COMPUTING H_∞-NORM OF DISCRETE-TIME PERIODIC SYSTEMS—A QUADRATICALLY CONVERGENT ALGORITHM*

J. Sreedhar$^\diamond$, **Paul Van Dooren**† , **Bassam Bamieh**$^\diamond$

$^\diamond$ Department of Electrical Computer Engineering
University of Illinois at Urbana-Champaign,
IL 61801-2307, USA.
Fax number: 1-217-244-1653
email: jsree@gayatri.csl.uiuc.edu
† Department of Mathematical Engineering,
Université Catholique de Louvain,
1348 Louvain-la-Neuve, Belgium.
Fax number: 32-10-47-2180
email: Vandooren@anma.ucl.ac.be

Research partially supported by NSF, grants CCR-9619596 and ECS-9624152

Keywords : Linear systems, Numerical methods, Discrete time

Abstract

We study the l^2-induced norm, or H_∞-norm, of discrete-time periodically time varying (PTV) systems and propose a quadratically convergent algorithm to compute it. Our method involves numerically stable computations (periodic Schur decomposition) and works even for the descriptor case. This work has connections to *stability radii* problems. Among other things, we clarify the relationship between various TI representations of a PTV system.

1 Introduction

We consider the periodically time varying (PTV) system

$$\Sigma : \quad \begin{cases} E_k x_{k+1} = A_k x_k + B_k u_k, \\ y_k = C_k x_k + D_k u_k, \end{cases} \qquad k \in \mathbb{Z}, \qquad (1)$$

where $\mathbb{Z}$ is the set of integers, x_k is an n-dimensional vector of descriptor variables, u_k is an m-dimensional vector of input variables, y_k is a p-dimensional vector of output variables, and K is the smallest positive integer for which $E_k = E_{k+K}, A_k = A_{k+K}, B_k = B_{k+K}, C_k = C_{k+K}, D_k = D_{k+K}, \forall k \in \mathbb{Z}$. The matrices E_k, A_k, B_k, C_k and D_k are real, with sizes $n \times n$, $n \times n$, $n \times m$, $p \times n$ and $p \times m$, respectively. Throughout, we shall represent vectors by lower case characters, and matrices by upper case. Our aim in this note is to compute the H_∞-norm of Σ.

First of all, what do we mean by the H_∞-norm of Σ? The answer is that, as long as the map from the inputs u_k to the outputs y_k in (1) is well defined and l^2-stable, Σ has a finite l^2-induced norm—this is what we call its H_∞-norm. We remind the reader that, for any integer multiple M of K, the M-lifting [1, 2] of Σ has the same l^2-induced norm as Σ—lifting is an isometry. Moreover, because the lifted system is *time invariant* (TI), the usual definition of H_∞-norm holds for it. Thus, one way to think about the H_∞-norm of Σ is to go through its lifted system.

2 Lifting and Stacked-forms

The foregoing discussion seems to suggest that, before we start, we need a suitable realization of the lifted system. Towards this end, we introduce some notation. Given r_ℓ, Q_ℓ, $\ell \in \mathbb{Z}$, and two integers k, h, let $\mathbf{r}_k(h)$ and $\mathbf{Q}_k$ denote, respectively, a concatenated vector and a block-diagonal matrix:

$$\mathbf{r}_k^{\mathrm{T}}(h) := \left[r_{k+hK}^{\mathrm{T}}, r_{k+hK+1}^{\mathrm{T}}, \ldots, r_{k+hK+K-1}^{\mathrm{T}}\right],$$
$$\mathbf{Q}_k := \mathrm{diag}\left\{Q_k, Q_{k+1}, \ldots, Q_{k+K-1}\right\}.$$

We reserve bold face characters for vectors and matrices of 'big' dimension (a multiple of K). Using this, we can rewrite (1) as

$$\mathbf{\Sigma} : \quad \begin{cases} \mathbf{E}_k \mathbf{x}_{k+1}(h) = \mathbf{A}_k \mathbf{x}_k(h) + \mathbf{B}_k \mathbf{u}_k(h), \\ \mathbf{y}_k(h) = \mathbf{C}_k \mathbf{x}_k(h) + \mathbf{D}_k \mathbf{u}_k(h), \end{cases} \qquad (2)$$

*This paper presents research results of the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with its authors.

one of $k, h \in \mathbb{Z}$ held constant. If $h = h_0$ is held constant, $\mathbf{\Sigma}$ is still PTV in k with period K; in fact, (2) is nothing but K copies of (1) running in parallel. On the other hand, if $k = k_0$ is held constant, $\mathbf{\Sigma}$ becomes time-invariant in h since $\mathbf{E}_k$, $\mathbf{A}_k$, etc., do not depend on h. For each fixed $k = k_0$, (2) is a non-minimal realization (involving polynomial matrices, as we shall see soon) of the corresponding K-lifting of Σ and is termed the *stacked form at* k_0; we denote it by $\mathbf{\Sigma}^{\mathrm{s}}_{k_0}$ to emphasize that k_0 is constant. Because Σ is periodic, it has only K different K-liftings and hence stacked forms, corresponding to $k_0 = 0, 1, \ldots, K-1$. The stacked form representation was first proposed by Grasselli and Longhi [3] in the context of standard state space systems ($E_k \equiv I_n$ in (1)).

