# Exponential Ranking: taking into account negative links*

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**Abstract.** Networks have attracted a great deal of attention the last decade, and play an important role in various scientific disciplines. Ranking nodes in such networks, based on for example PageRank or eigenvector centrality, remains a hot topic. Not only does this have applications in ranking web pages, it also allows peer-to-peer systems to have effective notions of trust and reputation and enables analyses of various (social) networks. Negative links however, confer distrust or dislike as opposed to positive links, and are usually not taken into account. In this paper we propose a ranking method we call *exponential ranking*, which allows for negative links in the network. We show convergence of the method, and demonstrate that it takes into account negative links effectively.

## 1 Introduction

The ranking of nodes, or assigning some 'importance' or 'trust' scores to nodes, has attracted a great deal of attention when networks are being studied. Already in the 1970s, various researchers from the social sciences have introduced concepts such as betweenness [1], closeness [2] and eigenvector centrality [3,4] to measure how central or important a node in the network was. For example, centrality-like measures are shown to have an influence on spreading processes on networks, such as failing cascades [5], or the infection process of sexually transmitted diseases [6,7]. Furthermore, it helps to identify different roles nodes might play in a network [8].

In the 1990s several alternative ranking measures were added, notably Kleinbergs HITS-algorithm [9], and Googles PageRank [10]. When filesharing and especially peer-to-peer applications grew, these measures, and variants thereof, became popular to keep 'good' peers in the sharing network, and exclude 'bad' peers [11,12]. Reputation and trust also plays a vital role in online markets such as eBay [13].

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Negative links however, are usually not taken into account by these ranking measures, or worse, they break down when negative entries appear as weights of the links. However, the signs of links (positive or negative) should not be ignored, since they may bear important consequences for the structure of the network, not in the least for the ranking of nodes. Proposals have been made to include such semantic information in hyperlinks on the World Wide Web [14]. Negative links are also present in various other settings such as reputation networks [15], sharing networks [11], social networks [16] and international networks [17], and play a key, if not vital, role in these networks. Studying how negative links influence the 'importance' of nodes may help the understanding of such systems, and such a concept of 'importance' might facilitate the analyses of such networks again.

Recently there has been more attention to negative links in ranking measures, for example PageTrust [18]. The difference between PageTrust and PageRank is that in the random walk in PageTrust nodes that are negatively pointed to during the random walk are blacklisted, and are visited less often, thereby lowering their PageTrust score. Another suggestion was to first calculate some ranking using only the positive links (e.g. using PageRank), and then apply one step of distrust, so that the negative links are taken into account [19,14].

It was also suggested to introduce a propagation of distrust [19], implying that if $i$ distrusts $j$, and $j$ distrusts $k$, then $i$ should trust $k$ (the adagium that the enemy of my enemy is my friend). The authors noted that this could lead to situations in which a node is its own enemy (if one considers a cycle of three negative links). This phenomenon is studied in the social sciences under the denominator of 'social balance theory' [20,21]. A network is considered as balanced, if all triads (a cycle of three nodes) in the network are either completely positive, or have only one positive link, and more recently some models have tried to capture dynamics based on social balance [22]. Already in 1953 it was shown that a network is balanced in this sense if and only if it divides neatly in two clusters with negative links appearing only between the two clusters [20]. Later, this idea was extended to include the possibility to cluster nodes in more than one cluster, by only demanding there is not exactly one negative link in any cycle [21], although for practical clustering in networks with negative links other methods have been devised [23]. The triad with only negative links is found to appear more often than expected by social balance theory [16,17], so that they (potentially) divide in more than two clusters.

