Correspondence

Periodic Descriptor Systems: Solvability and Conditionability

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***Abstract*—The authors consider discrete-time linear periodic descriptor systems and study the concepts of *solvability* and *conditionability*, introduced by Luenberger. They prove that solvability is equivalent to conditionability, just as in the time-invariant case. We give a characterization of solvability/conditionability in terms of a cyclic matrix pencil and, furthermore, propose a simple test via the periodic Schur decomposition to check for either property. This could lead to further systematic study of these systems.**

***Index Terms*—Descriptor systems, periodic systems.**

I. Introduction

Linear descriptor systems represent a broad class of time evolutionary phenomena and are often the product of problem formulation in system theory, especially when the variables used are the natural describing variables of the underlying process. This topic has received a lot of attention over the last 20 years; see, in particular, [5] and [6]. Within the general class of linear descriptor systems, periodic systems form an important subclass—they are suitable models for many natural and man-made phenomena and are finding increasing use in control theory as well [1]. In this paper, we restrict attention to *solvability* and *conditionability* of periodic linear descriptor systems. Throughout, we adopt the basic framework and concepts defined by Luenberger [5], [6] and present new results which extend those obtained by him for the time-invariant (TI) case.

We begin by briefly recalling some definitions and prior work (see [5] for details). A linear discrete-time descriptor variable system defined on the time interval $[0, N]$ has the form

$$E_k x_{k+1} = A_k x_k + u_k, \qquad k = 0, 1, \cdots, N-1. \tag{1}$$

For each k, x_k is an n-dimensional vector of descriptor variables and u_k is an m-dimensional vector of input variables. The matrices E_k and A_k are $n \times n$ with complex entries, and possibly singular. The length of the time interval is N; likewise there are N equations in (1), and these can be written out in block matrix form as

$$\left[\begin{array}{c|cccc|c} -A_0 & E_0 & & & & \\ \cline{2-5} & -A_1 & E_1 & & & \\ & & \ddots & \ddots & & \\ & & & -A_{N-2} & E_{N-2} & \\ \cline{2-5} & & & & -A_{N-1} & E_{N-1} \end{array}\right] \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \tag{2}$$

showing that the dynamic system can be regarded as one (large) system of linear equations. Two fundamental concepts associated with such systems, which characterize their "well behavedness," are solvability and conditionability. The *solvability matrix* of (1), which we shall denote by $S(0; N)$, is the coefficient matrix in (2). It can be regarded as an $N \times (N+1)$ block matrix or, in ordinary terms, as an $nN \times n(N+1)$ matrix. System (1) is said to be *solvable* if $S(0; N)$ is of full rank. Notice that solvability is the standard nondegeneracy condition for (2); it means that (1) possesses an n-dimensional linear variety of solutions for any input sequence. The *conditionability matrix* of (1), denoted by $C(0; N)$, is the submatrix of $S(0; N)$ obtained by deleting the first and last block columns. It is shown enclosed by vertical lines in (2) and is an $N \times (N-1)$ block matrix. System (1) is said to be *conditionable* if $C(0; N)$ is of full rank. Conditionability is equivalent to the property that any two distinct solutions to (2) must differ at (at least) one end-point, zero or N. If the system is solvable, then conditionability implies that a unique solution (within the n-dimensional linear variety) can be specified by appending a total of n boundary conditions involving the values x_0 and x_N only. In general, these boundary conditions cannot all be specified at one end.

It is clear that solvability and conditionability of (1) depend only on the homogeneous system $E_k x_{k+1} = A_k x_k$, $k = 0, 1, \cdots, N-1$. Later in the paper, we shall look at descriptor systems defined over $[0, N]$, where N is an arbitrary positive integer. Furthermore, even if N is fixed, while studying the set of equations in (2) it is often convenient to consider a subset obtained by dropping some of the first and/or last (block) equations. This corresponds to restricting to a subinterval the full time interval over which the original system is defined. Of course, solvability and conditionability are preserved under such a time restriction. In other words, if either of these properties holds for the interval $[0, N]$, then it also holds for any subinterval $[k, l]$, $0 \leq k < l \leq N$; we term this the *inheritance* property. In our notation for solvability and conditionability matrices, the first argument denotes the left end-point, and the second argument the length of the time interval under consideration. For instance, the "boxed" submatrix in (2) is $S(1; N-2)$, the solvability matrix over the interval $[1, N-1]$.