The various stacked forms $\mathbf{\Sigma}^{\mathrm{s}}_{k_0}$ are related to one another in an interesting manner. Before we can state this result (Lemma 1), however, we must introduce the transfer function of $\mathbf{\Sigma}^{\mathrm{s}}_{k_0}$. Let λ be the one-step forward shift operator in k, defined by

$$\lambda x_k = x_{k+1}, \, \forall k \in \mathbb{Z},$$

and let $z := \lambda^K$. Thus $z x_k = x_{k+K}$, $\forall k \in \mathbb{Z}$. We can think of z as the one-step forward shift operator in h for the big vector $\mathbf{x}_{k_0}(h)$, because $z\mathbf{x}_{k_0}(h) = \mathbf{x}_{k_0}(h+1)$, $\forall h \in \mathbb{Z}$. Define the $jK \times jK$ matrix

$$\boldsymbol{R}_j(z) := \begin{bmatrix} 0 & I_{(K-1)j} \\ zI_j & 0 \end{bmatrix}, \quad j > 0, z \in \mathbb{C},$$

where $\mathbb{C}$ is the set of complex numbers. It is easily verified that

$$\mathbf{x}_{k_0+1}(h) = \boldsymbol{R}_n(z)\mathbf{x}_{k_0}(h).$$

Therefore, taking z-transforms in (2), we obtain the transfer function of $\mathbf{\Sigma}^{\mathrm{s}}_{k_0}$ as

$$\boxed{\mathbf{W}_{k_0}(z) = \mathbf{C}_{k_0}(\mathbf{E}_{k_0}\boldsymbol{R}_n(z) - \mathbf{A}_{k_0})^{-1}\mathbf{B}_{k_0} + \mathbf{D}_{k_0}.} \tag{3}$$

Lemma 1. *Let k_0 be a fixed but arbitrary integer. All TI systems $\mathbf{\Sigma}^{\mathrm{s}}_{k_0}$ have the same H_∞-norm. In other words, if $\mathbf{W}_{k_0}(z)$ denotes the transfer function of $\mathbf{\Sigma}^{\mathrm{s}}_{k_0}$, then*

$$\left\|\mathbf{W}_{k_0}(z)\right\|_\infty = \max_{\theta \in [0, 2\pi]} \sigma_{\max}\Big(\mathbf{W}_{k_0}(\mathrm{e}^{\mathrm{j}\theta})\Big)$$

is independent of k_0.

Proof. Since
$\mathbf{E}_{k_0+1} = \boldsymbol{R}_n(z)\mathbf{E}_{k_0}\boldsymbol{R}_n^{-1}(z)$,
$\mathbf{A}_{k_0+1} = \boldsymbol{R}_n(z)\mathbf{A}_{k_0}\boldsymbol{R}_n^{-1}(z)$,
$\mathbf{B}_{k_0+1} = \boldsymbol{R}_n(z)\mathbf{B}_{k_0}\boldsymbol{R}_m^{-1}(z)$,
$\mathbf{C}_{k_0+1} = \boldsymbol{R}_p(z)\mathbf{C}_{k_0}\boldsymbol{R}_n^{-1}(z)$,
$\mathbf{D}_{k_0+1} = \boldsymbol{R}_p(z)\mathbf{D}_{k_0}\boldsymbol{R}_m^{-1}(z)$,
for all $k_0 \in \mathbb{Z}$, $z \in \mathbb{C}$, we have the following simple relation between the transfer functions of $\mathbf{\Sigma}^{\mathrm{s}}_{k_0}$ and $\mathbf{\Sigma}^{\mathrm{s}}_{k_0+1}$.

$$\mathbf{W}_{k_0+1}(z) = \boldsymbol{R}_p(z)\mathbf{W}_{k_0}(z)\boldsymbol{R}_m^{-1}(z), \quad k_0 \in \mathbb{Z}, z \in \mathbb{C}.$$