In this short paper we introduce a ranking measure based on discrete choice theory, that can also be used when negative links are present in the network. The goal is to infer some global ranking of nodes, based on a particular given network (with possibly some negative links present). We do so in terms of reputation and trust, but the application of the measure need not be restricted to the domain of trust management. It might find also applications in collaborative or iterative filtering, where items such as movies or products need to be recommended[3] [24],

[3] Often recommendation is personalized based on existing preferences or purchases of movies or products. We do not currently consider such a personalization in this short paper, but there seem to be some possibilities for doing so.

or somewhat related, predict the sign of links [25]. However, such a measure might also be of interest for characterizing nodes in various networks, such as the international network of conflict and alliances [17], or in an online social network [16].

Given the considerations of social balance provided above—that indeed the enemy of your enemy need not be your friend—it would be undesirable to assume so in any ranking scheme. That is, if a node were to have a negative reputation, his links should not be distrusted, only trusted less. In other words, we should not assume a node with a negative reputation is not trustworthy (if he points negatively towards someone, we should not interpret it as positive, and vice versa), we should only trust his judgements less. This will actually follow from the derivation of the measure based on a discrete choice argument, which we will present in the following section. Most of the existing algorithms dealing with negative links do not apply distrust in such a recursive manner, thereby limiting their effect. Furthermore, none of the algorithms can actually deal with negative reputations, while this negativity can actually provide additional insight. For example, a negative reputation would signal that such a node should be blocked from the network.

The reputation of nodes is based on judgments by other nodes, which is detailed in Sec. 3. Convergence and uniqueness of the proposed measure is proven in Sec. 4. We give some conclusion and indications of further research in the final section.

## 2 Discrete choice

Let $G = (V, E)$ be a directed graph with $n = |V|$ nodes and $m = |E|$ edges. Each edge $(ij)$ has an associated weight $w_{ij}$ which can possibly be negative. By $A$ we denote the $n \times n$ adjacency matrix associated to the graph, such that $A_{ij} = w_{ij}$ if there is an $(ij)$ edge and zero otherwise. Furthermore, let $k_i$ be some reputation of node $i$ (we will make this explicit in the next section). We consider the links to indicate a certain trust: if node $i$ points positively (negatively) to node $j$, this indicates that $i$ trusts (distrusts) $j$. The goal is to infer some global trust values from the local trust links.

Suppose we are asked which node to trust, if we were to choose one. We assume that a higher reputation indicates some degree of trust, so we should preferably choose nodes which have a high reputation $k_i$. However, there might be some errors in choosing the one with the highest reputation. This is where the framework of discrete choice theory comes in.

The usual background for discrete choice theory is the following [26]. Suppose there are $n$ different choices (in our case, nodes), which have a different associated utility $u_i$. We observe the utility $o_i$ and have some error term $\epsilon_i$ such that

$$u_i = o_i + \epsilon_i. \tag{1}$$

We would like to choose the object with the maximum utility. However, since we only observe $o_i$, it is uncertain which item actually has the maximum real utility. So, the question becomes: what is the probability we will select a certain object? That is, what is the probability that $u_i \geq u_j$ for all $i \neq j$, or

$$\Pr(u_i = \max_j u_j), \tag{2}$$

depending on the observed utility $o_i$ and the error term $\epsilon_i$. In our case, we equate the observed utility $o_i$ with some reputation $k_i$. We assume the real reputation is then $u_i = k_i + \epsilon_i$, where $\epsilon_i$ is the error made in observing the reputation.

The probability of choosing the node with the highest reputation depends on the distribution of the error term $\epsilon_i$. Using the following assumption for the error term, we arrive at the well known multinomial logit model [26]. Suppose the $\epsilon_i$ are i.i.d. double exponentially distributed[4] according to

$$\Pr(\epsilon_i \leq x) = \exp - \left[\exp - \left(\frac{x}{\mu} + \gamma\right)\right], \tag{3}$$

where $\gamma \approx 0.5772$ is Euler's constant. The mean of (3) equals zero, and the variance equals $1/6\pi^2\mu^2$. With this error distribution it can be proven [26] that the probability node $i$ has the highest real reputation becomes

