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A LEVEL-SET IDEA TO COMPUTE THE REAL HURWITZ-STABILITY RADIUS

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ABSTRACT

We sketch an algorithm for the computation of the real Hurwitz-stability radius, based on a formula recently obtained by Qiu *et al.* (Automatica, vol. 31, pp. 879-890, 1995). Numerical evidence suggests that the rate of convergence is quadratic.

1. Preliminaries

Let us denote the singular values of a $p \times m$ matrix, ordered nonincreasingly, by $\sigma_k(\cdot)$, $k = 1, 2, \ldots, \min\{p, m\}$. The real structured Hurwitz-stability radius of $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$, with A Hurwitz-stable, is defined [1] as

$$r_{\mathbb{R}}(A, B, C) := \inf\{\sigma_1(\Delta) : A + B\Delta C \text{ is not Hurwitz-stable}\}.$$

Qiu *et al.* [2] proved that

$$r_{\mathbb{R}}(A, B, C) = \left\{ \max_{\omega\in\mathbb{R}} \mu_{\mathbb{R}}[C(\mathrm{j}\omega I - A)^{-1}B] \right\}^{-1}, \quad (1)$$

where, for any $M \in \mathbb{C}^{n\times p}$,

$$\mu_{\mathbb{R}}(M) = \inf_{\gamma\in(0,1]} \sigma_2 \left(\begin{bmatrix} \mathrm{Re}\, M & -\gamma \mathrm{Im}\, M \\ \gamma^{-1}\mathrm{Im} M & \mathrm{Re} M \end{bmatrix} \right). \quad (2)$$

In this note, we propose an efficient algorithm for the computation of $r_{\mathbb{R}}(A, B, C)$ via (1) and (2). Let a transfer function matrix be given through $G(s) = C(sI - A)^{-1}B$ and let A be Hurwitz-stable. For $\xi > 0$, define the Hamiltonian matrix

$$H(\xi, A, B, C) := \begin{bmatrix} A & BB^{\mathrm{T}}/\xi \\ -C^{\mathrm{T}}C/\xi & -A^{\mathrm{T}} \end{bmatrix}. \quad (3)$$

The following result is well-known:

Proposition 1 *For all $\omega \in \mathbb{R}$, ξ is a singular value of $G(\mathrm{j}\omega)$ iff $\mathrm{j}\omega$ is an eigenvalue of $H(\xi, A, B, C)$.*

We seek to adapt this result to compute $r_{\mathbb{R}}(A, B, C)$. For $\gamma \in (0, 1]$, let us use G to define a new transfer function matrix

$$P(\gamma, G) := \begin{bmatrix} \mathrm{Re}\, G & \gamma \mathrm{j}\, \mathrm{Im}\, G \\ \gamma^{-1}\mathrm{j}\, \mathrm{Im}\, G & \mathrm{Re}\, G \end{bmatrix} \quad (4)$$

which is unitarily equivalent to the matrix in (2) with $M = G(\mathrm{j}\omega)$, so the two have the same singular values and we can limit our attention to $P(\gamma, G)$. Using

$$T_\gamma := \frac{1}{\sqrt{2}} \begin{bmatrix} I & \gamma I \\ \gamma^{-1} I & -I \end{bmatrix}, \quad (5)$$

it is easy to verify that

$$P(\gamma, G) = T_\gamma \begin{bmatrix} G & 0 \\ 0 & \overline{G} \end{bmatrix} T_\gamma, \quad (6)$$

where the left and right T_γ matrices are of dimensions $2p \times 2p$ and $2m \times 2m$, respectively. This leads to the promised adaptation of Proposition 1:

Theorem 1 *Let $\gamma \in (0, 1]$ and $\xi > 0$ be given. Then, for all $\omega \in \mathbb{R}$, ξ is a singular value of $P(\gamma, G(\mathrm{j}\omega))$ iff $\mathrm{j}\omega$ is an eigenvalue of $H(\xi, \widetilde{A}, \widetilde{B}_\gamma, \widetilde{C}_\gamma)$, where*

$$\widetilde{A} = \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}, \ \widetilde{B}_\gamma = \tfrac{1}{\sqrt{2}} \begin{bmatrix} B & \gamma B \\ -\gamma^{-1}B & B \end{bmatrix},$$
$$\widetilde{C}_\gamma = \tfrac{1}{\sqrt{2}} \begin{bmatrix} C & \gamma C \\ \gamma^{-1}C & -C \end{bmatrix}.$$