At first glance, this does not seem to have led us anywhere, but notice that z is restricted to the unit circle when computing H_∞-norms. And, because $\boldsymbol{R}_p(\mathrm{e}^{\mathrm{j}\theta})$ and $\boldsymbol{R}_m(\mathrm{e}^{\mathrm{j}\theta})$ are unitary matrices, $\mathbf{W}_{k_0+1}(\mathrm{e}^{\mathrm{j}\theta})$ and $\mathbf{W}_{k_0}(\mathrm{e}^{\mathrm{j}\theta})$ have the same singular values! It then follows trivially that $\mathbf{W}_{k_0+1}(z)$ and $\mathbf{W}_{k_0}(z)$ have the same H_∞-norm. This holds for all $k_0 \in \mathbb{Z}$, proving the claim. $\square$

Lemma 1 shows that we can use any $\mathbf{\Sigma}^{\mathrm{s}}_{k_0}$ to compute the H_∞-norm of Σ. We shall now examine $\mathbf{\Sigma}^{\mathrm{s}}_1$, given by putting $k = 1$ in (2), a bit more closely for later reference. Using $\mathbf{E}_1 \cdot \mathbf{x}_2(h) = \mathbf{E}_1 \cdot \boldsymbol{R}_n(\lambda^K)\mathbf{x}_1(h)$, we arrive at the following detailed equations for $\mathbf{\Sigma}^{\mathrm{s}}_1$:

$$\begin{bmatrix} & E_1 & & \\ & & \ddots & \\ & & & E_{K-1} \\ \lambda^K E_0 & & & \end{bmatrix} \mathbf{x}_1(h) = \begin{bmatrix} A_1 & & & \\ & \ddots & & \\ & & A_{K-1} & \\ & & & A_0 \end{bmatrix} \mathbf{x}_1(h) + \begin{bmatrix} B_1 & & & \\ & \ddots & & \\ & & B_{K-1} & \\ & & & B_0 \end{bmatrix} \mathbf{u}_1(h),$$

$$\mathbf{y}_1(h) = \begin{bmatrix} C_1 & & & \\ & \ddots & & \\ & & C_{K-1} & \\ & & & C_0 \end{bmatrix} \mathbf{x}_1(h) + \begin{bmatrix} D_1 & & & \\ & \ddots & & \\ & & D_{K-1} & \\ & & & D_0 \end{bmatrix} \mathbf{u}_1(h). \tag{4}$$

To give additional insight into our approach to computing the H_∞-norm of Σ, we mention the following: for standard state space systems, it can be proved [3, 4] that $\mathbf{\Sigma}^{\mathrm{s}}_{k_0}$ has the same transfer function as the well known associated system at k_0 [2], viz., the minimal TI representation of Σ involving the monodromy matrix. This is hardly surprising because both $\mathbf{\Sigma}^{\mathrm{s}}_{k_0}$ and the associated system realize the same input-output map—the K-lifting at k_0 of Σ. One could in fact start with $\mathbf{\Sigma}^{\mathrm{s}}_{k_0}$ and, by strict system equivalence transformations, obtain a minimal realization similar to the associated system at k_0. The advantage of this method [5] is increased numerical accuracy—only unitary operations and linear system solves are used, and explicit formation of matrix products is avoided. Recall that the associated systems require computation of the monodromy matrices.

When E_k are singular, the associated systems do not even exist and we must deal with the stacked forms. Unfortunately, it turns out that it is not easy to compute the H_∞-norm of stacked forms. An intuitive reason for this is that stacked forms involve *matrix polynomials* rather than *matrix pencils*—as (3) and (4) clearly show—and this makes them harder to handle. One could, of course, proceed along the lines of [5] and construct a minimal descriptor realization from $\mathbf{\Sigma}^{\mathrm{S}}_{k_0}$ by strict system equivalence transformations—this work is currently under progress [15]—but such methods to arrive at a TI matrix pencil representation of Σ involve a lot of computation and can be cumbersome. We shall see that the *extended form*, to be introduced next, ameliorates this difficulty and fits our requirements perfectly. It is the representation we will use to compute the H_∞-norm of Σ. It involves matrix pencils only and, moreover, can be written down directly (without any computation).

