\ T_k \end{bmatrix} = \begin{bmatrix} Q_{11k} \\ Q_{12k} \end{bmatrix} Q_{11k}^{-1}. \tag{49}$$

The invertibility of Q_{11k} and Q_{12k} is guaranteed when (A_k, C_k) is observable.

Assume now that we have done a preliminary reduction of A_k, F_k to Schur form (actually F_k can be *chosen* to have this form). One way of finding these transformation matrices Q_k is to first do a permutation to swap diagonal blocks:

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}^{*} \begin{bmatrix} A_k & 0 \\ H_k C_k & F_k \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} F_k & H_k C_k \\ 0 & A_k \end{bmatrix}. \tag{50}$$

This is already in Schur form, but with the incorrect order of eigenvalues. So we update the transformations to put the eigenvalues of Ψ_A on top of $S^{(k)}$ by reordering, and this yields a solution to (45). Note that we did not need to assume invertibility of A_k.

Extension of these ideas to the equation $D_k T_{k+1} A_k - F_k T_k B_k = C_k$, for periodic observers in a descriptor variable framework, can be considered by taking pairs of (periodic) matrices.

Another application:

An equation resembling (46) arises when one considers the *spectral projection* problem for periodic discrete-time systems. See [22] for details in the time-invariant case. Suppose one wishes to use a transformation method similar to [23] to solve for T_k in (46) — clearly, as indicated earlier, we can use our Schur method to implicitly put Ψ_A and Ψ_F in upper (or lower) triangular form, and solve for T_k by back-substitution. The other T_k can then be found recursively using (45), provided A_k is invertible.

5 Conclusion

In this paper, a Schur method was proposed for finding the steady-state solution of some periodic discrete-time matrix equations. On the whole, only unitary transformations were used. The DPRE in its various forms was solved by using the periodic Schur decomposition to (simultaneously) triangularize the matrices connected with a periodic pencil formulation of the Hamiltonian difference equations. The DPLE was tackled by two methods. In one, it was solved after simplification by a technique similar to the Hammarling method for a DALE. In the other, deflating subspace approach, it was treated as a special case of the DPRE. Similar techniques were used to study periodic Sylvester equations.

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