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Minimal Order Time Invariant Representation of Periodic Descriptor Systems

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Abstract

A large number of results from linear time invariant system theory can be extended to periodic systems provided an equivalent time invariant system can be found. This problem has been well investigated for periodic systems which have a standard state space representation. This paper presents a numerical procedure to achieve the same for descriptor periodic systems with possibly singular descriptor matrix, in which case the monodromy matrix is not defined. It is shown that using a *stacked* representation of periodic systems a minimal order generalized state space description can always be obtained under system equivalence.

Keywords: Periodic systems, system equivalence, generalized state space systems.

1 Introduction

In this paper, we consider periodic systems represented by linear difference equations of the form,

$$\begin{array}{rcl} E_k x(k+1) & = & A_k x(k) + B_k u(k) \\ y(k) & = & C_k x(k) + D_k u(k) \end{array} \qquad (1.1)$$

where, $k \in \mathcal{Z}^+$, $x \in \mathcal{C}^n$, $u \in \mathcal{C}^m$, $y \in \mathcal{C}^p$, E_k, A_k, B_k, C_k and D_k, $k = 0, 1, \ldots, (\omega - 1)$ are (possibly) complex periodic matrices of compatible dimensions with period ω. We will refer to the systems characterized by (1.1) as ω-periodic descriptor systems.

For the case when the descriptor matrices E_k in (1.1) are identity matrices, there are three possible approaches to obtain a time invariant representation of discrete periodic systems. The first approach makes use of the monodromy matrix [7]. Its determination involves the computation of products of matrices that can lead to numerical difficulties. Hence, apart from being computationally expensive, the use of monodromy matrices for time invariant representations can conceivably introduce significant numerical errors. The second approach is based on a discrete version of the *Floquet transform* [2], [5] and [11]. But this transformation deals only with homogeneous differential equations and hence does not make the input and output matrices (B_k), (C_k) and (D_k) time-invariant. The third approach is based on a lifted representation of the periodic system as developed by Grasselli and coworkers in [4], by Verriest and coworkers in [12], [9] and by Flamm in [3]. The concept of *lifting* is quite fundamental to the study of discrete periodic systems. By lifting it is possible to represent a periodic system in a time invariant form that preserves most properties of the original system. This transformation permits one to use well established ideas from linear time invariant system theory to study periodically varying linear systems.

It should be noted that the monodromy matrix is not defined for singular E_k's, hence the first approach can not be carried through as such. In [8], it was shown that an alternative time invariant representation of periodic systems with E_k's being identity matrices, may be obtained from the polynomial description of the lifted system, using Rosenbrock's strict system equivalence transformations. In the present paper, we extend those results to the case when E_k are arbitrary and possibly singular matrices, with the assumption that the lifted pencil

$$\begin{bmatrix} O & \text{diag}[E_i] \\ \lambda E_{\omega-1} & O \end{bmatrix} = \text{diag}[A_j] \qquad (1.2)$$

$i = 0, \ldots, (\omega - 2)$, $j = 0, \ldots, (\omega - 1)$ formed from the matrices defined in (1.1), is regular.

2 Background

The key idea in deriving time-invariant representations of dynamical systems with periodic coefficients, is to consider the input/output behavior of the system over a time intervals of length ω, rather than 1. The corresponding input and output vectors of this modified system will thus be "compound" vectors of the type:

$$\begin{array}{rcl} u_k(h) & := & \left[u^T(k+h\omega) \quad \cdots \quad u^T(k+h\omega+\omega-1) \right]^T, \\ y_k(h) & := & \left[y^T(k+h\omega) \quad \cdots \quad y^T(k+h\omega+\omega-1) \right]^T. \end{array} \qquad (2.1)$$

A first way to derive a time invariant representation is based on a state space representation using the state

$$\hat{x}_k(h) := x(k+h\omega),$$

which is also of dimension n. This leads to the so-called *monodromy* form which we give below for the case when the descriptor matrices E_k in (1.1) are identity matrices of order n :

$$\begin{array}{rcl} \hat{x}_k(h+1) &=& \hat{F}_k\hat{x}_k(h) + \hat{G}_k u_k(h) \\ y_k(h) &=& \hat{H}_k\hat{x}_k(h) + \hat{J}_k u_k(h) \end{array} . \quad (2.2)$$