3 H_∞-norm and extended form

We now leave stacked forms and introduce another TI representation of Σ, using a method due to Park and Verriest [7]. Once again we start with $\mathbf{\Sigma}$ given by (2), but this time treat it as a PTV system in k. We fix $h = 0$, and refer to $\mathbf{r}_k(0)$ as simply $\mathbf{r}_k$. Equation (2) then reads

$$\mathbf{E}_k\mathbf{x}_{k+1} = \mathbf{A}_k\mathbf{x}_k + \mathbf{B}_k\mathbf{u}_k, \tag{5a}$$
$$\mathbf{y}_k = \mathbf{C}_k\mathbf{x}_k + \mathbf{D}_k\mathbf{u}_k, \tag{5b}$$

$k \in \mathbb{Z}$, and is merely K copies of Σ running in parallel, successively offset in time-index by one. Next, we define $\boldsymbol{M}_j$ to be the $jK \times jK$ generator of a cyclic group of order K:

$$\boldsymbol{M}_j := \begin{bmatrix} 0 & I_{(K-1)j} \\ I_j & 0 \end{bmatrix}, \quad j > 0,$$

and perform the following transformations:

1. $\mathbf{x}_k = \boldsymbol{M}_n^{(k-1)}\hat{\mathbf{x}}_k$,
2. premultiply (5a) by $\boldsymbol{M}_n^{-k}$,
3. $\mathbf{u}_k = \boldsymbol{M}_m^{k}\hat{\mathbf{u}}_k$, and
4. premultiply (5b) by $\boldsymbol{M}_p^{-k}$.

Due to the identities

$$\begin{aligned} &\boldsymbol{M}_n\mathbf{E}_k\boldsymbol{M}_n^{-1} = \mathbf{E}_{k+1}, \\ &\boldsymbol{M}_n\mathbf{A}_k\boldsymbol{M}_n^{-1} = \mathbf{A}_{k+1}, \boldsymbol{M}_n\mathbf{B}_k\boldsymbol{M}_m^{-1} = \mathbf{B}_{k+1}, \\ &\boldsymbol{M}_p\mathbf{C}_k\boldsymbol{M}_n^{-1} = \mathbf{C}_{k+1}, \boldsymbol{M}_p\mathbf{D}_k\boldsymbol{M}_m^{-1} = \mathbf{D}_{k+1}, \end{aligned}$$

this yields, rather pleasantly, a TI system

$$\mathbf{\Sigma}^{\mathrm{E}}: \quad \begin{cases} \boldsymbol{\mathcal{E}}\hat{\mathbf{x}}_{k+1} = \boldsymbol{\mathcal{A}}\hat{\mathbf{x}}_k + \boldsymbol{\mathcal{B}}\hat{\mathbf{u}}_k, \\ \hat{\mathbf{y}}_k = \boldsymbol{\mathcal{C}}\hat{\mathbf{x}}_k + \boldsymbol{\mathcal{D}}\hat{\mathbf{u}}_k, \end{cases} \quad k \in \mathbb{Z},$$

where

$$\begin{aligned} \boldsymbol{\mathcal{E}} &:= \boldsymbol{M}_n^{-k}\mathbf{E}_k\boldsymbol{M}_n^{k} = \mathbf{E}_0, \\ \boldsymbol{\mathcal{A}} &:= \boldsymbol{M}_n^{-k}\mathbf{A}_k\boldsymbol{M}_n^{(k-1)} = \boldsymbol{M}_n^{-1}\mathbf{A}_1, \ \boldsymbol{\mathcal{B}} := \boldsymbol{M}_n^{-k}\mathbf{B}_k\boldsymbol{M}_m^{k} = \mathbf{B}_0, \\ \boldsymbol{\mathcal{C}} &:= \boldsymbol{M}_p^{-k}\mathbf{C}_k\boldsymbol{M}_n^{(k-1)} = \boldsymbol{M}_n^{-1}\mathbf{C}_1, \ \boldsymbol{\mathcal{D}} := \boldsymbol{M}_p^{-k}\mathbf{D}_k\boldsymbol{M}_m^{k} = \mathbf{D}_0. \end{aligned}$$

Let $\boldsymbol{\mathcal{W}}(\lambda) := \boldsymbol{\mathcal{C}}(\lambda\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{A}})^{-1}\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{D}}$ denote its transfer function. We call this new TI representation the extended form (of Σ) and write it in detail below to facilitate comparison with $\mathbf{\Sigma}^{\mathrm{S}}_1$ in (4):

$$\begin{aligned} \begin{bmatrix} \lambda E_0 & & & \\ & \lambda E_1 & & \\ & & \ddots & \\ & & & \lambda E_{K-1} \end{bmatrix} \hat{\mathbf{x}}_k &= \begin{bmatrix} & & & A_0 \\ A_1 & & & \\ & \ddots & & \\ & & A_{K-1} & \end{bmatrix} \hat{\mathbf{x}}_k \\ &+ \begin{bmatrix} B_0 & & & \\ & B_1 & & \\ & & \ddots & \\ & & & B_{K-1} \end{bmatrix} \hat{\mathbf{u}}_k, \\ \hat{\mathbf{y}}_k &= \begin{bmatrix} & & & C_0 \\ C_1 & & & \\ & \ddots & & \\ & & C_{K-1} & \end{bmatrix} \hat{\mathbf{x}}_k \\ &+ \begin{bmatrix} D_0 & & & \\ & D_1 & & \\ & & \ddots & \\ & & & D_{K-1} \end{bmatrix} \hat{\mathbf{u}}_k. \end{aligned} \tag{6}$$

Remark 1. All the four transformations involved in going from (5) to (6) are K-periodic and reversible—we can switch back and forth between the two systems. While transformations 1 & 2 affect only internal book-keeping and don't change the inputs or outputs, 3 & 4 permute the inputs and outputs themselves. Together, they conspire to produce a TI system from a PTV one.

