FORWARD/BACKWARD DECOMPOSITION OF PERIODIC DESCRIPTOR SYSTEMS

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Abstract

We propose a Forward/Backward (F/B) decomposition of periodic discrete time descriptor systems and describe a numerically reliable method to compute it. We mention a couple of applications of this F/B canonical form; namely, the solution of two-point boundary value problems, and discrete time Floquet theory.

1 Introduction

We consider the periodically time varying (PTV) system

$$\Sigma : \quad E(k)x(k+1) = A(k)x(k), \quad k \in \mathbb{Z}^+, \tag{1}$$

where $\mathbb{Z}^+$ is the set of non-negative integers, $x(k)$ is an n-dimensional vector of descriptor variables, and K is the smallest positive integer for which $E(k) = E(k+K), A(k) = A(k+K), \forall k \in \mathbb{Z}^+$. We assume that the matrices $E(k)$ and $A(k)$ are real, $n \times n$, and satisfy the condition

$$\lambda \begin{bmatrix} E_0 & & & \\ & E_1 & & \\ & & \ddots & \\ & & & E_{K-1} \end{bmatrix} - \begin{bmatrix} & & & A_0 \\ A_1 & & & \\ & \ddots & & \\ & & A_{K-1} & \end{bmatrix} \tag{2}$$

is regular.
Recently [1] we have shown that statement (2) is equivalent to *solvability* or *conditionability* of Σ, an assumption commonly made in connection with the Two-point Boundary Value Problem (TPBVP)

$$\begin{bmatrix} -A(0) & E(0) & & & \\ & -A(1) & E(1) & & \\ & & \ddots & \ddots & \\ & & & -A(N-1) & E(N-1) \\ \hdashline W(0) & & & & W(N) \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \\ x(N) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ w \end{bmatrix}. \tag{3}$$

Here N simply specifies a time interval of interest. For a discussion of the general time-varying TPBVP, see Luenberger [2, 3]. When studying equation (1), it is common to write

$$V(k)\,E(k)\,W(k+1)\,x(k+1) = V(k)\,A(k)\,W(k)\,x(k), \tag{4}$$

where $V(k) = V(k+K)$, $W(k) = W(k+K)$ are nonsingular for all k, and try to achieve as much simplification

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in form as possible for $E(k), A(k)$. For instance, in the time-invariant (TI) case, where none of the above matrices depend on k, one could reduce E, A to Schur form [4] or decompose the system into "forward" and "backward" subsystems [5]. In this note, for the periodic case, we take the latter approach and demonstrate that it is possible to obtain a canonical forward/backward (F/B) form in (4):

$$E(k) \leftarrow \begin{bmatrix} I & 0 \\ 0 & E_b(k) \end{bmatrix}, \quad A(k) \leftarrow \begin{bmatrix} A_f(k) & 0 \\ 0 & I \end{bmatrix},$$

where the subscripts 'f' and 'b' refer to forward and backward components. Such a F/B form appears to be quite useful in the general study of periodic systems, e.g., in realization [6]. As an immediate application, we demonstrate that the decomposition into F/B form greatly simplifies the *boundary recursion* formulae of Luenberger [3] and also helps to extend an earlier result in discrete time Floquet theory [7] to descriptor systems.

2 A spectral decomposition theorem and the F/B form

We begin by introducing some notation. For Σ, it is well-known that the spectrum, i.e., the set of eigenvalues counting multiplicities, of the periodic pencil $\lambda E(k) - A(k)$ plays an important role; we shall denote it by $\boldsymbol{\lambda}([E, A])$. The special situations where either $E(k) = I_n$ or $A(k) = I_n$ afford some simplification and we mention these separately. When $E(k) = I_n$, $k \in \mathbb{Z}^+$, we have the *forward transition matrix*

$$\begin{aligned} &\phi_A(k, k_0) = \\ &\begin{cases} A(k-1)A(k-2)\ldots A(k_0+1)A(k_0) & \text{if } k > k_0 \geq 0, \\ I & \text{if } k = k_0 \geq 0, \end{cases} \\ &\phi_A(k, k_0) \text{ undefined for } k < k_0, \end{aligned} \tag{5}$$

