Dynamical Models Explaining Social Balance and Evolution of Cooperation

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Social networks with positive and negative links often split into two antagonistic factions. Examples of such a split abound: revolutionaries versus an old regime, Republicans versus Democrats, Axis versus Allies during the second world war, or the Western versus the Eastern bloc during the Cold War. Although this structure, known as social balance, is well understood, it is not clear how such factions emerge. An earlier model could explain the formation of such factions if reputations were assumed to be symmetric. We show this is not the case for non-symmetric reputations, and propose an alternative model which (almost) always leads to social balance, thereby explaining the tendency of social networks to split into two factions. In addition, the alternative model may lead to cooperation when faced with defectors, contrary to the earlier model. The difference between the two models may be understood in terms of the underlying gossiping mechanism: whereas the earlier model assumed that an individual adjusts his opinion about somebody by gossiping about that person with everybody in the network, we assume instead that the individual gossips with that person about everybody. It turns out that the alternative model is able to lead to cooperative behavior, unlike the previous model.

I. INTRODUCTION

Why do two antagonistic factions emerge so frequently in social networks? This question was already looming in the 1940s, when Heider [1] examined triads of individuals in networks, and postulated that only balanced triads are stable. A triad is balanced when friends agree in their opinion of a third party, while foes disagree, see Fig. 1. The individuals in an unbalanced triad have an incentive to adjust their opinions so as to reduce the stress experienced in such a situation [2]. Once an adjustment is made, the triad becomes balanced, and the stress disappears.

A decade later, Harary [3] showed that a complete social network splits in at most two factions if and only if all its triads are balanced, see also [4]. Such networks are called (socially) balanced as well. Since then, the focus of much of the research has been on detecting such factions in signed networks [5, 6]. Many signed networks show evidence of social balance, although the split into factions might not be exact, that is, they are only nearly socially balanced [7, 8, 9, 10].

What has been lacking until fairly recently, are dynamical models that explain *how* social balance emerges. The purpose of this paper is to analyze two such models. One of these models, proposed first in [11], was proved to exhibit social balance in [12]. However, this was done under a restrictive symmetry assumption for the reputation matrix. Here, we continue the analysis of this model and show that it generically does not lead to social balance when the symmetry assumption is dropped. In contrast, we propose a second model that is based on a different underlying gossiping mechanism, and show that it generically does lead to social balance, even when reputations are not symmetric.

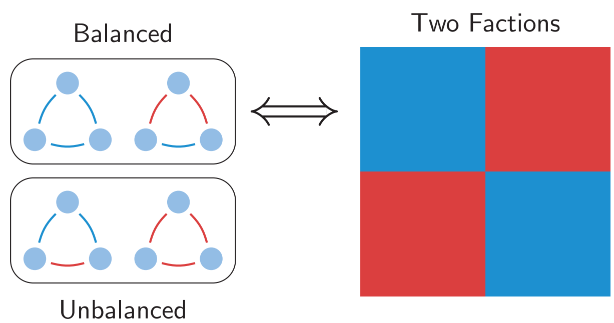

FIG. 1 **Social Balance.** The two upper triads are balanced, while the two lower triads are unbalanced. According to the structure theorem [3], a complete graph can be split into (at most) two opposing factions, if and only if all triads are balanced. This is represented by the colored matrix on the right, where blue indicates positive entries, and red negative entries.

Moreover, there is a natural connection between negative links and the evolution of cooperation: we consider positive links as indicating cooperation and negative links as defection. We will show that our alternative model is able to lead to cooperation, whereas the earlier model cannot.

II. EARLIER MODEL

Certain discrete-time, stochastic dynamics have been investigated [13, 14], but they exhibit so-called jammed states [15]: no change in the sign of a reputation improves the degree of social balance, as measured by the total number of balanced triads in the network. A surprisingly simple continuous-time model [11] was proved to converge to social balance for certain symmetric initial conditions [12]. The authors assume that the social network is described by a complete graph (everybody is connected to everybody), with weighted links representing reputations that change continuously in time. Let X denote the real-valued matrix of the reputations, so that X_{ij} represents the opinion i has about j. It is positive whenever i considers j a friend, and negative if i thinks of j as an enemy. The network is balanced, if, up to a possible relabeling of the individuals, the sign structure of X takes one of two possible block forms:

$$(+) \text{ or } \begin{pmatrix} + & - \\ - & + \end{pmatrix}. \tag{1}$$

Changes in the reputations are modeled as follows:

$$\dot{X} = X^2, \text{ or } \dot{X}_{ij} = \sum_k X_{ik} X_{kj}, \tag{2}$$

where $\dot{X}$ denotes the derivative with respect to time of the matrix X. The idea behind this model is that reputations are adjusted based on the outcome of a particular gossiping process. More specifically, suppose that Bob (individual i) wants to revise his opinion about John (individual j). Bob then asks everybody else in the network what they think of John. If one such opinion X_{kj} has the same sign as the opinion Bob has about his gossiping partner, i.e. as X_{ik}, then Bob will increase his opinion about John. But if these opinions differ in sign, then Bob will decrease his opinion about John.