Solvability and conditionability are in a very natural sense dual concepts. Corresponding to a *primal* set of dynamic equations (1), there exists the *subdual* set

$$E_{k-1}^{\mathrm{H}} \lambda_{k-1} = A_k^{\mathrm{H}} \lambda_k + v_k, \qquad k = 1, 2, \cdots, N-1. \tag{3}$$

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We shall denote the solvability matrix and conditionability matrix of the subdual set by $S_D(0;\, N)$ and $C_D(0;\, N)$, respectively; although, strictly speaking, (3) is defined over the interval $[0,\, N-1]$ only.[1] On writing out (3) in block matrix form like (2), we see that

$$S_D(0;\, N) = C(0;\, N)^{\mathrm{H}} \tag{4a}$$
$$C_D(0;\, N) = S(1;\, N-2)^{\mathrm{H}}. \tag{4b}$$

Thus, a primal set of dynamic equations is conditionable iff its subdual is solvable; and, if a primal is solvable, then its subdual is conditionable. This concludes our review.

Next we turn to the periodically time-varying (PTV) descriptor variable system

$$\Sigma\colon E_k x_{k+1} = A_k x_k, \qquad k = 0,\, 1,\, 2,\, \cdots \tag{5}$$

of fundamental period K, i.e., K is the smallest positive integer for which $E_k = E_{k+K}$, $A_k = A_{k+K}$, $\forall k \geq 0$. According to our basic definition (1), a set of dynamic equations is defined with respect to a specific time interval of finite length. Systems of infinite duration are considered [5] to be solvable or conditionable if the corresponding finite sets of equations, terminating at a fixed N, are solvable or conditionable, respectively, for every value of N.

Definition 1: A PTV descriptor system Σ given by (5) is said to be solvable if its solvability matrix $S(0;\, N)$ is of full rank for every $N > 0$. It is said to be conditionable if its conditionability matrix $C(0;\, N)$ is of full rank for every $N > 0$.

This is analogous to the TI situation, where E_k and A_k are independent of k. Just as in the TI case, the presence here of additional structure has interesting implications. For instance, due to periodicity and the inheritance property, we notice straightaway that, for Σ

$$S(0;\, N) \text{ is of full rank for all } N > 0 \tag{6a}$$
$$\Updownarrow$$
$$\text{For any } i > 0,\ S(i;\, N) \text{ is of full rank for all } N > 0. \tag{6b}$$

Hence solvability of Σ is also equivalent to (6b).

Since Σ is of infinite duration it has a true *dual,* Σ_D, given by $E_{k-1}^{\mathrm{H}}\lambda_{k-1} = A_k^{\mathrm{H}}\lambda_k$, $k = 1,\, 2,\, \cdots$. For each $N > 0$, identities (4) continue to hold, so it is straightforward that

$$\Sigma \text{ is conditionable} \Longleftrightarrow \Sigma_D \text{ is solvable} \tag{7a}$$
$$\Sigma \text{ is solvable} \Longrightarrow \Sigma_D \text{ is conditionable.} \tag{7b}$$

In addition

$$\begin{aligned} & C_D(0;\, N) \text{ is of full (column) rank for all } N > 0 \\ & \quad \Longleftrightarrow S(1;\, N-2) \text{ is of full (row) rank for all } N > 0, \\ & \qquad \text{by (4b),} \\ & \quad \Longleftrightarrow S(0;\, N) \text{ is of full (row) rank for all } N > 0, \text{ by (6)} \end{aligned}$$

which proves the reverse implication in (7b) as well.

Lemma 1: A PTV descriptor system Σ is solvable (conditionable) iff its dual Σ_D is conditionable (solvable).

II. Solvability and Conditionability

Luenberger [5], [6] showed that solvability and conditionability are identical concepts in the TI case. It turns out that this is true in the periodic case also. We prove this first using simple rank arguments. An indirect proof will be given later.

Theorem 1: A PTV descriptor system Σ given by (5) is solvable iff it is conditionable.

[1] This is the reason why the term *subdual* rather than *dual* is employed for (3).