$$p_i = \frac{\exp \frac{k_i}{\mu}}{\sum_j \exp \frac{k_j}{\mu}}. \tag{4}$$

The probability a node $i$ has the highest reputation, increases with higher reputation $k_i$, depending on the amount of noise characterized by $\mu$. There are two extreme scenarios depending on $\mu$. If $\mu \to \infty$ the variance goes to infinity, and the contribution of the observed reputation in $u_i = k_i + \epsilon_i$ becomes negligibly small. In that case, the probability a node has the highest real reputation becomes uniform, or $p_i = 1/n$. In the other extreme, $\mu \to 0$, there is essentially no error, and we will always be correct in choosing nodes with a maximum $k_i$. That is, if there is a set of nodes $M$ with $k_i = \max_j k_j$ for $i \in M$, then $p_i = 1/|M|$ for $i \in M$, and zero otherwise.

The probabilities $p$ shows how much we should trust nodes. Nodes with a higher reputation are more trustworthy than nodes with a lower reputation. The difference in trust becomes more pronounced with decreasing $\mu$, up to the point where we only trust nodes with the highest reputation. We shall call these probabilities the trust probabilities.

## 3 Reputation and judges

The trust probabilities defined in the previous section depend on the reputation $k_i$, which we will define now. We will ask a certain node $j$ to provide the reputation values of the other nodes. That is, we ask node $j$ to be the judge of his peers. Since we consider $A_{ji}$ to be the trust placed by node $j$ in node $i$, we

[4] This distribution is also known as the Gumbel distribution

| | |
|---|---|
| $A$ | Adjacency matrix of given network |
| $k_i$ | Reputation of node $i$ |
| $\epsilon_i$ | Error term in reputation |
| $\mu$ | Parameter influencing variance of error |
| $p_i$ | Trust probability of node $i$ |
| $k_i^*$ | Final (fixed point) reputation |
| $p_i^*$ | Final (fixed point) trust |

**Table 1.** Overview of variables used in this paper.

will assume that if node $j$ is the judge, he would simply say that $k_i = A_{ji}$. The general idea is that the probability to be a judge depends on the reputation, which then influences that probability again.

To choose a judge, we will again consider a discrete choice argument. We would like to select a judge $j$ for which $k_j + \epsilon_j$ is maximal, where $\epsilon_j$ is again distributed according to (3). Obviously, the probability to select a judge will then be the same as in (4).

Using those probabilities $p_i$, we select a judge at random, and let him give his opinion on the reputation of his peers. We thus allow trustworthy nodes a higher probability to judge their peers. The expected reputation can then be written as

$$k_i = \sum_j A_{ji} p_j, \tag{5}$$

or in matrix notation, the column vector $k$ can be expressed as

$$k = A^\mathsf{T} p, \tag{6}$$

where $A^\mathsf{T}$ is the transpose of $A$ and $p$ is a column probability vector (i.e. $\|p\|_1 = 1$ and $p_i \geq 0$). If we plug this formulation of the reputation into (4) we obtain a recursive formulation of trust probabilities

$$p(t+1) = \frac{\exp \frac{1}{\mu} A^\mathsf{T} p(t)}{\| \exp \frac{1}{\mu} A^\mathsf{T} p(t) \|_1}, \tag{7}$$

for some initial condition $p(0)$, with $\exp(\cdot)$ the elementwise exponential. We will prove in the next section that this iteration actually converges to a unique fixed point $p^*$, i.e. independent of the initial conditions, for some range of values for $\mu$. The final values of the trust probabilities can thus be defined as the limiting vector $p^* = \lim_{t\to\infty} p(t)$ or, equivalently, the fixed point $p^*$ for which

$$p^* = \frac{\exp \frac{1}{\mu} A^\mathsf{T} p^*}{\| \exp \frac{1}{\mu} A^\mathsf{T} p^* \|_1}, \tag{8}$$

and the final reputation values as

$$k^* = A^\mathsf{T} p^*. \tag{9}$$

**Table 2.** Trust probabilities for the example network in Fig. 1. Decreasing $\mu$ also decreases the trust for $d$ and $e$, but as $\mu \to 0$ we obtain cyclic behavior. We also provide the ordinary PageRank (PR) with a zapping factor of 0.85 for comparison. We removed the negative link for the calculation of the PageRank.