Here,

$$\begin{array}{rcl} \hat{F}_k &:=& \Phi(k+\omega,k) \\ \hat{G}_k &:=& [G_{k,0} \quad G_{k,1} \quad \cdots \quad G_{k,\omega-1}] \\ \hat{H}_k &:=& \begin{bmatrix} H_{k,0} \\ H_{k,1} \\ \vdots \\ H_{k,\omega-1} \end{bmatrix} \\ \hat{J}_k &:=& \begin{bmatrix} J_{k,0,0} & O & \cdots & O \\ J_{k,1,0} & J_{k,1,1} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ J_{k,\omega-1,0} & J_{k,\omega-1,1} & \cdots & J_{k,\omega-1,\omega-1} \end{bmatrix} \end{array} \quad (2.3)$$

and,

$$\begin{array}{rcl} \Phi(i,j) &:=& A_{i-1}\cdots A_{j+1}A_j, \quad \forall i,j \in \mathcal{Z}, i>j, \\ \Phi(j,j) &:=& I_n, \quad \forall j \in \mathcal{Z}, \\ G_{k,j} &:=& \Phi(k+\omega, k+j+1)B_{k+j}, \\ && j = 0,\ldots,(\omega-1) \\ H_{k,j} &:=& C_{k+j}\Phi(k+j,k), \quad j=0,\ldots,(\omega-1) \\ J_{k,j,j} &:=& D_{k+j}, \quad j=0,\ldots,(\omega-1), \\ J_{k,i,j} &:=& C_{k+i}\Phi(k+i,k+j+1)B_{k+j}, \\ && j=0,\ldots,(\omega-2),\ i=j+1,\ldots,(\omega-1) \end{array} \quad (2.4)$$

The system (2.3), with the matrices and vectors as defined in (2.2) and (2.4) is known as the *associated system* at the initial time k of the given ω-periodic system [7]. The monodromy matrix $\Phi(i,j)$ as well as the other system matrices of this model have the disadvantage of being quite involved expressions of the original coefficient matrices of (1.1). In [8], it was shown that an alternative time invariant representation may be obtained by using a generalized state-space model with input $u_k(h)$ and output $y_k(h)$ as defined earlier, but using a state of length $n \times \omega$ rather than n:

$$\tilde{x}_k(h) := [x^T(k+h\omega) \quad \cdots \quad x^T(k+h\omega+\omega-1)]^T . \quad (2.5)$$

This new system has the form

$$\begin{array}{rcl} \mathcal{R}(\lambda)\tilde{x}_k(h) &=& \mathcal{A}_k\tilde{x}_k(h) + \mathcal{B}_k u_k(h) \\ y_k(h) &=& \mathcal{C}_k\tilde{x}_k(h) + \mathcal{D}_k u_k(h) \end{array} \quad (2.6)$$

where

$$\mathcal{R}(\lambda) \quad := \quad \begin{bmatrix} O & I_{(\omega-1)n} \\ \lambda I_n & O \end{bmatrix}, \quad (2.7)$$

with λ denoting the one step forward time operator in the variable h or ω-step forward time operator in the variable k. Further, the matrices $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$ and $\mathcal{D}_k$ are block diagonal matrices defined directly in terms of the matrices of (1.1) as follows :

$$\begin{array}{rcl} \mathcal{A}_k &:=& \text{block diag}\{A_k, A_{k+1},\ldots,A_{k+\omega-1}\} \\ \mathcal{B}_k &:=& \text{block diag}\{B_k, B_{k+1},\ldots,B_{k+\omega-1}\} \\ \mathcal{C}_k &:=& \text{block diag}\{C_k, C_{k+1},\ldots,C_{k+\omega-1}\} \\ \mathcal{D}_k &:=& \text{block diag}\{D_k, D_{k+1},\ldots,D_{k+\omega-1}\}. \end{array} \quad (2.8)$$

Equations (2.6)–(2.8) define the *ω-stacked form* at the initial time $k(=k_0)$ of the given ω-periodic system [4].

Since both models describe the same input/output behavior, it is natural to expect that one model can be derived from the other. Such a derivation was given in [12] and an efficient computational algorithm was developed in [8], showing that if $\mathcal{S}_k(\lambda)$ denotes the system matrix for the ω-stacked realization,

$$\mathcal{S}_k(\lambda) := \begin{bmatrix} \mathcal{A}_k - \mathcal{R}(\lambda) & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{bmatrix}, \quad \det[\mathcal{A}_k - \mathcal{R}(\lambda)] \neq 0. \quad (2.9)$$

then,

$$\begin{bmatrix} U(\lambda) & O \\ Y(\lambda) & I_{p\omega} \end{bmatrix} \mathcal{S}_k(\lambda) \begin{bmatrix} V(\lambda) & X(\lambda) \\ O & I_{m\omega} \end{bmatrix} =$$

$$\left[\begin{array}{cc|c} -I_{n(\omega-1)} & O & O \\ O & \tilde{F}_k - \lambda I_n & \tilde{G}_k \\ \hline O & \tilde{H}_k & \tilde{J}_k \end{array}\right] =: \tilde{\mathcal{S}}_k(\lambda). \quad (2.10)$$