Proof—Only If: Suppose Σ is not conditionable. Then there exists an $\hat{N} > 0$ such that $C(0;\, \hat{N})$ is not of full rank. It follows from the inheritance property that $C(0;\, N)$ is not of full rank for any $N \geq \hat{N}$. Let us choose the smallest $\tilde{N} \geq \hat{N}$ such that $\tilde{N} - 1$ is an integer multiple of the period K, say $\tilde{N} - 1 = sK$. Then

$$\begin{aligned} C(0;\, \tilde{N}) &= \begin{bmatrix} E_0 & & & \\ -A_1 & E_1 & & \\ & \ddots & \ddots & \\ & & \ddots & E_{\tilde{N}-2} \\ & & & -A_{\tilde{N}-1} \end{bmatrix} = \begin{bmatrix} E_0 & & & \\ -A_1 & E_1 & & \\ & \ddots & \ddots & \\ & & \ddots & E_{sK-1} \\ & & & -A_{sK} \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} E_0 & & & \\ -A_1 & E_1 & & \\ & \ddots & \ddots & \\ & & \ddots & E_{K-1} \\ & & & -A_0 \end{bmatrix}}_{\tilde{N} \text{ block rows, } \tilde{N}-1 \text{ block columns}} \end{aligned} \tag{8}$$

has rank no greater than $n(\tilde{N}-1)-1$, since it contains $n(\tilde{N}-1)$ columns. Observe that a matrix identical to $C(0;\, K+1)$ is repeated s times in (8) and forms a "building block" for $C(0;\, \tilde{N})$. In the same way, we can run together many copies of $C(0;\, \tilde{N})$ to form conditionability matrices whose columns number even bigger multiples of K. (Like $C(0;\, \tilde{N})$, none of these matrices will have full rank.) For our purpose, let us define

$$\overline{N} := (n+1)(\tilde{N}-1)+1.$$

The solvability matrix over the time interval $[0,\, \overline{N}]$ has the structure

$$S(0;\, \overline{N}) = \begin{bmatrix} -A_0 & \overset{1}{\boxed{C(0;\, \tilde{N})}} & & & & \\ & & \overset{2}{\boxed{C(0;\, \tilde{N})}} & & & \\ & & & \ddots & & \\ & & & & \overset{n+1}{\boxed{C(0;\, \tilde{N})}} & E_0 \end{bmatrix}$$

where we have marked (with boxes) and numbered the repeating patterns. Some simple algebra now shows that this matrix cannot be of full rank, proving that Σ is not solvable. Indeed, the submatrix composed of the boxes, which is nothing but $C(0;\, \overline{N})$, has rank no greater than $(n+1)[n(\tilde{N}-1)-1]$; therefore, the maximum rank that $S(0;\, \overline{N})$ can have, with its $2n$ additional columns in $-A_0$ and E_0, is $(n+1)[n(\tilde{N}-1)-1]+2n = n(n+1)(\tilde{N}-1)+n-1$. This is less than the number of rows it contains, which is $n\overline{N} = n(n+1)(\tilde{N}-1)+n$.

If: The foregoing discussion shows that if a PTV descriptor system is solvable, then it is also conditionable. One can prove the reverse implication using similar arguments; it is easiest to see this by invoking duality. Thus

$$\begin{aligned}
&\Sigma \text{ is conditionable}\\
&\quad\Longrightarrow \Sigma_D \text{ is solvable}, && \text{by Lemma 1,}\\
&\quad\Longrightarrow \Sigma_D \text{ is conditionable}, && \text{just proved,}\\
&\quad\Longrightarrow \Sigma \text{ is solvable}, && \text{by Lemma 1 again.}
\end{aligned}$$

□

Remark 1: For $K = 1$, our proof of Theorem 1 reduces essentially to Luenberger's proof for the TI case [5]. The only subtlety for the $K > 1$ case is that $\tilde{N} - 1$ must be chosen to be a multiple of K, so as to get a repeatable block.

Since the length N of the time interval in question is irrelevant for solvability (or conditionability) of Σ, it is to be expected that this property depends only on the periodic matrix sequence $\{E_k, A_k\}$. We will see that a useful characterization of solvability/conditionability is that a particular cyclic matrix pencil be *regular*. Furthermore, this condition can be recast in terms of the periodic Schur decomposition of $\{E_k, A_k\}$. We state our findings as Theorem 2. First, we recall a couple of facts.