The analysis for symmetric initial conditions $X(0) = X^T(0)$ was carried out in [12]: First, $X(0)$ is diagonalized by an orthogonal transformation $X(0) = U\Lambda(0)U^T$, where the columns of U are orthonormal eigenvectors $u_1, \ldots, u_n$ of $X(0)$ so that $UU^T = I_n$, and $\Lambda(0)$ is a diagonal matrix whose diagonal entries are the corresponding real eigenvalues $\lambda_1(0) \geq \lambda_2(0) \geq \cdots \geq \lambda_n(0)$ of $X(0)$. Direct substitution of the matrix function $U\Lambda(t)U^T$ shows that it is the solution of Eq. 2 with initial condition $X(0)$. Here, $\Lambda(t)$ is a diagonal matrix, solving the uncoupled matrix equation $\dot{\Lambda} = \Lambda^2$ with initial condition $\Lambda(0)$. The diagonal entries of $\Lambda(t)$ are obtained by integrating the scalar first order equations $\dot{\lambda}_i = \lambda_i^2$:

$$\lambda_i(t) = \frac{\lambda_i(0)}{1 - \lambda_i(0)t}, \quad t \in \begin{cases} [0, +\infty) \text{ if } \lambda_i(0) \leq 0 \\ [0, 1/\lambda_i(0)) \text{ if } \lambda_i(0) > 0 \end{cases} \tag{3}$$

Hence, the solution $X(t)$ blows up in finite time if and only if $\lambda_1(0) > 0$. Moreover, if $\lambda_1(0) > 0$ is a simple eigenvalue, then the solution $X(t)$, normalized by its Frobenius norm, satisfies:

$$\lim_{t \to 1/\lambda_1(0)} \frac{X(t)}{|X(t)|_F} = u_1 u_1^T. \tag{4}$$

Assuming that u_1 has no zero entries, and up to a suitable permutation of its components, the latter limit takes one of the forms in Eq. 1. In other words, if the initial reputation matrix is symmetric and has a simple, positive eigenvalue, then the normalized reputation matrix becomes balanced in finite time.

Our first main result is that this conclusion remains valid for normal initial conditions, i.e. for initial conditions that satisfy the equality $X(0)X^T(0) = X^T(0)X(0)$, see SI Theorem 2. Whereas the real eigenvalues behave similar to the symmetric case, the complex eigenvalues show circular behavior, which results in small "bumps" in the dynamics as shown in Fig. 2 (see Fig. S3 for more detail). More precisely, if $X(0)$ is normal and if $\lambda_1(0)$ is a real, positive and simple eigenvalue which is larger than every other real eigenvalue (if any), then the solution $X(t)$ of Eq. 2 satisfies Eq. 4. Hence, once again, the normalized reputation matrix converges to a balanced state.

Our second main result is that this conclusion does not carry over to the case where $X(0)$ is not normal, see SI Theorem 3. This general case is analyzed by first transforming $X(0)$ into its real Jordan-canonical form $J(0)$: $X(0) = TJ(0)T^{-1}$, where T consists of a basis of (the real and imaginary parts of) generalized eigenvectors of $X(0)$. It can then be shown that the solution $X(t)$ of Eq. 2 is given by $TJ(t)T^{-1}$, where $J(t)$ solves the matrix equation $\dot{J} = J^2$, an equation which can still be solved explicitly. Hence, $X(t)$ can still be determined. It turns out that if $X(0)$ has a real, positive and simple eigenvalue $\lambda_1(0)$ which is larger than every other real eigenvalue (if any), then the normalized reputation matrix satisfies:

$$\lim_{t \to 1/\lambda_1(0)} \frac{X(t)}{|X(t)|_F} = \frac{u_1 v_1^T}{|u_1 v_1^T|_F}, \tag{5}$$

where u_1 and v_1^T are left and right eigenvectors of $X(0)$ respectively, that correspond to the eigenvalue $\lambda_1(0)$. If we assume that none of the entries of u_1 and v_1 are zero, then we can always find a suitable permutation of the components of u_1 and v_1 such that they have the following sign structure:

$$u_1 = \begin{pmatrix} + \\ + \\ \hline - \\ - \end{pmatrix} \text{ and } v_1^T = \begin{pmatrix} + & - & \vert & + & - \end{pmatrix}$$