We justify our introduction of the extended form $\mathbf{\Sigma}^{\mathrm{E}}$ through the following lemma which clarifies its connection to lifting.

Lemma 2. *The TI system $\mathbf{\Sigma}^{\mathrm{E}}$, described by (6), has the same H_∞-norm as any of the TI systems $\mathbf{\Sigma}^{\mathrm{S}}_{k_0}$, and thus the same H_∞-norm as Σ.*

Proof. It is well known that the transfer function $\boldsymbol{\mathcal{W}}(\lambda)$ of the extended form $\mathbf{\Sigma}^{\mathrm{E}}$ is the Schur complement of its system matrix [6],

$$\left[\begin{array}{c|c} \boldsymbol{\mathcal{A}} - \lambda\boldsymbol{\mathcal{E}} & \boldsymbol{\mathcal{B}} \\ \hline \boldsymbol{\mathcal{C}} & \boldsymbol{\mathcal{D}} \end{array}\right].$$

Similarly, for the stacked form $\mathbf{\Sigma}^{\mathrm{S}}_1$, the transfer function $\mathbf{W}_1(\lambda^K)$ is the Schur complement of its system matrix,

$$\left[\begin{array}{c|c} \mathbf{A}_1 - \mathbf{E}_1\boldsymbol{R}_n(\lambda^K) & \mathbf{B}_1 \\ \hline \mathbf{C}_1 & \mathbf{D}_1 \end{array}\right].$$

It can be verified that

$$\left[\begin{array}{c|c} \boldsymbol{M}_n^{-1}(\mathbf{A}_1 - \mathbf{E}_1 \boldsymbol{R}_n(\lambda^K)) & \boldsymbol{M}_n^{-1}\mathbf{B}_1 \\ \hline \mathbf{C}_1 & \mathbf{D}_1 \end{array}\right] =$$

$$\left[\begin{array}{ccccc|ccccc}
I_n &&&&&&&&& \\
& \lambda^{-(K-1)}I_n &&&&&&&& \\
&& \ddots &&&&&&& \\
&&& \lambda^{-2}I_n &&&&&& \\
&&&& \lambda^{-1}I_n &&&&& \\ \hline
&&&&&& \lambda^{-(K-1)}I_p &&& \\
&&&&&&& \ddots && \\
&&&&&&&& \lambda^{-2}I_p & \\
&&&&&&&&& \lambda^{-1}I_p \\
&&&&& I_p &&&&
\end{array}\right]$$

$$\cdot \left[\begin{array}{c|c} \boldsymbol{\mathcal{A}} - \lambda\boldsymbol{\mathcal{E}} & \boldsymbol{\mathcal{B}} \\ \hline \boldsymbol{\mathcal{C}} & \boldsymbol{\mathcal{D}} \end{array}\right] \cdot$$

$$\left[\begin{array}{ccccc|ccccc}
\lambda^{K-1}I_n &&&&&&&&& \\
& \ddots &&&&&&&& \\
&& \lambda^2 I_n &&&&&&& \\
&&& \lambda I_n &&&&&& \\
&&&& I_n &&&&& \\ \hline
&&&&&&&&& I_m \\
&&&&& \lambda^{K-1}I_m &&&& \\
&&&&&& \ddots &&& \\
&&&&&&& \lambda^2 I_m && \\
&&&&&&&& \lambda I_m &
\end{array}\right].$$

Taking Schur complements, we can therefore relate $\mathbf{W}_1(\lambda^K)$ and $\boldsymbol{\mathcal{W}}(\lambda)$:

$$\mathbf{W}_1(\lambda^K) = \begin{bmatrix}
& \lambda^{-(K-1)}I_p &&& \\
&& \ddots && \\
&&& \lambda^{-2}I_p & \\
&&&& \lambda^{-1}I_p \\
I_p &&&&
\end{bmatrix}$$

$$\cdot \boldsymbol{\mathcal{W}}(\lambda) \cdot \begin{bmatrix}
&&&& I_m \\
\lambda^{K-1}I_m &&&& \\
& \ddots &&& \\
&& \lambda^2 I_m && \\
&&& \lambda I_m &
\end{bmatrix}.$$