Here $U(\lambda)$ and $V(\lambda)$ are unimodular matrices, and $\tilde{F}_k$, $\tilde{G}_k$, $\tilde{H}_k$ and $\tilde{J}_k$ are constant matrices of appropriate dimensions. In fact, it is clear from the procedure developed in [8] that the matrices $U(\lambda)$, $V(\lambda)$ $X(\lambda)$ and $Y(\lambda)$ in (2.10) are indeed constant matrices. Then, (2.10) represents Rosenbrock's strict system equivalence relation. Finally, the time invariant systems described by the 4-tuples

$$(\hat{F}_k, \hat{G}_k, \hat{H}_k, \hat{J}_k) \quad \text{and} \quad (\tilde{F}_k, \tilde{G}_k, \tilde{H}_k, \tilde{J}_k)$$

in equations (2.2) and (2.10), respectively, are system equivalent [8].

Notice that this reduction process can also be described entirely in terms of a block elimination on (2.9). The following matrix

$$M := \begin{bmatrix} -I & O & \cdots & O \\ A_{k+1} & -I & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & A_{k+\omega-2} & -I \end{bmatrix}$$

is indeed a sub-matrix of $\mathcal{S}_k(\lambda)$ in (2.9). Eliminating the matrices below this block and to its left and right yields immediately the decomposition

$$\left[\begin{array}{cc|c} O & M & O \\ \tilde{F}_k - \lambda I_n & O & \tilde{G}_k \\ \hline \tilde{H}_k & O & \tilde{J}_k \end{array}\right] \quad (2.11)$$

which is essentially (2.10) up to a scaling and a column permutation. This will be discussed in more details in the next section.

3 Time Invariant Representation: Singular Periodic Systems

It is obvious that if the descriptor matrices E_k in (1.1) are invertible then the problem of minimal order representation of periodic descriptor systems can be reduced to finding minimal order representation of standard periodic systems by setting $A_k := E_k^{-1}A_k$; $B_k := E_k^{-1}B_k$ and following the steps in [8]. However, if one or more descriptor matrices E_k are singular, then the results from [8] are no longer applicable. It might still be possible to proceed by eliminating the algebraic relations of the descriptor periodic system. However, in doing this, the structure of the problem is lost. In this section, we will use the periodic Schur decomposition and the forward backward decomposition to obtain the minimal order representation.

Stacked Representation of Descriptor Periodic Systems: The periodic system (1.1) can be expressed as

$$\begin{aligned} \mathcal{R}(\lambda)\tilde{x}_k(h) &= \mathcal{A}_k\tilde{x}_k(h) + \mathcal{B}_k u_k(h) \\ y_k(h) &= \mathcal{C}_k\tilde{x}_k(h) + \mathcal{D}_k u_k(h) \end{aligned} \tag{3.1}$$

where $u_k(h)$, $y_k(h)$, $\tilde{x}_k(h)$, $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$ and $\mathcal{D}_k$ are as defined in (2.1), (2.5), (2.8) and

$$\mathcal{R}(\lambda) := \left[\begin{array}{c|ccc} O & E_k & & \\ & & \ddots & \\ & & & E_{(k+\omega-2)} \\ \hline \lambda E_{(k+\omega-1)} & & O & \end{array}\right] \tag{3.2}$$

with λ denoting the one step forward time operator in the variable h or ω-step forward time operator in the variable k.

Periodic Schur Decomposition: The following result was established in [1]. Let A_i, E_i, $i = 0,\ldots,\omega-1$ be $n\times n$ complex matrices. Then there exist unitary matrices Q_i and Z_i, $i = 0,\ldots,\omega-1$ such that

$$\begin{array}{lllll} \hat{E}_0 & = Z_0^*E_0Q_1, & \hat{A}_0 & = Z_0^*A_0Q_0 \\ \hat{E}_1 & = Z_1^*E_1Q_2, & \hat{A}_1 & = Z_1^*A_1Q_1 \\ \vdots & & \vdots & \\ \hat{E}_{\omega-1} & = Z_{\omega-1}^*E_{\omega-1}Q_0, & \hat{A}_{\omega-1} & = Z_{\omega-1}^*A_{\omega-1}Q_{\omega-1} \end{array} \tag{3.3}$$

where Z_i^* denotes the conjugate transpose of Z_i and the matrices $\hat{A}_i$, $\hat{E}_i$, $i = 0,\ldots,\omega-1$ are complex upper triangular. The above result is a generalization of the QZ-algorithm to solve the generalized eigenvalue problem.