Regular Matrix Pencils: A matrix pencil is a polynomial matrix of the form $z\mathcal{E} - \mathcal{A}$, where z is a complex variable. It is said to be regular if $\mathcal{E}$ and $\mathcal{A}$ are square and $\det(z\mathcal{E} - \mathcal{A})$ does not vanish identically [3]. In other words, a square pencil $z\mathcal{E} - \mathcal{A}$ is regular iff it is of full rank with respect to all polynomial combinations of its rows (or columns). Let $\rho(z) := \rho_0 + \rho_1 z + \ldots + \rho_{\ell-1} z^{\ell-1}$ be a row vector of dimension n. By equating coefficients of powers of z, we can check that

$$\rho(z)\cdot[z\mathcal{E}-\mathcal{A}] = 0 \iff [\rho_0 \quad \rho_1 \quad \cdots \quad \rho_{\ell-1}] \cdot \underbrace{\begin{bmatrix} -\mathcal{A} & \mathcal{E} & & & \\ & -\mathcal{A} & \mathcal{E} & & \\ & & \ddots & \ddots & \\ & & & -\mathcal{A} & \mathcal{E} & \\ & & & & -\mathcal{A} & \mathcal{E} \end{bmatrix}}_{\ell \text{ block rows, } \ell+1 \text{ block columns}} = 0. \tag{9}$$

Therefore, a square pencil $z\mathcal{E} - \mathcal{A}$ is regular iff the block matrix in (9) is of full row rank for any $\ell > 0$. It is appropriate to remark here that these full-rank conditions are equivalent to absence of left Kronecker indexes of $z\mathcal{E} - \mathcal{A}$. In exactly the same way, by considering the column vector $\nu(z) := \nu_0 + \nu_1 z + \cdots + \nu_{\ell-1} z^{\ell-1}$ of dimension n, we can verify that

$$(z\mathcal{E}-\mathcal{A})\cdot\nu(s) = 0 \quad \iff \quad \underbrace{\begin{bmatrix} \mathcal{E} & & & \\ -\mathcal{A} & \mathcal{E} & & \\ & \ddots & \ddots & \\ & & -\mathcal{A} & \mathcal{E} \\ & & & -\mathcal{A} \end{bmatrix}}_{\ell+1 \text{ block rows, } \ell \text{ block columns}} \cdot \begin{bmatrix} \nu_{\ell-1} \\ \nu_{\ell-2} \\ \vdots \\ \nu_1 \\ \nu_0 \end{bmatrix} = 0. \tag{10}$$

Therefore a square pencil $z\mathcal{E} - \mathcal{A}$ is regular iff the block matrix in (10) is of full column rank for any $\ell > 0$, the full-rank conditions being equivalent to absence of right Kronecker indices of $z\mathcal{E} - \mathcal{A}$. These observations are well known and can be found, for instance, in [3]. Luenberger [6] was the first to relate regularity of $z\mathcal{E} - \mathcal{A}$ to solvability and conditionability of the TI matrix pair $\{\mathcal{E}, \mathcal{A}\}$, a connection made clear by (9) and (10).

Periodic Schur Decomposition: We quote a well-known result [2], [4]. For a proof, and an $O(Kn^3)$-algorithm implementation, please refer to [2].

Proposition 1: Let $H_i, G_i,\ i = 0, 1, \ldots, K-1$, be $n \times n$ complex matrices. Then there exist $n \times n$ unitary matrices $Q_i, Z_i,\ i = 0, 1, \cdots, K-1$, such that

$$\begin{aligned}
\hat{G}_0 &= Z_0^{\mathrm{H}} \cdot G_0 \cdot Q_1 & \hat{H}_0 &= Z_0^{\mathrm{H}} \cdot H_0 \cdot Q_0\\
\hat{G}_1 &= Z_1^{\mathrm{H}} \cdot G_1 \cdot Q_2 & \hat{H}_1 &= Z_1^{\mathrm{H}} \cdot H_1 \cdot Q_1\\
\hat{G}_2 &= Z_2^{\mathrm{H}} \cdot G_2 \cdot Q_3 & \hat{H}_2 &= Z_2^{\mathrm{H}} \cdot H_2 \cdot Q_2\\
&\vdots & &\vdots\\
\hat{G}_{K-1} &= Z_{K-1}^{H} \cdot G_{K-1} \cdot Q_0 & \hat{H}_{K-1} &= Z_{K-1}^{\mathrm{H}} \cdot H_{K-1} \cdot Q_{K-1}
\end{aligned}$$

all are upper-triangular. □

Now we come to the main result of this paper.

Theorem 2: The following statements are equivalent.