and the characteristic multipliers $\boldsymbol{\lambda}_f([A]) := \boldsymbol{\lambda}([I, A])$, which are nothing but the eigenvalues of the monodromy matrix $\phi_A(k_0 + K, k_0)$. Similarly, when $A(k) = I_n$, $k \in \mathbb{Z}^+$, we have the *backward transition matrix*

$$\begin{aligned} &\varphi_E(k_0, k) = \\ &\begin{cases} E(k_0)E(k_0+1)\ldots E(k-1)E(k-1) & \text{if } k > k_0 \geq 0, \\ I & \text{if } k = k_0 \geq 0, \end{cases} \\ &\varphi_E(k_0, k) \text{ undefined for } k < k_0, \end{aligned} \tag{6}$$

and the characteristic multipliers $\boldsymbol{\lambda}_b([E]) := \boldsymbol{\lambda}([E, I])$, which are nothing but the eigenvalues of the monodromy matrix $\varphi_E(k_0, k_0 + K)$.

We now state a spectral decomposition theorem, which extends a well known result for time invariant pencils to the periodic case.

Theorem 1. *Given $E(k), A(k)$ satisfying* (1), (2), *and an arbitrary disjoint partition Λ_1, Λ_2 of the spectrum $\boldsymbol{\lambda}([E, A])$, i.e., $\Lambda_1, \Lambda_2 \subsetneq \boldsymbol{\lambda}([E, A])$, $\Lambda_1 \cup \Lambda_2 = \boldsymbol{\lambda}([E, A])$, $\Lambda_1 \cap \Lambda_2 = \varnothing$, it is always possible to find K-periodic nonsingular matrices $V(k)$ and $W(k)$ such that*

$$V(k)\,E(k)\,W(k+1) = \begin{bmatrix} E_{11}(k) & 0 \\ 0 & E_{22}(k) \end{bmatrix},$$
$$V(k)\,A(k)\,W(k) = \begin{bmatrix} A_{11}(k) & 0 \\ 0 & A_{22}(k) \end{bmatrix},$$

where all matrices on the right are upper-triangular, and

$$\boldsymbol{\lambda}([E_{11}, A_{11}]) = \Lambda_1, \quad \boldsymbol{\lambda}([E_{22}, A_{22}]) = \Lambda_2.$$

Sketch of proof. Step 1. Put $E(k)$ and $A(k)$ in upper-triangular form using the periodic Schur algorithm [8], which always exists (and defines the spectrum unambiguously when (2) is satisfied). Arrange, by reordering diagonal elements if necessary, that

$$E(k) = \begin{bmatrix} E_{11}(k) & E_{12}(k) \\ 0 & E_{22}(k) \end{bmatrix}, \quad A(k) = \begin{bmatrix} A_{11}(k) & A_{12}(k) \\ 0 & A_{22}(k) \end{bmatrix},$$

with $\boldsymbol{\lambda}([E_{11}, A_{11}]) = \Lambda_1$ and $\boldsymbol{\lambda}([E_{22}, A_{22}]) = \Lambda_2$. This step uses unitary periodic transformations only.

Step 2. Since $\boldsymbol{\lambda}([E_{11}, A_{11}]) \cap \boldsymbol{\lambda}([E_{22}, A_{22}]) = \varnothing$, zero each off-diagonal block using a further (nonunitary) periodic transformation. This involves solving the periodic Sylvester equation

$$\begin{bmatrix} I & L(k) \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} E_{11}(k) & E_{12}(k) \\ 0 & E_{22}(k) \end{bmatrix} \begin{bmatrix} I & R(k+1) \\ 0 & I \end{bmatrix} = \begin{bmatrix} E_{11}(k) & 0 \\ 0 & E_{22}(k) \end{bmatrix}$$
$$\begin{bmatrix} I & L(k) \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} A_{11}(k) & A_{12}(k) \\ 0 & A_{22}(k) \end{bmatrix} \begin{bmatrix} I & R(k) \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11}(k) & 0 \\ 0 & A_{22}(k) \end{bmatrix}.$$