Forward/Backward Decomposition: Given the periodic system matrices $\hat{E}_i$ and $\hat{A}_i$, satisfying (1.2), as described above it is always possible reduce them to upper triangular matrices through unitary transformation matrices Q_i, Z_i. The transformed matrices will have the following structure:

$$\begin{aligned} E_i &:= Z_i^*E_iQ_{i+1}\begin{bmatrix} E_{11i} & E_{12i} \\ & E_{22i}\end{bmatrix}, \\ A_i &:= Z_i^*A_iQ_i\begin{bmatrix} A_{11i} & A_{12i} \\ & A_{22i}\end{bmatrix} \end{aligned}$$

where E_{11i} and A_{22i} are nonsingular matrices.

Assuming that there exists a non-trivial disjoint partition Λ_1, Λ_2 of the set of eigenvalues of the periodic pencil $\lambda E_i - A_i$ counting multiplicities (denoted $\lambda[E_i, A_i]$). Further, $\lambda[E_{11i}, A_{11i}] = \Lambda_1$ and $\lambda[E_{22i}, A_{22i}] = \Lambda_2$. Then using elementary periodic transformations, we can block diagonalize the periodic pencils as illustrated below.

$$\begin{aligned} \begin{bmatrix} E_{11i} & \\ & E_{22i}\end{bmatrix} &:= \begin{bmatrix} I & L_i \\ & I\end{bmatrix}\begin{bmatrix} E_{11i} & E_{12i} \\ & E_{22i}\end{bmatrix}\begin{bmatrix} I & R_{i+1} \\ & I\end{bmatrix} \\ \begin{bmatrix} A_{11i} & \\ & A_{22i}\end{bmatrix} &:= \begin{bmatrix} I & L_i \\ & I\end{bmatrix}\begin{bmatrix} A_{11i} & A_{12i} \\ & A_{22i}\end{bmatrix}\begin{bmatrix} I & R_i \\ & I\end{bmatrix}. \end{aligned} \tag{3.4}$$

by solving the periodic Sylvester equations:

$$\left.\begin{aligned} E_{11i}R_{i+1} + E_{12i} + L_iE_{22i} &= O \\ A_{11i}R_i + A_{12i} + L_iA_{22i} &= O \end{aligned}\right\},\ i = 0,\ldots,(\omega-1).$$

Comment: Note that by premultiplying the block diagonal matrices in (3.4) with a block diagonal matrix $\operatorname{diag}\left((E_{11i})^{-1}, (A_{22i})^{-1}\right)$ and defining $A_i^f = (E_{11i})^{-1}A_{11i}$ and $E_i^b = (A_{22i})^{-1}E_{22i}$, the periodic pencil may be written in its familiar forward/backward form:

$$\left(\lambda\begin{bmatrix} I & \\ & E_{Bi}\end{bmatrix} - \lambda\begin{bmatrix} A_{Fi} & \\ & I\end{bmatrix}\right). \tag{3.5}$$

By suitable permutations, it is possible to ensure that the eigenvalues of A_{Fi} lie on or inside the unit circle and those of E_{Bi} lie outside the unit circle. Additional details on forward/backward decomposition and its applications may be found in [10].

Time Invariant Representation: Consider the $(n\omega\times n\omega)$ matrix pencil $[\mathcal{A}_k - \mathcal{R}(\lambda)]$ of the ω-stacked form. When expanded, the pencil has the following structure:

$$\begin{bmatrix} O & \operatorname{diag}[E_i] \\ \lambda E_{\omega-1} & O\end{bmatrix}\tilde{x}_k(h) = \operatorname{diag}[A_j]\tilde{x}_k(h) \tag{3.6}$$

where $\tilde{x}_k(h)$ is defined in (2.7) and matrices $\mathcal{B}_k$, $\mathcal{C}_k$ and $\mathcal{D}_k$ are block diagonal matrices. For simplicity of presentations, we will restrict the representation of the matrices $\mathcal{B}_k$, $\mathcal{C}_k$ and $\mathcal{D}_k$ only where it is necessary. Additionally, for clarity, the procedure will be illustrated for $\omega = 4$ and $k = 1$.