Proof In view of Proposition 1, it is sufficient to verify that

$$P(\gamma, G(\mathrm{j}\omega)) = \widetilde{C}_\gamma(\mathrm{j}\omega I - \widetilde{A})^{-1}\widetilde{B}_\gamma,$$

which follows, after some algebraic manipulations, from (6). ∎

2. Outline of iterative algorithm

The Hamiltonian matrix $H(\xi, \widetilde{A}, \widetilde{B}_\gamma, \widetilde{C}_\gamma)$, which we shall denote by $\widetilde{H}(\xi, \gamma)$ to save space, plays an important role in our algorithm. Define

$$\Xi := \max_{\omega\in\mathbb{R}} \mu_{\mathbb{R}}[G(\mathrm{j}\omega)] \quad \text{and} \quad \Omega := \arg\max_{\omega\in\mathbb{R}} \mu_{\mathbb{R}}[G(\mathrm{j}\omega)],$$

assuming that such a unique maximizer exists. Suppose that at each iteration, $k = 0, 1, \ldots$, we are given ω_k, the current trial frequency, and a "maximizing set", S_k, containing Ω. We maintain two quantities Ξ_k and Ω_k, our best approximations thus far of Ξ and Ω, respectively, with $\Xi_k = \mu_{\mathbb{R}}[G(\mathrm{j}\Omega_k)]$. (We can begin with an arbitrary ω_0, and $S_0 = (-\infty, \infty)$, $\Omega_0 = \omega_0$.) We perform the following sequence of steps at iteration k. First, determine $\mu_{\mathbb{R}}[G(\mathrm{j}\omega_k)]$; assume that the inf is achieved at $\gamma = \gamma^*_{\omega_k}$ and call it ξ_k:

$$\xi_k := \min_\gamma \sigma_2(P(\gamma, G(\mathrm{j}\omega_k))) = \sigma_2(P(\gamma^*_{\omega_k}, G(\mathrm{j}\omega_k))).$$

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If $\xi_k > \Xi_{k-1}$, take $\Xi_k = \xi_k$, $\Omega_k = \omega_k$, otherwise keep the old estimates, i.e., $\Xi_k = \Xi_{k-1}$, $\Omega_k = \Omega_{k-1}$. Next, find the "level set" $R_k = \{\omega : \sigma_2(P(\gamma^*_{\omega_k}, G(\mathrm{j}\omega))) > \Xi_k\}$. By Theorem 1, the pure imaginary eigenvalues of $\widetilde{H}(\Xi_k, \gamma^*_{\omega_k})$ are exactly those ω for which some singular value of $P(\gamma^*_{\omega_k}, G(\mathrm{j}\omega))$ equals Ξ_k; the endpoints of the frequency intervals where $\sigma_2(P(\gamma^*_{\omega_k}, G(\mathrm{j}\omega)))$ exceeds Ξ_k must be among these and can be identified using the signs of their derivatives[1] with respect to ξ. As illustrated in Fig. 1, the derivative-signs allow us to *index* the singular values equal to Ξ_k and also to identify the intervals where $\sigma_2(\cdot) > \Xi_k$. Since any frequency point $\hat{\omega}$ not in R_k satisfies

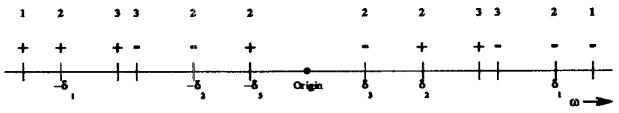

Figure 1: Using derivatives of imaginary Hamiltonian eigenvalues to find $R_k = (-\delta_1, -\delta_2) \cup (-\delta_3, +\delta_3) \cup (\delta_2, \delta_1)$.

$$\mu_{\mathbb{R}}(G(\mathrm{j}\hat{\omega})) \leq \sigma_2(P(\gamma^*_{\omega_k}, G(\mathrm{j}\hat{\omega}))) \leq \Xi_k,$$

the global maximizer Ω of $\mu_{\mathbb{R}}(G(\mathrm{j}\omega))$ cannot lie outside R_k, if $R_k \neq \emptyset$. Thus, by setting $S_{k+1} = R_k \cap S_k$, we can bracket Ω at every iteration. As for the next trial frequency ω_{k+1}, if S_{k+1} contains more than one interval we set ω_{k+1} equal to the midpoint of the largest such interval; otherwise we follow a more complicated rule which sometimes involves cubic-fit (two function values and two derivatives) of $\mu_{\mathbb{R}}$. A more detailed description of the algorithm can be found in [3]. In our numerical tests we have observed that, eventually, S_k always reduces to a single interval and that the cubic-fit rule appears to lead to quadratic convergence (of Ξ_k to Ξ).