The left and right transformations from Steps 1 and 2 can be combined to get $V(k)$ and $W(k)$. $\square$

Remark 1. By choosing Λ_1, Λ_2 appropriately in Theorem 1, it is possible to have $\boldsymbol{\lambda}([E_{11}, A_{11}])$ lie on or inside the unit circle and $\boldsymbol{\lambda}([E_{22}, A_{22}])$ outside the unit circle. Note that $E_{11}(k)$ and $A_{22}(k)$ are then nonsingular. Such a form would be useful, for instance, in finding a minimal order TI description of Σ using strict system equivalence [9] operations on its stacked [10] representation. This work is currently under progress [11]. See [12] for details of this procedure for standard state space systems, viz., when $E(k) = I_n, k \in \mathbb{Z}^+$.

Remark 2. Theorem 1 would be useful when one considers the *additive decomposition* problem for periodic discrete-time systems. See [13] for details in the time-invariant case.

Remark 3. Theorem 1 shows the possibility of diagonalizing $E(k)$ and $A(k)$ when the eigenvalues $\boldsymbol{\lambda}([E, A])$ are distinct.

Theorem 1 quickly leads to the promised F/B form, as shown next.

Corollary 1. *Given $E(k)$ and $A(k)$ satisfying (1), (2), it is always possible to find K-periodic nonsingular matrices $V(k)$ and $W(k)$ such that*

$$
\begin{aligned}
V(k)\, E(k)\, W(k+1) &= \begin{bmatrix} I & 0 \\ 0 & E_b(k) \end{bmatrix}, \\
V(k)\, A(k)\, W(k) &= \begin{bmatrix} A_f(k) & 0 \\ 0 & I \end{bmatrix},
\end{aligned} \tag{7}
$$

where $E_b(k)$ and $A_f(k)$ are upper-triangular, $\boldsymbol{\lambda}_f([A_f])$ lies on or inside the unit circle, and $\boldsymbol{\lambda}_b([E_b])$ lies inside the unit circle.

Sketch of proof. Use Theorem 1 (in particular, see Remark 1). Premultiply with $\operatorname{diag}\big(E_{11}^{-1}(k), A_{22}^{-1}(k)\big)$ to get the final result in (7), with

$$
A_f(k) = E_{11}^{-1}(k) A_{11}(k), \quad E_b(k) = A_{22}^{-1}(k) E_{22}(k). \quad \square
$$

We conclude this note by mentioning two uses of the F/B canonical form, namely, periodic TPBVPs and discrete time Floquet theory.

3 Two-point boundary value problems

Clearly, Corollary 1 decomposes the original system Σ into a forward and a backward part:

$$
x_f(k+1) = A_f(k) x_f(k), \quad A_f(k+K) = A_f(k), \tag{8f}
$$

$$
E_b(k) x_b(k+1) = x_b(k), \quad E_b(k+K) = E_b(k), \tag{8b}
$$

Given $x_f(0)$, we can iterate (??f) forward in time to obtain $x_f(k)$ for all $k \geq 0$, and, similarly, given $x_b(N)$, we can iterate (??b) backwards in time to obtain $x_b(k)$ for all $0 \leq k \leq N$. The solution will, of course, involve the forward and backward transition matrices introduced in (5) and (6).

Alternatively, if the boundary conditions are specified as in (3), then the F/B decomposition yields very simple expressions for Luenberger's *boundary recursion* procedure [3] applied to Σ, as we show next. The idea behind the procedure is to express the inherent linear restrictions on $x(0)$ and $x(N)$ in the form of a *boundary mapping* equation $Z_0(0,N)x(0) + Z_N(0,N)x(N) = 0$. Likewise, one could write

$$
Z_0(0,k)x(0) + Z_k(0,k)x(k) = 0, \quad k = 1, 2, \ldots, N, \tag{9}
$$

where $Z_0(0,k)$ and $Z_k(0,k)$ are some appropriate $n \times n$ matrices, to make explicit the linear restrictions on $x(0)$ and $x(k)$ that are imposed implicitly by (1). The basic step is boundary recursion, wherein $Z_0(0,k)$ and $Z_k(0,k)$ are found recursively as follows:

$$
\begin{aligned}
&\begin{bmatrix} F(k) & G(k) \\ H(k) & J(k) \end{bmatrix} \begin{bmatrix} Z_0(0,k) & Z_k(0,k) & 0 \\ 0 & -A(k) & E(k) \end{bmatrix} \\
&\qquad = \begin{bmatrix} Z_0(0,k+1) & 0 & Z_{k+1}(0,k+1) \\ X(k) & I & Y(k) \end{bmatrix}.
\end{aligned} \tag{10}
$$

It is easy to see that $F(k) = A(k), G(k) = Z_k(0,k)$ will produce the desired zero block on the right hand side of (10). Furthermore, when $E(k)$ and $A(k)$ are in the F/B form, we have

$$
\begin{aligned}
&\left[\begin{array}{cc|cc} A_f(k) & & I & \\ & I & & \varphi_{E_b}(0,k) \\ \hline I & & 0 & \\ & 0 & & -I \end{array}\right] \\
&\quad \cdot \left[\begin{array}{cc|cc|cc} -\phi_{A_f}(k,0) & & I & & & \\ & -I & & \varphi_{E_b}(0,k) & & \\ \hline & & -A_f(k) & & I & \\ & & & -I & & E_b(k) \end{array}\right] \\
&\quad = \left[\begin{array}{cc|cc|cc} -\phi_{A_f}(k+1,0) & & & & I & \\ & -I & & & & \varphi_{E_b}(0,k+1) \\ \hline -\phi_{A_f}(k,0) & & I & & 0 & \\ & 0 & & I & & -E_b(k) \end{array}\right].
\end{aligned} \tag{11}
$$

Comparing (10) with (11), we get all the necessary formulae for boundary recursion of periodic systems. Finally, the boundary conditions $W(0)x(0) + W(N)x(N) = w$, with $W(0) = [W_{0f}, W_{0b}]$, $W(N) = [W_{Nf}, W_{Nb}]$ partitioned according to (8), are independent of the boundary mapping of Σ if

$$
\left[W_{0f} + \phi_{A_f}(N,0) W_{Nf}, \; W_{Nb} + \varphi_{E_b}(0,N) W_{0b} \right] \tag{12}
$$

is nonsingular. Equations (11) and (12) are analogous to the TI case discussed in [3].

4 Discrete time Floquet theory

Since time invariant linear system theory is much more developed than its periodic counterpart, it is natural to wonder under what conditions can the PTV system Σ be converted into a TI system by a nonsingular transformation as shown in equation (4). A transformation of the kind

$$
E(k) \leftarrow Z(k)\, E(k)\, Q(k+1), \quad A(k) \leftarrow Z(k)\, A(k)\, Q(k),
$$

where the matrices $Q(k)$, $Z(k)$ are nonsingular for all k, is known as a *kinematic similarity*, and a *K-periodic similarity* when $Q(k)$, $Z(k)$ are, in addition, periodic of period K.

In this section, we give a necessary and sufficient condition for a K-periodic pair $\{A(k),\ E(k)\}$ to be K-periodically similar to a constant pair $\{E, A\}$.

We need a couple of lemmas first. Lemma 1 states when a forward PTV system

$$x(k+1) = A_f(k)x(k), \quad A_f(k+K) = A_f(k),$$

can be transformed into a TI system

$$\hat{x}(k+1) = A_f\hat{x}(k)$$

under an invertible, K-periodic, change of variable $\hat{x}(k) := T_f^{-1}(k)x(k)$. See also [7], [14].