It now follows, by an argument similar to that used to prove Lemma 1, that $\mathbf{W}_1(\mathrm{e}^{\mathrm{j}\theta})$ and $\boldsymbol{\mathcal{W}}(\mathrm{e}^{\mathrm{j}\theta})$ have the same singular values; hence $\mathbf{\Sigma}^{\mathrm{E}}$ and $\mathbf{\Sigma}_1^{\mathrm{S}}$ have the same H_∞-norm. Lemma 2 then follows trivially from Lemma 1. □

4 Main result

We shall derive our result for the H_∞-norm of the periodic system Σ by making use of the TI system $\mathbf{\Sigma}^{\mathrm{E}}$. Computation of the H_∞-norm, or its reciprocal the complex stability radius, for TI systems has been well studied, in both the discrete-time [8] and continuous-time [9, 10, 11, 12] cases.

First, we need the following TI result.

Proposition 1. *Consider $(E, A, B, C, D) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m} \times \mathbb{C}^{p\times n} \times \mathbb{C}^{p\times m}$ and assume that $\lambda E - A$ has no eigenvalues on the unit circle $\lambda = e^{\mathrm{j}\theta}$, $\theta \in [0, 2\pi]$. Let $W(\lambda) = C(\lambda E - A)^{-1}B + D$. For ξ not a singular value of D,*

$$\lambda \begin{bmatrix} E & \xi B R^{-1} B^{\mathrm{H}} \\ 0 & -A^{\mathrm{H}} + C^{\mathrm{H}} D R^{-1} B^{\mathrm{H}} \end{bmatrix} - \begin{bmatrix} A - BR^{-1}D^{\mathrm{H}}C & 0 \\ -\xi C^{\mathrm{H}} S^{-1} C & -E^{\mathrm{H}} \end{bmatrix}$$
has eigenvalue $e^{\mathrm{j}\theta}$
$$\Updownarrow$$
$W(e^{\mathrm{j}\theta})$ has singular value ξ,

where $R := D^{\mathrm{H}}D - \xi^2 I$ and $S := DD^{\mathrm{H}} - \xi^2 I$.

Proof. We prove the *if* part first. Let u and v be singular vectors of $W(e^{\mathrm{j}\theta})$ corresponding to the singular value ξ,

$$\begin{aligned} \left[C\left(e^{\mathrm{j}\theta}E - A\right)^{-1}B + D\right]u &= \xi v, \\ \left[B^{\mathrm{H}}\left(e^{-\mathrm{j}\theta}E^{\mathrm{H}} - A^{\mathrm{H}}\right)^{-1}C^{\mathrm{H}} + D^{\mathrm{H}}\right]v &= \xi u. \end{aligned} \tag{7}$$

Define

$$\begin{aligned} r &:= \left(e^{\mathrm{j}\theta}E - A\right)^{-1}Bu, \\ s &:= e^{-\mathrm{j}\theta}\left(e^{-\mathrm{j}\theta}E^{\mathrm{H}} - A^{\mathrm{H}}\right)^{-1}C^{\mathrm{H}}v. \end{aligned} \tag{8}$$

Equation (7) then becomes

$$\begin{aligned} Cr + Du &= \xi v, \\ e^{\mathrm{j}\theta}B^{\mathrm{H}}s + D^{\mathrm{H}}v &= \xi u, \end{aligned}$$

or

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} -D & \xi I \\ \xi I & -D^{\mathrm{H}} \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & e^{\mathrm{j}\theta}B^{\mathrm{H}} \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}, \\ &= \begin{bmatrix} -R^{-1}D^{\mathrm{H}}C & -\xi R^{-1}e^{\mathrm{j}\theta}B^{\mathrm{H}} \\ -\xi S^{-1}C & -DR^{-1}e^{\mathrm{j}\theta}B^{\mathrm{H}} \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}. \end{aligned} \tag{9}$$