Consequently, in general, the matrix limit in Eq. 5 has the sign structure:

$$\left(\begin{array}{cc|cc} + & - & + & - \\ - & + & - & + \end{array}\right),$$

as illustrated in Fig. 2. Clearly, this configuration doesn't correspond to social balance any longer.

III. ALTERNATIVE MODEL

Let us briefly reconsider the gossiping process underlying model $\dot{X} = X^2$. In our example of Bob and John,

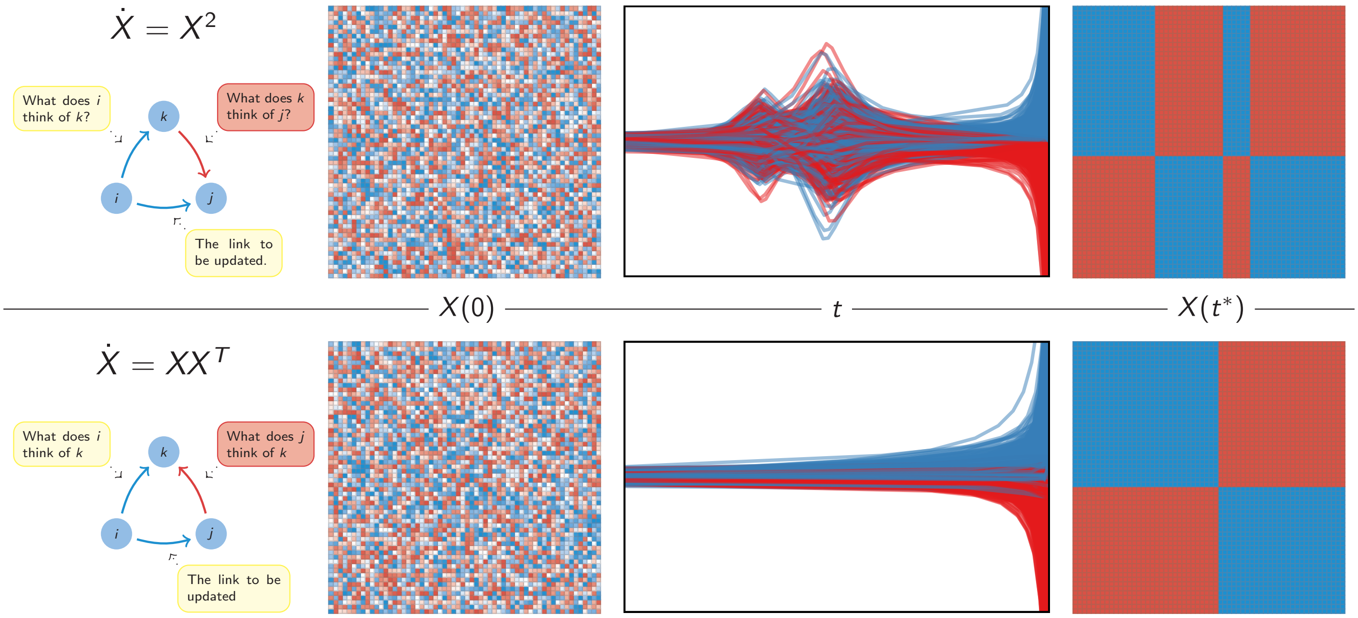

FIG. 2 **The two models compared.** The first row illustrates what happens generically for the model $\dot{X} = X^2$, while the second row displays the results for $\dot{X} = XX^T$. Each row contains from left to right: (1) an illustration of the model; (2) the random initial state; (3) the dynamics of the model; and (4) the final state to which the dynamics converge. Blue indicates positive entries, and red negative entries. Although the first model converges to a rank one matrix, it is not socially balanced. The second model does converge generically to social balance. The small bumps in the dynamics for $\dot{X} = X^2$ are due to complex eigenvalues that show circular behavior (see Fig. S3).