3. Numerical example

Let us take the example given in [2],

$$n = 4, m = 2, p = 2,$$

$$A = \begin{bmatrix} 79 & 20 & -30 & -20 \\ -41 & -12 & 17 & 13 \\ 167 & 40 & -60 & -38 \\ 33.5 & 9 & -14.5 & -11 \end{bmatrix}, B = \begin{bmatrix} 0.2190 & 0.9347 \\ 0.0470 & 0.3835 \\ 0.6789 & 0.5194 \\ 0.6793 & 0.8310 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.0346 & 0.5297 & 0.0077 & 0.0668 \\ 0.0535 & 0.6711 & 0.3834 & 0.4175 \end{bmatrix},$$

and choose $\omega_0 = 0$ ($\gamma^*_0 = 1$). In Fig. 2, the solid curve denotes $\mu_{\mathbb{R}}[G(\mathrm{j}\omega)]$ and the horizontal (dotted) line the level $\Xi_0 = \mu_{\mathbb{R}}(G(\mathrm{j}\omega_0)) = 0.932$. The first two steps ($k = 0, 1$) of our algorithm can be understood by studying the dashed curves in Fig. 2 which represent $\sigma_2(P(\gamma^*_{\omega_0}, G(\mathrm{j}\omega)))$ and $\sigma_2(P(\gamma^*_{\omega_1}, G(\mathrm{j}\omega)))$. For the first step, $R_0 = (6 \times 10^{-8}, 12.05) = S_1$ and the next trial frequency, marked with an 'x', is $\omega_1 = 6.02$ (midpoint of S_1). Optimizing with respect to γ at ω_1 yields $\gamma^*_{\omega_1} = 0.1365$ and R_1 is found to be the union of two "σ_2-intervals". Intersecting R_1 with S_1 gives $S_2 = (0.131, 2.037) \cup (9.255, 12.05)$; the next trial frequency ω_2 is 10.65 (midpoint of the larger interval in S_2, marked with an 'o') and so on. Some quantities of interest at each iteration are shown in Table 1. As can be seen, the number of significant digits of Ξ_k doubles at each step for $k = 2, \ldots, 5$, suggesting that the algorithm is quadratically convergent. The maximum (1.9450) of $\mu_{\mathbb{R}}(G(\mathrm{j}\omega))$ occurs at $\omega = 1.38$ and $r_{\mathbb{R}}(A, B, C) = 0.5141$.

[1] More precisely, the imaginary part of their derivatives.

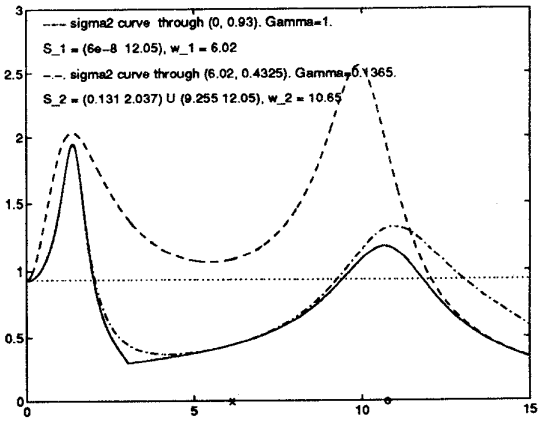

Figure 2: Iterations 1 and 2 for Example, with $\omega_0 = 0$.

Table 1: Convergence of S_{k+1} for Example with $\omega_0 = 0$.

k	ω_k	Ω_k	Ξ_k [digits]	S_{k+1}	size of S_{k+1}
0	0	0	0.932 [0]	(6e−8,12.05)	12.05
1	6.02	0	0.932 [0]	(0.131, 2.037), (9.255, 12.05)	2.79
2	10.65	10.65	1.177 [1]	(0.699, 1.922), (10.65, 10.66)	1.224
3	1.311	1.311	1.919 [2]	(1.311, 1.440)	0.13
4	1.376	1.376	1.945 [6]	(1.376, 1.378)	0.002
5	1.377	1.377	1.945 [11]	(1.377, 1.377)	1e−7

4. Conclusion

We have sketched an algorithm for the computation of the real Hurwitz-stability radius. Preliminary computational evidence suggests that its local convergence rate is quadratic. The discrete-time version of this problem can be solved in much the same way by considering symplectic pencils rather than Hamiltonian matrices and using level sets on the unit circle rather than on the $\mathrm{j}\omega$ axis [3]. Finally, a related algorithm can be used for the computation of an upper bound to the μ-norm of a transfer matrix. This is discussed in [4] where a proof of global convergence is also given.

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