(a) Trust for various values of $\mu$

| | PR | $\mu = 1$ | $\mu = 1/5$ | $\mu = 1/8$ |
|---|---|---|---|---|
| $a$ | 0.183 | 0.223 | 0.384 | 0.424 |
| $b$ | 0.184 | 0.213 | 0.179 | 0.142 |
| $c$ | 0.263 | 0.223 | 0.384 | 0.424 |
| $d$ | 0.184 | 0.171 | 0.026 | 0.005 |
| $e$ | 0.186 | 0.171 | 0.026 | 0.005 |

(b) Cyclic behavior for $\mu = 0$

| | $t = 0$ | $t = 1$ | $t = 2$ | $t = 3$ |
|---|---|---|---|---|
| $a$ | 0.20 | 0.50 | 0.3 | 0.5 |
| $b$ | 0.20 | – | 0.3 | – |
| $c$ | 0.20 | 0.50 | 0.3 | 0.5 |
| $d$ | 0.20 | – | – | – |
| $e$ | 0.20 | – | – | – |

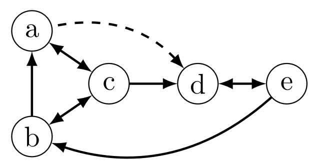


**Fig. 1.** A small example with one negative link (the dashed one), showing trust values for various values of $\mu$. The weights for the positive links are $+1$ and for the negative link $-1$.

Notice that these reputation values are also a fixed point of the equation

$$k^* = A^\mathsf{T} \frac{\exp \frac{1}{\mu} k^*}{\| \exp \frac{1}{\mu} k^* \|_1} \tag{10}$$

and that the trust probabilities are related to the reputation values as

$$p^* = \frac{\exp \frac{1}{\mu} k^*}{\| \exp \frac{1}{\mu} k^* \|_1}. \tag{11}$$

In this sense, the trust probabilities and the reputation values can be seen as a dual formulation of each other.

Upon closer examination of (10), a certain node $j$ might indeed get a negative reputation, but his judgements are taken less into account, they are not reversed. That is, as soon as a node has a negative reputation, we do not assume he his completely untrustworthy, and that his negative judgements should be taken positive, but only that he is less trustworthy. This means we indeed do not assume that the enemy of my enemy is my friend. A node could get a negative reputation for example if he is negatively pointed to by trustworthy nodes. This approach can be summarized in the idea that the reputation of a node depends on the reputation of the nodes pointing to him, or stated differently, a node is only as trustworthy as the nodes that trust him. Notice that this idea is similar to that of eigenvector centrality [4] namely that nodes are as central as the neighbours pointing to him, a recursive notion also present in PageRank [10].

Let us take a look at a small example to see what the effect is of negative links in the network as shown in Fig. 2. There is only one negative link, from $a$ to $d$. The effect of the negative link becomes more penalizing when $\mu$ is decreased, as shown in Table 2(a). That has also consequences for node $e$, who is only pointed to by $d$, who receives little trust, which then also leads to little trust for $e$. The PageRank for these nodes (for which we did not take into account the negative link, and used a zapping factor of 0.85) are provided as comparison, which assigns nodes $d$ and $e$ actually higher rankings.

We will now show that indeed this limit converges (for some range of $\mu$) and is unique, i.e. does not depend on the actual initial condition $p(0)$.