the following happens. Bob asks others what they think of John. Bob takes into account what he thinks of the people he talks to, and adjusts his opinion of John accordingly. An alternative approach is to consider a type of homophily process [16, 17, 18]: people tend to befriend people who think alike. When Bob seeks to revise his opinion of John, he talks to John about everybody else (instead of talking to everybody else about John). For example, suppose that Bob likes Alice, but that John dislikes her. When Bob and John talk about Alice, they notice they have opposing views about her, and as a result the relationship between Bob and John deteriorates. On the other hand, should they share similar opinions about Alice, their relationship will improve. Thus, our alternative model for the update law of the reputations is:

$$\dot{X} = XX^T, \text{ or } \dot{X}_{ij} = \sum_k X_{ik} X_{jk}. \tag{6}$$

Although there apparently is only a subtle difference in the gossiping processes underlying the models in Eq. 2 and 6, these models turn out to behave quite differently, as we discuss next.

Our third main result is that for generic initial conditions, the normalized solution of system Eq. 6 converges to a socially balanced state in finite time. To show this, we decompose the solution $X(t)$ into its symmetric and skew-symmetric parts: $X(t) = S(t) + A(t)$, where $S(t) = S^T(t)$ and $A(t) = -A^T(t)$. Since $\dot{X} = \dot{X}^T$, the skew-symmetric part remains constant, and therefore $A(t) = A(0) \equiv A_0$. The symmetric part then obeys the matrix Riccati differential equation $\dot{S} = (S+A_0)(S-A_0)$. We introduce $Z(t) = e^{-A_0 t} S(t) e^{A_0 t}$ to eliminate the linear terms in this equation, and obtain

$$\dot{Z} = Z^2 + A_0 A_0^T. \tag{7}$$

The latter matrix Riccati differential equation can be integrated, yielding the solution $Z(t)$ explicitly, and hence $S(t)$, as well as $X(t)$, can be calculated.

It turns out that if $A_0 \neq 0$, then $X(t)$ always blows up in finite time. Moreover, using a perturbation argument, it can be shown there is a dense set of initial conditions $X(0)$ such that the normalized solution of Eq. 6 converges to

$$\lim_{t \to t^*} \frac{X(t)}{|X(t)|_F} = ww^T, \tag{8}$$

for some vector w, as t approaches the blow-up time t^*, see SI Theorem 5. If w has no zero entries, this implies that the normalized solution becomes balanced in finite time. Hence, the alternative model in Eq. 6 generically evolves to social balance, see Fig. 2.

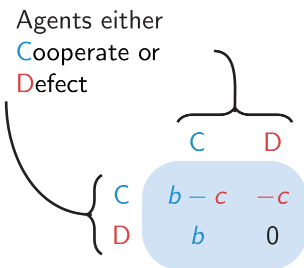

FIG. 3 **Prisoner's Dilemma.** Both players have the option to either Cooperate or Defect. Whenever an agent cooperates, it costs him c while his partners receives a benefit $b > c$, leading to the indicated payoffs.

IV. EVOLUTION OF COOPERATION

Positive and negative links have a natural interpretation in the light of cooperation: positive links indicate cooperation and negative links indicate defection. There is then also a natural motivation for the alternative model in terms of cooperation. Again, suppose Bob wants to revise his opinion of John. For Bob it is important to know whether John is cooperative in order to determine whether he should cooperate with John or not. So, instead of asking Alice whether she has cooperated with John, Bob would like to know whether John has cooperated with her. In other words, Bob is not interested in X_{kj} but in X_{jk}, consistent with Eq. 6, illustrated in Fig. 2. This is also what is observed in studies on gossip: it often concerns what others did, not what one thinks of others [19, 20]

Indeed gossiping seems crucial in explaining the evolution of human cooperation through indirect reciprocity [21]. It has even been suggested that humans developed larger brains in order to gossip, so as to control the problem of cooperation through social interaction [22]. In general, the problem is that if defection allows individuals to gain more, why then do individuals cooperate? This is usually modeled in the form of a prisoner's dilemma, in which each agent has the possibility to give his partner some benefit b at some cost $c < b$. So, if an agent's partner cooperates (he gives the agent b) but the agent doesn't cooperate (he doesn't pay the cost c) his total payoff will be b. Considering the other possibilities results in the payoff matrix detailed in Fig. 3.

Irrespective of the choice of the other player, it is better to defect in a single game. Suppose that the second player cooperates. Then if the first player cooperates he gains $b - c$, while if he defects he gains b, so defecting is preferable. Now suppose that the second player defects. The first player then has to pay c, but doesn't have to pay anything when defecting. So indeed, in a single game, it is always better to defect, yet the payoff is higher if both cooperate, whence the dilemma.