1) The PTV descriptor system Σ given by (5) is solvable (or conditionable).
2) The TI descriptor system

$$\Sigma^{\mathrm{E}}\colon \mathcal{E}y_{k+1} = \mathcal{A}y_k, \qquad k = 0, 1, 2, \ldots, \tag{11a}$$

is solvable (or conditionable), where

$$\mathcal{E} := \begin{bmatrix} E_0 & 0 & \cdots & \cdots & 0 \\ 0 & E_1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & E_{K-2} & 0 \\ 0 & 0 & \cdots & 0 & E_{K-1} \end{bmatrix}$$

$$\mathcal{A} := \begin{bmatrix} 0 & 0 & \cdots & 0 & A_0 \\ A_1 & 0 & 0 & \cdots & 0 \\ \vdots & A_2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & A_{K-1} & 0 \end{bmatrix}. \tag{11b}$$

3) The cyclic matrix pencil $z\mathcal{E} - \mathcal{A}$ defined by (11b) is regular.
4) There are no "zero by zero divides on the diagonal" in the periodic Schur decomposition of $\{E_k, A_k\}$. More precisely, if $e_{j,j}^{(i)}$ and $a_{j,j}^{(i)}$ denote the jth diagonal elements of the triangular matrices $\hat{E}_i$ and $\hat{A}_i$, respectively, in Proposition 1 applied to $\{E_k, A_k\}$, then

$$\frac{a_{j,j}^{(K-1)} \cdots a_{j,j}^{(1)} a_{j,j}^{(0)}}{e_{j,j}^{(K-1)} \cdots e_{j,j}^{(1)} e_{j,j}^{(0)}}, \qquad j = 1, 2, \cdots, n \tag{12}$$

is well defined (can be zero or $\pm\infty$, but not indeterminate).

Proof $1) \Leftrightarrow 2)$*:* Over an interval of length ℓ, the solvability matrix of Σ^{E} is

$$\left.\left[\begin{array}{c|ccccc|c} -\mathcal{A} & \mathcal{E} & & & & & \\ & -\mathcal{A} & \mathcal{E} & & & & \\ & & \ddots & \ddots & & & \\ & & & -\mathcal{A} & \mathcal{E} & & \\ & & & & -\mathcal{A} & & \mathcal{E} \end{array}\right]\right\}\ell \text{ block rows.} \tag{13}$$

Modulo row and column permutations, it is identical to

$$\operatorname{diag}\{S(0;\,\ell),\, S(1;\,\ell),\, \cdots,\, S(K-1;\,\ell)\} \tag{14}$$

where the $S(i;\,\ell)$ refer to solvability matrices of the PTV system Σ. To convince ourselves that this is indeed the case, we need only to look at the arrangement, relative to one another, of the periodic matrices E_k, A_k in (13) and (14). In each block row, $-A_k$ is followed by E_k (same index), while in each block column, E_k is followed by $-A_{k+1}$ (one index higher). By beginning at the top of (13) with one of $-A_0, -A_1, \cdots, -A_{K-1}$ and descending the "steps," we can form K chains each of length $\ell+1$; these are precisely the diagonal blocks in (14). This shows that (13) is of full row rank iff (14) is. Since this holds for all $\ell > 0$, we conclude that Σ^{E} is solvable iff Σ is.

Similar reasoning can be used to prove that Σ^{E} is conditionable iff Σ is. As a matter of fact, solvability and conditionability are one and the same concept for both PTV and TI systems; therefore, we do not actually need to repeat the argument here. We do so only to illustrate how the conditionability matrices of Σ and Σ^{E} are related. Over an interval of length ℓ, the conditionability matrix of Σ^{E} is the submatrix of (13) enclosed by vertical lines, which on rearranging rows and columns as before becomes $\mathrm{diag}\{C(0;\ell), C(1;\ell), \cdots, C(K-1;\ell)\}$, where the $C(i;\ell)$ refer to conditionability matrices of Σ. One is of full column rank iff the other is, for all $\ell > 0$.

2) $\Leftrightarrow$ 3): This follows from the facts about regular matrix pencils mentioned earlier; see (9) and (10) and also [6].

3) $\Leftrightarrow$ 4): Perform a periodic Schur decomposition on, i.e., apply Proposition 1 to, $\{E_k, A_k\}$. Using the unitary matrices Q_i, Z_i so obtained, define $\mathcal{Z} := \mathrm{diag}\{Z_0, Z_1, \cdots, Z_{K-2}, Z_{K-1}\}$, $\mathcal{Q} := \mathrm{diag}\{Q_1, Q_2, \cdots, Q_{K-1}, Q_0\}$. Update the cyclic matrix pencil $z\mathcal{E} - \mathcal{A}$ given by (11b) as follows: $z\mathcal{E} - \mathcal{A} \longleftarrow \mathcal{Z}^{\mathrm{H}}(z\mathcal{E} - \mathcal{A})\mathcal{Q}$, so that it has the following block structure:

$$z\mathcal{E} - \mathcal{A} = z\begin{bmatrix} U & & & \\ & U & & \\ & & \ddots & \\ & & & U \end{bmatrix} - \left.\begin{bmatrix} & & & U \\ U & & & \\ & \ddots & & \\ & & U & \end{bmatrix}\right\} K \text{ block rows and columns} \tag{15}$$

where U is an upper triangular matrix. Since this updating involves unitary transformations only, it does not change $\det(z\mathcal{E} - \mathcal{A})$ except by a factor of modulus one. Hence regularity of $z\mathcal{E} - \mathcal{A}$ is not affected. Next, do a *perfect shuffle* on (15) with the permutation vector

$$p = [1:n:1+(K-1)n, 2:n:2+(K-1)n, \cdots, n:n:nK]$$

to get (using Matlab notation)

$$\begin{aligned} z\mathcal{E}[p,p] - \mathcal{A}[p,p] &= z\begin{bmatrix} D & D & \cdots & D \\ & D & \cdots & D \\ & & \ddots & \vdots \\ & & & D \end{bmatrix} \\ &\quad - \left.\begin{bmatrix} C & C & \cdots & C \\ & C & \cdots & C \\ & & \ddots & \vdots \\ & & & C \end{bmatrix}\right\} n \text{ block rows and columns} \end{aligned} \tag{16}$$

where D is diagonal and C is circulant, and where the (i,j)th element of (16) is the $(p(i), p(j))$th element of (15). Again, this does not affect regularity. Now, (16) is block upper-triangular, and its jth diagonal block

$$\begin{bmatrix} ze_{j,j}^{(0)} & & & -a_{j,j}^{(0)} \\ -a_{j,j}^{(1)} & ze_{j,j}^{(1)} & & \\ & \ddots & \ddots & \\ & & -a_{j,j}^{(K-1)} & ze_{j,j}^{(K-1)} \end{bmatrix}$$

has determinant

$$(e_{j,j}^{(K-1)} \cdots e_{j,j}^{(1)} e_{j,j}^{(0)}) z^K - (a_{j,j}^{(K-1)} \cdots a_{j,j}^{(1)} a_{j,j}^{(0)}). \tag{17}$$

Clearly, (17) must not vanish identically for the pencil in (16), and hence $z\mathcal{E} - \mathcal{A}$, to be regular. Thus $z\mathcal{E} - \mathcal{A}$ is regular iff $\prod_{i=0}^{K-1} e_{j,j}^{(i)}$ and $\prod_{i=0}^{K-1} a_{j,j}^{(i)}$ are not simultaneously zero for $j = 1, 2, \cdots, n$; in other words, iff (12) is determinate.

Remark 2: Theorem 2 shows that solvability and conditionability of the PTV system Σ can be directly linked to the corresponding properties of a particular TI system, namely Σ^{E} given by (11). This TI system is the so-called *extended form*; it arises naturally in a variety of contexts in periodic systems and control theory. Other important properties of Σ such as stability and l_2-induced norm are intimately related to those of Σ^{E} as well [2], [8].

Remark 3: For $K = 1$, i.e., when $E_i \equiv E$, $A_i \equiv A$, the extended form Σ^{E} is identical to Σ, and Theorem 2 states that Σ is solvable (or conditionable) iff $z\mathcal{E} - \mathcal{A} = zE - A$ is regular. Thus we recover Luenberger's result [6] for the TI case. Furthermore, since a TI system can be thought of as being PTV with period equal to any integer $K > 1$, Theorem 2 shows that $zE - A$ is regular iff the K-cyclic matrix pencil

$$z\begin{bmatrix} E & & & \\ & E & & \\ & & \ddots & \\ & & & E \end{bmatrix} - \begin{bmatrix} & & & A \\ A & & & \\ & \ddots & & \\ & & A & \end{bmatrix}$$

is regular (for arbitrary K).

Remark 4: Statements 3) and 4) of Theorem 2 generalize, for $K > 1$, the corresponding statements in the classical QZ-algorithm theory [7]. Also notice that the generalized eigenvalues of the cyclic matrix pencil $z\mathcal{E} - \mathcal{A}$ in (11b), which are given by the zeros of (17), are the Kth roots of (12). Of course, (12) gives the eigenvalues of the *monodromy matrix* [1] $(E_K^{-1}A_K) \cdots (E_2^{-1}A_2)(E_1^{-1}A_1)$ when it exists, i.e., when the E_i matrices are invertible [2].

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