The required inverses exist because of our choice of ξ, i.e., it is not a singular value of D. Plugging (9) into (8), we get

$$\begin{aligned} &\left(e^{\mathrm{j}\theta}E - A\right)r = B \cdot u = -BR^{-1}D^{\mathrm{H}}Cr - \xi e^{\mathrm{j}\theta}BR^{-1}B^{\mathrm{H}}s, \\ &\left(e^{-\mathrm{j}\theta}E^{\mathrm{H}} - A^{\mathrm{H}}\right)s = \\ &\qquad e^{-\mathrm{j}\theta}C^{\mathrm{H}} \cdot v = -\xi e^{-\mathrm{j}\theta}C^{\mathrm{H}}S^{-1}Cr - C^{\mathrm{H}}DR^{-1}B^{\mathrm{H}}s, \end{aligned}$$

which, on rearranging, is nothing but the eigen-equation

$$\begin{aligned} &\begin{bmatrix} A - BR^{-1}D^{\mathrm{H}}C & 0 \\ -\xi C^{\mathrm{H}}S^{-1}C & -E^{\mathrm{H}} \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} \\ &\qquad = e^{\mathrm{j}\theta} \begin{bmatrix} E & \xi BR^{-1}B^{\mathrm{H}} \\ 0 & -A^{\mathrm{H}} + C^{\mathrm{H}}DR^{-1}B^{\mathrm{H}} \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}. \end{aligned} \tag{10}$$

This concludes the proof of the *if* part. For the *only if* part, we can essentially reverse the above steps and go from (10), (9) to (8), (7). In that case, u, v would be *defined* by (9), and (8) would follow from (10) because of (9). □

For the special case of $E = I, D = 0$, Proposition 1 appeared in [8] in the context of complex stability radius of TI systems. It was also used in [13] for the special case of $D = 0$ to find the real stability radius of TI systems.

The usual SVD enables us to find the singular values of a transfer function matrix at a given frequency point $e^{\mathrm{j}\theta}$. Proposition 1, on the other hand, enables us to find the *frequencies* corresponding to a certain singular value (or level) ξ. This, alongwith knowledge of "level sets" on the unit circle leads to quadratically convergent algorithms for the H_∞-norm. We can expect a similar convergence rate when using these ideas for H_∞-norm of the TI system $\boldsymbol{\Sigma}^{\mathrm{E}}$.

Our main result in this paper is the periodic version of Proposition 1.

Theorem 1. *Consider a set of K-periodic matrices $(E_k, A_k, B_k, C_k, D_k)$ as in (1) and assume that $\lambda\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{A}}$ has no eigenvalues when $\lambda = e^{\mathrm{j}\theta}$, $\theta \in [0, 2\pi]$. Then, for ξ not a singular value of D_k, $k = 0, 1, \ldots, K-1$,*

the periodic pencil $\lambda G_k - H_k$
has eigenvalue $e^{\mathrm{j}\theta} \iff \boldsymbol{\mathcal{W}}(e^{\mathrm{j}\theta})$ has singular value ξ,

where

$$\begin{aligned} G_k &:= \begin{bmatrix} E_k & \xi B_k R_k^{-1} B_k^{\mathrm{H}} \\ 0 & -A_k^{\mathrm{H}} + C_k^{\mathrm{H}} D_k R_k^{-1} B_k^{\mathrm{H}} \end{bmatrix}, \\ H_k &:= \begin{bmatrix} A_k - B_k R_k^{-1} D_k^{\mathrm{H}} C_k & 0 \\ -\xi C_k^{\mathrm{H}} S_k^{-1} C_k & -E_k^{\mathrm{H}} \end{bmatrix}, \\ R_k &:= D_k^{\mathrm{H}} D_k - \xi^2 I, \quad S_k := D_k D_k^{\mathrm{H}} - \xi^2 I. \end{aligned} \tag{11}$$

Proof. As explained previously, for the purpose of computing the H_∞-norm, we can work with $\boldsymbol{\Sigma}^{\mathrm{E}}$ instead of Σ. Invoking Proposition 1 for the TI system $\boldsymbol{\Sigma}^{\mathrm{E}}$, we see that the pencil of interest is

$$\lambda \begin{bmatrix} \boldsymbol{\mathcal{E}} & \xi \boldsymbol{\mathcal{B}} (\boldsymbol{\mathcal{D}}^{\mathrm{H}} \boldsymbol{\mathcal{D}} - \xi^2 I)^{-1} \boldsymbol{\mathcal{B}}^{\mathrm{H}} \\ 0 & -\boldsymbol{\mathcal{A}}^{\mathrm{H}} + \boldsymbol{\mathcal{C}}^{\mathrm{H}} \boldsymbol{\mathcal{D}} (\boldsymbol{\mathcal{D}}^{\mathrm{H}} \boldsymbol{\mathcal{D}} - \xi^2 I)^{-1} \boldsymbol{\mathcal{B}}^{\mathrm{H}} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{B}} (\boldsymbol{\mathcal{D}}^{\mathrm{H}} \boldsymbol{\mathcal{D}} - \xi^2 I)^{-1} \boldsymbol{\mathcal{D}}^{\mathrm{H}} \boldsymbol{\mathcal{C}} & 0 \\ -\xi \boldsymbol{\mathcal{C}}^{\mathrm{H}} (\boldsymbol{\mathcal{D}} \boldsymbol{\mathcal{D}}^{\mathrm{H}} - \xi^2 I)^{-1} \boldsymbol{\mathcal{C}} & -\boldsymbol{\mathcal{E}}^{\mathrm{H}} \end{bmatrix}.$$