## 4 Convergence and uniqueness

More formally, let us define the map $V : S^n \to S^n$, which maps

$$V(p) = \frac{\exp \frac{1}{\mu} A^\mathsf{T} p}{\| \exp \frac{1}{\mu} A^\mathsf{T} p\|_1}, \tag{12}$$

where $S^n = \{y \in \mathbb{R}^n_+ : \|y\|_1 = 1\}$, the $n$-dimensional unit simplex. For the proof of convergence we rely on mixed matrix norms, or subordinate norms, which are defined as

$$\|A\|_{p,q} = \max_{\|x\|_q = 1} \|Ax\|_p. \tag{13}$$

Denoting by $\|A\|_{\max} = \max_{ij} |A_{ij}|$, we have the following useful inequality

$$\|Ax\|_\infty = \max_i \|e_i^\mathsf{T} Ax\| \leq \|A\|_{\max} \cdot \|x\|_1, \tag{14}$$

hence

$$\|A\|_{\infty,1} \leq \|A\|_{\max} \tag{15}$$

where $e_i$ is the $i$-th coordinate vector. Let us now take a look at the Jacobian of $V$, which can be expressed as

$$\frac{\partial V(p)_i}{\partial p_j} = \frac{\exp(\frac{1}{\mu} A^\mathsf{T} p)_i \frac{1}{\mu} A_{ji}}{\sum_l \exp(\frac{1}{\mu} A^\mathsf{T} p)_l} - \frac{\exp(\frac{1}{\mu} A^\mathsf{T} p)_i \sum_l \exp(\frac{1}{\mu} A^\mathsf{T} p)_l \frac{1}{\mu} A_{jl}}{\left(\sum_l \exp(\frac{1}{\mu} A^\mathsf{T} p)_l\right)^2}. \tag{16}$$

Now let $u = \exp(\frac{1}{\mu} A^\mathsf{T} p)$, and $q = \|u\|_1$. Then $V(p) = u/q$, and $\frac{\partial V(p)_i}{\partial p_j}$ can be simplified to

$$\frac{\partial V(p)_i}{\partial p_j} = \frac{1}{\mu} \left( \frac{u_i}{q} A_{ji} - \frac{1}{q^2} \sum_l u_i u_l A_{jl} \right) \tag{17}$$

or in matrix notation

$$V'(p) = \frac{1}{\mu} \left( \frac{1}{q} \mathrm{diag}(u) - \frac{1}{q^2} u u^\mathsf{T} \right) A^\mathsf{T} \tag{18}$$

at which point the following lemma is useful.

**Lemma 1.** *Denote by* $M(p)$ *the matrix* $M(p) = \mathrm{diag}(p) - pp^\mathsf{T}$ *where* $p \in S^n$*, then* $\|M(p)\|_{1,\infty} \leq 1$*.*

*Proof.* Note that $\|M(p)x\|_1 = \sum_{i=1}^n p_i|x_i - p^\mathsf{T}x|$. We need to find the maximum of this function on the unit box (that is, where $\|x\|_\infty = 1$). Clearly this is attained at some vector $\sigma \in R^n$ with coordinates $\pm 1$. Denoting by $I_+ = \{i : \sigma_i = 1\}$ the set of positive entries, and by $S_1 = \sum_{i \in I_+} p_i$ and $S_2 = 1 - S_1$. Then $p^\mathsf{T}\sigma = S_1 - S_2$, and we have

$$\begin{aligned} \|M(p)\sigma\|_1 &= \sum_{i=1}^n p_i|\sigma_i - S_1 + S_2| = \sum_{i \in I_+} p_i|1 - S_1 + S_2| + \sum_{i \notin I_+} p_i|1 + S_1 - S_2| \\ &= S_1(1 - S_1 + S_2) + S_2(1 + S_1 - S_2) = 1 - (S_1 - S_2)^2. \end{aligned}$$

Since $(S_1 - S_2)^2 \geq 0$, $\|M(p)\sigma\|_1 \leq 1$. □

This immediately leads to the following proof that the map $V$ converges.