In reality, we do observe cooperation, and various mechanisms for explaining the evolution of cooperation have been suggested [23], such as kin selection [24, 25], reciprocity [26] or group selection [27]. Humans have a tendency however to also cooperate in contexts beyond kin, group or repeated interactions. It is believed that some form of indirect reciprocity can explain the breadth of human cooperation [21]. Whereas in direct reciprocity the favor is returned by the interaction partner, in indirect reciprocity the favor is returned by somebody else, which usually involves some reputation. It has been theorized that such a mechanism could even form the basis of morality [28]. Additionally, reputation (and the fear of losing reputation) seems to play an important role in maintaining social norms [29, 30, 31].

In general, the idea is the following: agents obtain some good reputation by helping others, and others help those with a good reputation. Initially a strategy known as image scoring was introduced [32]. Shortly after, it was argued that a different strategy, known as the standing strategy, should actually perform better [33], although experiments showed people tend to prefer the simpler image scoring strategy [34]. This led to more systematic studies of how different reputation schemes would perform [35, 36, 37]. Although much research has been done on indirect reciprocity, only few theoretical works actually study how gossiping shapes reputations [38, 39]. Nonetheless, most studies (tacitly) assume that reputations are shaped through gossip. Additionally, it was observed experimentally that gossiping is an effective mechanism for promoting cooperation [40, 41, 42].

Moreover, these reputations are usually considered as objective. That is, all agents know the reputation X_j of some agent j, and all agents have the same view of agent j. Private reputations—so that we have X_{ij}, the reputation of j in the eyes of i—have usually been considered by allowing a part of the population to "observe" an interaction, and update the reputation accordingly. If too few agents are allowed to "observe" an interaction, the reputations X_{ij} tend to become uncorrelated and incoherent. This makes reputation unreliable for deciding whether to cooperate or defect. The central question thus becomes how to model private reputations such that they remain coherent and reliable for deciding whether to cooperate or not.

Dynamical models of social balance might provide an answer to this question. Although it allows to have private reputations—that is X_{ij}—the dynamics could also lead to some coherence in the form of social balance. In addition, it models more explicitly the gossiping process, commonly suggested to be the foundation upon which reputations are forged.

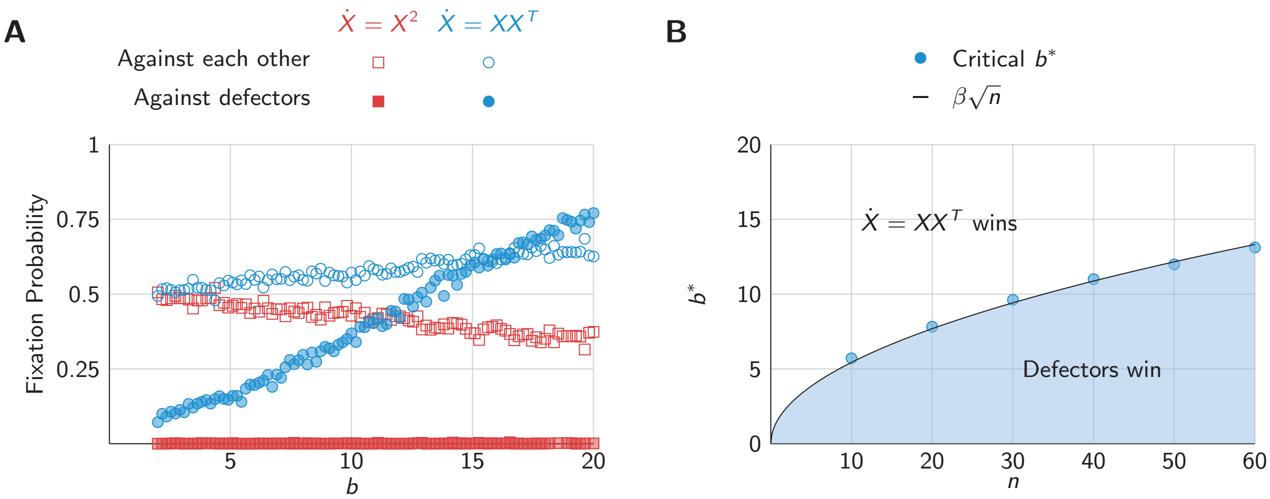

FIG. 4 **Evolution of Cooperation.** (A) The fixation probability (probability to be the sole surviving species) is higher for model $\dot{X} = XX^T$ than $\dot{X} = X^2$. This implies that the model $\dot{X} = XX^T$ is more viable against defectors, and has an evolutionary advantage compared to $\dot{X} = X^2$. (B) The point b^* at which the model $\dot{X} = XX^T$ has an evolutionary advantage against defectors (i.e. the fixation probability $\rho > 1/2$) depends on the number of agents n. The condition for the model $\dot{X} = XX^T$ to defeat defectors can be approximated by $b > b^* = \beta\sqrt{n}$, with $\beta \approx 1.72$.