The first step in reduction process is transformation of the matrix pairs $[E_i, A_i]$ to a periodic Schur form. To this end using the periodic Schur decomposition described earlier, unitary matrices Q_i and Z_i are found such that

$$\operatorname{diag}[Z_i^*]\begin{bmatrix} A_1 & -E_1 & & \\ & A_2 & -E_2 & \\ & & A_3 & -E_3 \\ -\lambda E_4 & & & A_4 \end{bmatrix}\operatorname{diag}[Q_i]$$

After multiplication, define $A_i = Z_i^* A_i Q_i$ and $E_i = Z_i^* A_i Q_{i+1}$, such that

$$A_i = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}_i, \quad E_i = \begin{bmatrix} E_{11} & E_{12} \\ O & E_{22} \end{bmatrix}_i. \tag{3.7}$$

Note that each of the transformed matrices is an upper triangular matrix. Further, the elements along the diagonal of the upper triangular matrices can be reordered such that the E_{11} blocks and A_{22} blocks are invertible.

Next, using the periodic Sylvester equation discussed earlier, the matrices E_i and A_i transformed to upper triangular forms can be block diagonalized. This is accomplished as described next. Determine elementary transformation matrices L_i and R_i such that

$$[\lambda\mathcal{E} - \mathcal{A}] := \operatorname{diag}[L_i]\begin{bmatrix} A_1 & -E_1 & & \\ & A_2 & -E_2 & \\ & & A_3 & -E_3 \\ -\lambda E_4 & & & A_4 \end{bmatrix}\operatorname{diag}[R_i]$$

where,

$$A_i = \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}_i, \quad E_i = \begin{bmatrix} E_{11} & O \\ O & E_{22} \end{bmatrix}_i. \tag{3.8}$$

The above two transformations are also applied to the block diagonal input and output matrices to get $\mathcal{B}$ and $\mathcal{C}$, respectively. Next, shuffling the matrix blocks, we can block diagonalize the system into its forward and backward decomposition. Specifically, the forward subsystem is given by

$$\left[\begin{array}{cccc|c} A_{111} & -E_{111} & O & O & \operatorname{diag}[B_{1i}] \\ O & A_{112} & -E_{112} & O & \\ O & O & A_{113} & -E_{113} & \\ -\lambda E_{114} & O & O & A_{113} & \\ \hline & \operatorname{diag}[C_{1i}] & & & \operatorname{diag}[D_2] \end{array}\right] \tag{3.9}$$

where, the matrices E_{11k}, $k = 1,2,3,4$ are invertible. Since the block

$$M_1 := \begin{bmatrix} -E_{111} & O & O \\ A_{112} & -E_{112} & O \\ O & A_{113} & -E_{113} \end{bmatrix} \tag{3.10}$$

is invertible, using M_1 as pivot, corresponding block rows and column of the system matrix can be eliminated using system equivalent transformations to yield

$$\left[\begin{array}{cc|c} O & M_1 & O \\ \mathcal{A}_F - \lambda\mathcal{E}_F & O & \mathcal{B}_F \\ \hline \mathcal{C}_F & O & \mathcal{D}_F \end{array}\right] \tag{3.11}$$

where, the matrices in (3.11) are defined as:

$$\begin{aligned} \mathcal{E}_F &= E_{114} \\ \mathcal{A}_F &= A_{114}E_{113}^{-1}A_{113}E_{112}^{-1}A_{112}E_{111}^{-1}A_{111} \\ \mathcal{B}_F &= \begin{bmatrix} A_{114}E_{113}^{-1}A_{113}E_{112}^{-1}A_{112}E_{111}^{-1}B_{11} \\ A_{114}E_{113}^{-1}A_{113}E_{112}^{-1}B_{12} \\ A_{114}E_{113}^{-1}B_{13} \\ B_{14} \end{bmatrix}^T \\ \mathcal{C}_F &= \begin{bmatrix} C_{11} \\ C_{12}E_{111}^{-1}A_{111} \\ C_{13}E_{112}^{-1}A_{112}E_{111}^{-1}A_{111} \\ C_{14}E_{113}^{-1}A_{113}E_{112}^{-1}A_{112}E_{111}^{-1}A_{111} \end{bmatrix} \end{aligned}$$

and structure of $\mathcal{D}$ is easily seen.