**Theorem 1.** *For* $\mu > \frac{1}{2}(\max_{ij} A_{ij} - \min_{ij} A_{ij})$ *the map* $V$ *has a unique fixed point* $p \in S^n$*.*

*Proof.* By the Banach fixed point theorem, this map has a unique fixed point if it is contractive. That is, there should be a $c \leq 1$ such that

$$\frac{\|V(p) - V(u)\|_1}{\|p - u\|_1} < c, \tag{19}$$

for $p, u \in S^n$. That is, if $\|V'(p)\|_{1,1} < c$. Since we can write $V'(p) = \frac{1}{\mu}M(V(p))A$, using the lemma and (15) we arrive at

$$\|V'(p)\|_{1,1} = \frac{1}{\mu}\|M(V(p))A\|_{1,1} \leq \frac{1}{\mu}\|M(V(p))\|_{1,\infty}\|A\|_{\infty,1} \leq \frac{1}{\mu}\|A\|_{\max}.$$

Since adding a constant to our matrix $A$ does not change the vector $V(p)$, we can subtract $\frac{1}{2}(\min_{ij} A_{ij} + \max_{ij} A_{ij})$, and arrive at

$$\|V'(p)\|_{1,1} \leq \frac{1}{2\mu}(\max_{ij} A_{ij} - \min_{ij} A_{ij}).$$

Hence, if

$$\mu > \frac{1}{2}(\max_{ij} A_{ij} - \min_{ij} A_{ij}),$$

the map $V$ is contractive and by the Banach fixed point theorem, it will have a unique fixed point, and iterates will converge to that point. □

For this lower bound on $\mu$, we can guarantee convergence of the iteration. Below this lower bound, we choose nodes with more and more certainty. As we said in Sec. 2, when $\mu \to 0$ the probabilities $p_i = 1/|M|$ for $i$ in some set $M$ of nodes with maximal reputation $k_i$. In the iteration this means only nodes with the highest reputation can become judges. Since we completely trust his judgments, to whatever node(s) he assigns the highest reputation will be the next judge. Unless everyone always agrees on the node with the highest reputation, cycles of judges pointing to the next judge will emerge.

For example, if we take $\mu \to 0$ for the example network given in Fig. 1, we cycle as follows. We start out with $p(0) = 1/n$, and the average reputation will be highest for nodes $a$ and $c$, and they will be chosen as judge with probability $1/2$. In the next iteration the average reputation will be $1/2$ for nodes $a$, $b$ and $c$ and zero for $d$ and $e$. Hence, one of the nodes $a$, $b$ and $c$ will be selected as judge, and the average reputation is $2/3$ for $a$ and $c$, and $1/3$ for $b$. Now we are back where we were after the first iteration, since $a$ and $c$ both have the same maximal reputation, and they are chosen as judge each with probability $1/2$, as summarized in Table 2(b).

## 5 Conclusions and further work

In this short paper we have suggested a new measure to compute global trust values and reputation, which can be used on networks that have negative links. We have shown that it converges linearly for some parameter range. The measure takes into account negative links effectively, penalizing nodes which are negatively pointed to, thereby decreasing their trust value. This might have applications in peer-to-peer systems [11], but also in online markets such as eBay. Furthermore, it might be used to analyze networks where negative links are present, such as social networks [16] and international networks [17]. In that sense, it is an alternative to measures such as betweenness [1] and eigenvector centrality [3].

The analysis offered here is rudimentary, and further experiments are need to investigate the performance of exponential ranking. We would for example need to compare its performance with other ranking methods [18,11,19]. One possible way to test performance is to create test networks with both good and bad nodes, where the methods would need to predict whether the nodes are good or bad based on some positive and negative link topology. A short preliminary analysis shows that the suggested method ought to perform well. Extending this method by including some personalization could possibly allow for prediction of signs of links, and we could thus test performance by replicating earlier experiments [19,25].