A. Simulation Results

The reputations of the agents are determined by the dynamics of the two models. We call agents using $\dot{X} = X^2$ dynamics type A, and those using $\dot{X} = XX^T$ dynamics type B. We assume that agent i cooperates with j whenever $X_{ij} > 0$ and defects otherwise. Agents reproduce proportional to their fitness, determined by their payoff. Agents that do well (have a high payoff) have a higher chance of reproduction, and we are interested in knowing the probability that a certain type becomes fixated in the population (i.e. takes over the whole population), known as the fixation probability ρ. All simulations start off with an equal amount of agents, so if some type wins more often than his initial relative frequency, it indicates it has an evolutionary advantage. For the results presented here this comes down to $\rho > 1/2$. More details on the simulations are provided in the Materials and Methods section at the end of the paper.

The results are displayed in Fig. 4 using a normalized cost of $c = 1$ (the ratio b/c drives the evolutionary dynamics, see Materials and Methods and [23]). When directly competing against each other, type B has an evolutionary advantage (its fixation probability $\rho_B > 1/2$) compared to type A, already for relatively small benefits. When each type is playing against defectors (agents that always defect), type A seems unable to defeat defectors ($\rho_A < 1/2$) for any $b < 20$, while type B performs quite well against them. When all three types are playing against each other results are similar (see Fig. S1). When varying the number of agents, the critical benefit b^* at which type B starts to have an evolutionary advantage changes (i.e. where the fixation probability $\rho_B = 1/2$). For $b > b^*$ agents using the model $\dot{X} = XX^T$ have a higher chance to become fixated, while for $b < b^*$ defectors tend to win. The inequality for type B to have an evolutionary advantage can be relatively accurately approximated by $b > b^* = \gamma\sqrt{n}$ where γ is estimated to be around $\gamma \approx 1.72 \pm 0.037$ (95% confidence interval). Varying the intensity of selection does not alter the results qualitatively (see Fig. S2). Summarizing, type B is able to lead to cooperation and defeats type A. Based on these results, if a gossiping process evolved during the course of human history in order to maintain cooperation, the model $\dot{X} = XX^T$ seems more likely to have evolved than $\dot{X} = X^2$. For smaller groups a smaller benefit is needed for the model $\dot{X} = XX^T$ to become fixated. This dependence seems to scale only as $\sqrt{n}$, so that larger groups only need a marginally larger benefit in order to develop cooperation.

V. CONCLUSION

To conclude, we have shown that the alternative model $\dot{X} = XX^T$ generically converges to social balance, whereas the model $\dot{X} = X^2$ did not. The current models exhibit several unrealistic features, we would like to address: (1) an all-to-all topology; (2) dynamics that blow-up in finite time; and (3) homogeneity of all agents. Although most of these issues can be addressed by specifying different dynamics, the resulting models are much more difficult to analyze, thereby limiting our understanding. Although the two models are somewhat simple, they are also tractable, and what we lose in truthfulness, we gain in deeper insights: in simplicity lies progress. Our current analysis offers a quite complete understanding, and we hope it provides a stepping stone to more realistic models, which we would like to analyze in the future.

The difference between the two models can be understood in terms of gossiping: we assume that people who wish to revise their opinion about someone talk to that person about everybody else, while the earlier model assumed that people talk about that person to everybody else. Both gossiping and social balance are at the center of many social phenomena [22, 29, 43, 44], such as norm maintenance [30], stereotype formation [45] and social conflict [46]. For example, a classic work [29] on the established and outsiders found that gossiping was the fundamental driving force for the maintenance of the cohesive network of the established at the exclusion of the outsiders. Understanding how social balance may emerge might help to understand the intricacies of these social phenomena.

Moreover, in light of the evolution of cooperation it appears that agents using $\dot{X} = XX^T$ dynamics perform well against defectors, and have an evolutionary advantage compared to agents using $\dot{X} = X^2$ dynamics. Contrary to other models of indirect reciprocity, not everybody might end up cooperating with everybody, and the population may split into two groups. This provides an interesting connection between social balance theory, gossiping and the evolution of cooperation. Our results improve our understanding of gossiping as a mechanism for group formation and cooperation, and as such contributes to the study of indirect reciprocity.