Clearly, the minimal order representation for the forward subsystem is given by

$$\left[\begin{array}{c|c} \mathcal{A}_F - \lambda\mathcal{E}_F & \mathcal{B}_F \\ \hline \mathcal{C}_F & \mathcal{D}_F \end{array}\right]. \tag{3.12}$$

The second block diagonal sub-pencil after block permutation defines the backward subsystem:

$$\left[\begin{array}{cccc|c} A_{221} & -E_{221} & O & O & \operatorname{diag}[B_{2i}] \\ O & A_{222} & -E_{222} & O & \\ O & O & A_{223} & -E_{223} & \\ -\lambda E_{224} & O & O & A_{223} & \\ \hline & \operatorname{diag}[C_{2i}] & & & O \end{array}\right] \tag{3.13}$$

where, the matrices A_{22k}, $k = 1,2,3,4$ are invertible.

Comment: It should be noted that to make the decomposition unique, the input output matrix $\mathcal{D}$ is only retained in the forward subsystem.

Similar to the discussion for the forward subsystem, we note that the block

$$M_2 := \begin{bmatrix} A_{221} & -E_{221} & O \\ O & A_{222} & -E_{222} \\ O & O & A_{223} \end{bmatrix} \tag{3.14}$$

is invertible. Again, using the block in (3.14) as pivot, and eliminating the corresponding block rows and columns of the system matrix yields

$$\left[\begin{array}{cc|c} M_2 & O & O \\ O & \mathcal{A}_B - \lambda\mathcal{E}_B & \lambda\mathcal{B}_B \\ \hline O & \mathcal{C}_B & \mathcal{D}_B \end{array}\right] \tag{3.15}$$

where, the matrices in (3.15) are defined as:

$$\begin{aligned} \mathcal{E}_B &= E_{224}A_{221}^{-1}E_{221}A_{222}^{-1}E_{222}A_{223}^{-1}E_{223} \\ \mathcal{A}_B &= A_{224} \end{aligned}$$

$$\mathcal{B}_B = \begin{bmatrix} -\lambda E_{224}A_{221}^{-1}B_{21} \\ \lambda E_{224}A_{221}^{-1}E_{221}A_{222}^{-1}B_{22} \\ -\lambda E_{122}A_{221}^{-1}E_{221}A_{222}^{-1}E_{222}A_{223}^{-1}B_{23} \\ O \end{bmatrix}$$

$$\mathcal{C}_B = \begin{bmatrix} C_{21}A_{221}^{-1}E_{221}A_{222}^{-1}E_{222}A_{223}^{-1}E_{223} \\ C_{22}A_{222}^{-1}E_{222}A_{223}^{-1}E_{223} \\ C_{23}A_{223}^{-1}E_{223} \\ C_{24} \end{bmatrix}$$

Again, due to space limitation, we are not reporting the $\mathcal{D}$ matrix, but its is easily determined.

The minimal order representation for the backward subsystem may be written as

$$\left[\begin{array}{c|c} \mathcal{A}_B - \lambda\mathcal{E}_B & \lambda\mathcal{B}_B \\ \hline \mathcal{C}_B & \mathcal{D}_B \end{array}\right]. \qquad (3.16)$$

It will be appropriate to divide the state equation by λ to get the following equivalent minimal order representation of the backward subsystem:

$$\left[\begin{array}{c|c} \lambda^{-1}\mathcal{A}_B - \mathcal{E}_B & \mathcal{B}_B \\ \hline \mathcal{C}_B & \mathcal{D}_B \end{array}\right]. \qquad (3.17)$$

Comment: In the expanded equation for $\mathcal{B}_B$, the last block column has been reduced to zero. The reason for it is that if it was not eliminated, then a delay would appear in the input matrix. It is more convenient to reflect that in the backward subsystem's input output matrix $\mathcal{D}_B$. In fact, the $(4,4)$ element of $\mathcal{D}_B$ is a consequence of the elimination discussed above.

4 Concluding Remarks

For linear periodic systems, there are several methods in the literature to express them as time-invariant systems. However, to the best of authors knowledge, the results presented in this paper are the first attempt to solve equivalent problem for descriptor periodic systems. If the descriptor matrices are non-singular, the descriptor periodic system can be reduced to standard periodic system. However, when the descriptor matrices are singular, the problem becomes considerably more complex. We demonstrated that by employing periodic Schur decomposition and solving periodic Sylvestor equation, it is possible to decompose the singular periodic system into time-invariant forward and backward subsystems.

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