Through simple block-row and block-column permutation operations, specifically, premultiplication with $\mathrm{diag}(I_{nK}, \boldsymbol{M}_n^{-1})$ followed by a *perfect shuffle*, this becomes

$$\lambda \begin{bmatrix} G_0 & & & \\ & G_1 & & \\ & & \ddots & \\ & & & G_{K-1} \end{bmatrix} - \begin{bmatrix} & & & H_0 \\ H_1 & & & \\ & \ddots & & \\ & & H_{K-1} & \end{bmatrix}, \tag{12}$$

where G_k and H_k are as given in (11). This completes the proof because the above pencil is equivalent to the periodic pencil $\lambda G_k - H_k$ [14]. □

We conclude this paper with a cautionary remark on implementing Theorem 1: the use of $\boldsymbol{\Sigma}^{\mathrm{E}}$ is chiefly for conceptual purposes; in practice, one would solve the periodic eigenvalue problem which arises at each step by using the periodic Schur algorithm [14] on the periodic pencil $\lambda G_k - H_k$, and not the usual QZ-algorithm on the pencil in (12).

References

[1] P. P. Khargonekar, K. Poolla, and A. R. Tannenbaum, "Robust control of linear time-invariant plants using periodic compensation," *IEEE Transactions on Automatic Control*, vol. AC-30, pp. 1088–1096, November 1985.

[2] R. A. Meyer and C. S. Burrus, "A unified analysis of multirate and periodically time-varying digital filters," *IEEE Trans. Circuits and Systems*, vol. 22, pp. 162–168, 1975.

[3] O. M. Grasselli and S. Longhi, "Finite zero structure of linear periodic discrete-time systems," *Int. J. of Systems Science*, vol. 22, no. 10, pp. 1785–1806, 1991.

[4] O.M. Grasselli, S. Longhi, and A. Tornambè. On the computation of the time-invariant associated system of a periodic system. In *Proc. Amer. Contr. Conf.*, Seattle, WA, 1995.

[5] P. Misra, "Time invariant representation of discrete periodic systems," *Automatica*, vol. 32, no. 2, pp. 267–272, 1996.

[6] H.H. Rosenbrock. *State Space and Multivariable Theory.* John Wiley, New York, 1970.

[7] B. Park and E.I. Verriest. Canonical forms on discrete linear periodically time-varying systems and a control application. In *Proc. 28th IEEE Conf. on Decision and Control*, (Tampa, FL), pp. 1220–1225, Dec. 1989.

[8] D. Hinrichsen and N. K. Son, "The complex stability radius of discrete-time systems and symplectic pencils," in *Proc. IEEE 28th Conference on Decision and Control*, (Tampa, FL), pp. 2265–2270, 1989.

[9] D. Hinrichsen, B. Kelb, and A. Linnemann, "An algorithm for the computation of the complex stability radius," *Automatica*, vol. 25, pp. 771–775, 1989.

[10] S. Boyd, V. Balakrishnan, and P. Kabamba. A bisection method for computing the H_∞ norm of a transfer matr ix and related problems. *Mathematics of Control, Signals, and Systems*, 2:207–219, 1989.

[11] S. Boyd and V. Balakrishnan. A regularity result for the singular values of a transfer matrix and a quadratically convergent algorithm for computing its L_∞ norm. *Systems and Control Letters*, 15:1–7, 1990.

[12] N. A. Bruinsma and M. Steinbuch, "A fast algorithm to compute the H_∞ norm of a transfer function matrix," *Systems & Control Letters*, vol. 14, pp. 287–293, 1990.

[13] J. Sreedhar, P. Van Dooren, and A. L. Tits, "A fast algorithm to compute the real structured stability radius," in *Stability Theory: Proceedings of Hurwitz Centenary Conference, Ticino, Switzerland, May 21–26, 1995*, Birkhauser Verlag AG, 1996.

[14] A. Bojanczyk, G. Golub, and P. Van Dooren. The periodic Schur decomposition. Algorithms and Applications. In Proceedings of SPIE, volume 1770, pages 31–42, 19-21 July, San Diego, CA, 1992, USA.

[15] J. Sreedhar, P. Van Dooren and P. Misra, “Minimal order time invariant representation of periodic descriptor systems,” *IEEE Intnl. Sympo. Computer-Aided Cntrl Syst. Design*, Dearborn, MI, USA, Sept 15–18, 1996.