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MATERIALS AND METHODS

In the simulations of the evolution of cooperation, the dynamics consist of two parts: (1) the interaction dynamics within each generation; and (2) the dynamics prescribing how the population evolves from generation to generation.

A. Interaction Dynamics

We include three possible types of agents in our simulations:

Type A: uses $\dot{X} = X^2$ dynamics,

Type B: uses $\dot{X} = XX^T$ dynamics, and

Defectors: have trivial reputation dynamics $\dot{X} = 0$, with negative constant reputations.

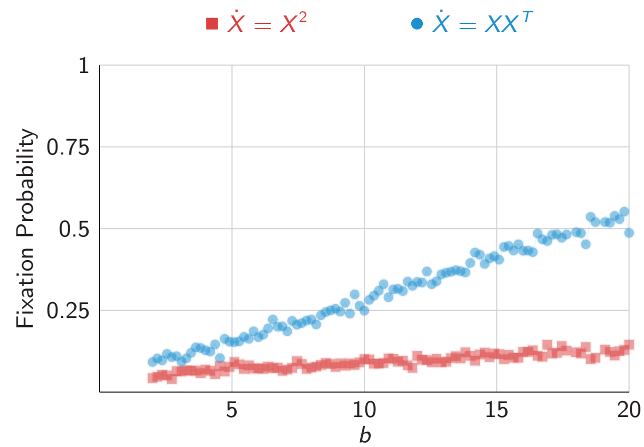


FIG. S1 Results including type A, B and defectors.

We can decompose the reputation matrix $X(t)$ accordingly into three parts:

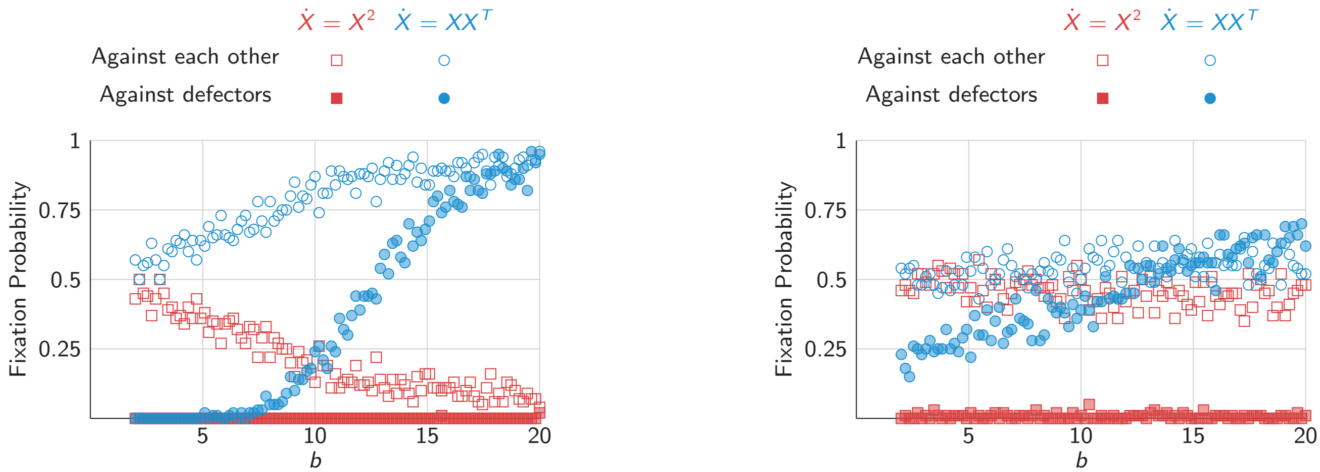
$$X(t) = \begin{pmatrix} X_A(t) \\ X_B(t) \\ X_D(t) \end{pmatrix},$$

where $X_A(t)$ are the reputations of all agents in the eyes of agents of type A, $X_B(t)$ for type B and $X_D(t)$ for defectors. The reputations $X_A(0)$ and $X_B(0)$ are initialized from a standard Gaussian distribution. The initial reputation for $X_D(0)$ will be set to a fixed negative value. To be clear, $X_D(0)$ is the reputation of all other agents in the eyes of defectors, which is negative initially. The initial reputation of the defectors themselves is of course not necessarily negative initially. For the results displayed here we have used $X_D(0) = -10$, but results remain by and large the same when varying this parameter, as long as it remains sufficiently negative.