## References


1. Freeman, L.: A set of measures of centrality based on betweenness. Sociometry **40(1)** (1977) 35–41
2. Freeman, L.: Centrality in social networks conceptual clarification. Soc. Networks **1** (1979) 215–239
3. Bonacich, P.: Some unique properties of eigenvector centrality. Soc. Networks **29** (2007) 555–564

4. Bonacich, P.: Factoring and weighting approaches to status scores and clique identification. J. Math. Sociol. **2** (1972) 113–120
5. Watts, D.J.: A simple model of global cascades on random networks. Proc. Nat. Acad. Sci. USA, **99(9)** (2002) 5766–5771
6. De, P., A. Singh, T. Wong, and W. Yacoub: Sexual network analysis of a gonorrhoea outbreak. Sex Transm. Infect., **80** (2004) 280–285
7. Christley R., G. Pinchbeck, and R. Bowers: Infection in social networks: using network analysis to identify high-risk individuals. Am. J. Epidemiol., **162(10)** (2005) 1024–1031
8. Perra, N. and S. Fortunato. Spectral centrality measures in complex networks. Phys. Rev. E, **78(3)** (2008) 036107+
9. Kleinberg, J.: Authoritative sources in a hyperlinked environment. J. ACM, **46(5)** (1999) 604–632
10. Brin, S. and Page, L.: The anatomy of a large-scale hypertextual web search engine. Comput. Networks ISDN **30** (1998) 107–117
11. Kamvar, SD., Schlosser, MT., and Garcia-Molina, H.: The eigentrust algorithm for reputation management in p2p networks. Proc. 12$^{th}$ Int. Conf. on WWW (2003) 640–651
12. Abrams, Z., McGrew, R., and Plotkin, S.: Keeping peers honest in eigentrust. 2nd Workshop on Eco. of P2P Syst. (2004)
13. Resnick, P., Zeckhauser, R., Swanson, J., and Lockwood, K.: The value of reputation on ebay: A controlled experiment. Exp. Econ. **9(2)** (2006) 79–101
14. Massa, P. and Hayes, C.: Page-reRank: using trusted links to re-rank authority Proc. IEEE/WIC/ACM Int. Conf. on Web Int. (2005) 614–617
15. Massa, P. and Avesani, P.: Controversial users demand local trust metrics: An experimental study on epinions.com community. Proc. of the Nat. Conf. on Artif. Int., **20(1)** (2005) 121–126
16. Szell, M., Lambiotte, R., and Thurner, S.: Trade, conflict and sentiments: Multi-relational organization of large-scale social networks. arXiv:1003.5137 (2010)
17. Maoz, Z., Terris, LG., Kuperman, RD., and Talmud, I.: What is the enemy of my enemy? causes and consequences of imbalanced international relations, 1816–2001. J. Polit., **69(1)** (2008) 100–115
18. De Kerchove C. and P. Van Dooren: The PageTrust algorithm: how to rank web pages when negative links are allowed. Proc. SIAM Int. Conf. on Data Mining (2008) 346–352
19. Guha, R., Kumar, R., Raghavan, P., and Tomkins, A.: Propagation of trust and distrust. Proc. 13$^{th}$ Int. Conf. on WWW (2004) 403–412
20. Harary, F.: On the notion of balance of a signed graph. Michigan Math J., **2** (1953) 143–146
21. Cartwright, D. and F. Harary: On the coloring of signed graphs. Elemente der Mathematik, 23:85–89, 1968.
22. Antal, T., P. L. Krapivsky, and S. Redner: Dynamics of social balance on networks. Phys. Rev. E, **72(3)** (2005) 036121+
23. Traag, V.A. and J. Bruggeman. Community detection in networks with positive and negative links. Phys. Rev. E, **80(3)** (2009) 36115
24. B Sarwar, G Karypis, and J Konstan: Item-based collaborative filtering recommendation algorithms. Proc. 10$^{th}$ Int. Conf. on WWW (2001) 285–295
25. Leskovec, J., D. Huttenlocher and J.M. Kleinberg. Predicting positive and negative links in online social networks. Proc. 19$^{th}$ Int. Conf. on WWW (2010)
26. Anderson, S.P., De Palma, A., and Thisse, J-C.: Discrete choice theory of product differentiation. MIT (1992)