Lemma 1. *Given K-periodic matrices $A_f(k)$, there exist K-periodic invertible matrices $T_f(k)$ satisfying*

$$T_f^{-1}(k+1)A_f(k)T_f(k) = A_f, \quad k \in \mathbb{Z}^+, \tag{13}$$

if and only if

$$\text{nullity} \begin{bmatrix} A_f(j+i-1) & & & & \\ & I & \ddots & & \\ & & \ddots & & \\ & & & I & A_f(j+1) \\ & & & & I & A_f(j) \end{bmatrix}$$

$$\textit{is independent of } j, \textit{ for } 1 \le i \le n,\ 1 \le j \le K. \tag{14}$$

Proof. In [7], we proved that (13) is satisfied if and only if

$$\text{nullity}\,\{A_f(j+i-1)\cdots A_f(j+1)A_f(j)\}$$

$$\text{is independent of } j, \text{ for } 1 \le i \le n,\ 1 \le j \le K. \tag{15}$$

But $A_f(j+i-1)\cdots A_f(j+1)A_f(j) = \phi_A(j+i,j)$ has the same nullity as the matrix in (14), because the latter matrix is, modulo right and left elementary operations, nothing but

$$\text{diag}\,\{I, \ldots, I, \phi_A(j+i,j)\}.$$

Therefore, (14) is the same as (4). ☐

We state the next lemma without proof; it gives a corresponding result for the backward PTV system

$$E_b(k)x(k+1) = x(k), \quad E_b(k+K) = E_b(k).$$

Lemma 2. *Given K-periodic matrices $E_b(k)$, there exist K-periodic invertible matrices $T_b(k)$ satisfying*

$$T_b^{-1}(k)E_b(k)T_b(k+1) = E_b, \quad k \in \mathbb{Z}^+, \tag{16}$$

if and only if

$$\text{nullity} \begin{bmatrix} E_b(j+i-1) & I & & \\ & \ddots & \ddots & \\ & & E_b(j+1) & I \\ & & & E_b(j) \end{bmatrix}$$

$$\textit{is independent of } j, \textit{ for } 1 \le i \le n,\ 1 \le j \le K. \tag{17}$$

We now state the conditions under which Σ is K-periodically similar to a TI descriptor system. The F/B form helps in deriving this result.

Theorem 2. *Given K-periodic matrices $E(k), A(k)$, there exist K-periodic invertible matrices $Z(k), Q(k)$ satisfying*

$$Z(k)\,E(k)\,Q(k+1) = E, \quad Z(k)\,A(k)\,Q(k) = A, \quad k \in \mathbb{Z}^+, \tag{18}$$

if and only if the nullities of

$$\begin{bmatrix} A(j+i-1) & & & \\ E(j+i-2) & \ddots & & \\ & \ddots & & \\ & & A(j+1) & \\ & & E(j) & A(j) \end{bmatrix}$$

$$\textit{and} \begin{bmatrix} E(j+i-1) & A(j+i-1) & & \\ & \ddots & \ddots & \\ & & E(j+1) & A(j+1) \\ & & & E(j) \end{bmatrix} \tag{19}$$

are independent of j, for $1 \le i \le n$, $1 \le j \le K$.

Sketch of proof. Necessity is easy to prove because (18) implies

$$\begin{bmatrix} \ddots & & \\ & Z(j+i-1) & \\ & & \ddots \\ & & & Z(j) \end{bmatrix} \cdot \begin{bmatrix} \ddots & \ddots & & \\ & E(j+i-1) & A(j+i-1) & \\ & & \ddots & \ddots \\ & & & E(j) & A(j) \end{bmatrix} \cdot \begin{bmatrix} \ddots & & \\ & Q(j+i) & \\ & & \ddots \\ & & & Q(j) \end{bmatrix} = \begin{bmatrix} \ddots & \ddots & & \\ & E & A & \\ & & \ddots & \ddots \\ & & & E & A \end{bmatrix}. \tag{20}$$

for all $j \in \mathbb{Z}^+$. To show sufficiency, we must invoke the F/B form and *perfect shuffle*, followed by Lemmas 1 and 2, We omit the details here due to lack of space. ☐

5 Conclusion

In this note, we studied spectral decomposition of periodic discrete time descriptor systems and described a numerically reliable method to compute it. This led to a Forward/Backward (F/B) form which is useful for both numerical computation and theory—as shown for instance in Sections 3 and 4, respectively. We expect that further applications for these decompositions will be found.

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