Since we are dealing with continuous dynamics in this paper, we assume all agents are involved in infinitesimally short games at each time instance t. Each agent i may choose to either cooperate or defect with another agent j, and this decision may vary from one agent to the next. For agents of type A and type B the decision to cooperate is based on the reputation: they defect whenever $X_{ij}(t) \leq 0$ and cooperate whenever $X_{ij}(t) > 0$. We define the cooperation matrix $C(t)$ accordingly

$$C_{ij}(t) = \begin{cases} 0 & \text{if } X_{ij} \leq 0 \\ 1 & \text{if } X_{ij} > 0 \end{cases}$$

Defectors will simply always defect. Whenever an agent i cooperates with j the latter receives a payoff of b at a cost of c to agent i. We integrate this payoff over all infinitesimally short games from time 0 to time t^*, which



A $\beta = 0.5$ **B** $\beta = 5$

FIG. S2 Results different intensities of selection.

can be represented as

$$P(g) = \frac{1}{n} \int_0^{t^*} bC(t)^T e - cC(t)edt,$$

where $e = (1, \ldots, 1)$ the vector of all ones for a certain generation g.

B. Evolutionary Dynamics

We have simulated the evolution of cooperation for $n = 10, 20, \ldots, 60$ agents, which stays constant throughout evolution. We consider four different schemes for initializing the first generation:

	$p_A(0)$	$p_B(0)$	$p_D(0)$
1) Type A vs Type B	1/2	1/2	-
2) Type A vs Defectors	1/2	-	1/2
3) Type B vs Defectors	-	1/2	1/2
4) Type A,B and Defectors	1/3	1/3	1/3

Here $p_A(0)$,$p_B(0)$ and $p_D(0)$ are respectively the proportion of agents of type A, type B and defectors in the first generation. We use the vector $T_i(g) \in \{A, B, D\}$ to denote the type of agent i in generation g, so that $T_i(g) = A$ if agent i is a type A player, $T_i(g) = B$ for a type B player, and $T_i(g) = D$ for a defector. We are interested in estimating the probability that a single type takes over the whole population, known as the fixation probability ρ_A, ρ_B and ρ_D for the three different types. If a type has no evolutionary advantage, it is said to be evolutionary neutral, and in that case its fixation probability is equal to its initial frequency, e.g. for type A $\rho_A = p_A(0)$.

We will keep the population constant at the initial n, and simply choose n new agents according to their payoff for the next generation. This can be thought of as choosing n times one of the n agents in the old generation for reproduction. Let ϕ_i denote the probability that an agent is selected for reproduction, which we define as

$$\phi_i = \frac{e^{\beta P_i(g)}}{\sum_i e^{\beta P_i(g)}}.$$

Since we are only interested in the number of agents of a certain type, we can also gather all payoffs for the same type of agents, and write

$$\Phi_q = \sum_{i: T_i(g) = q} \phi_i,$$

where $q \in \{A, B, D\}$ represents the type of agent. The probability to select a type A agent, a type B agent or a defector is then respectively Φ_A, Φ_B and Φ_D. In the next generation, the probability that agent i is of a specific type q can then be written as

$$\Pr(T_i(g + 1) = q) = \Phi_q.$$

This evolutionary mechanism can be seen as a Wright-Fisher process [47] with fitnesses $e^{\beta P_i(g)}$. It is well known that this process converges faster than a Moran birth-death process, since it essentially takes n time steps in a Moran process to reproduce the effect of one time step in a Wright-Fisher process [47]. Because of the high computational costs (solving repeatedly a non-linear system of differential equations of size n^2), this process is preferable.

Higher β signifies higher selective pressure, and leads to a higher reproduction of those with a high payoff, and in the case that $\beta \to \infty$ only those with the maximum payoff reproduce. On the other hand, for $\beta \to 0$ this tends to the uniform distribution $\phi_i = 1/n$, where payoffs no longer play any role. We have used $\beta = 0.5$ for the low selective pressure, $\beta = 5$ for the high selective pressure, reported in the SI. For the results in the main text we have used $\beta = 1$.

For an evolutionary neutral selection in where all $P_i(g) = P$ are effectively the same, β has no effect, and $\phi_i = 1/n$. Notice that if we rescale $P_i(g)$ by $1/c$ so that the payoff effectively becomes

$$\frac{1}{c}P_i(g) = \frac{1}{n}\int_0^{t^*} \frac{b}{c}C(t)^T e - C(t)edt,$$

and we rescale β by c, then the reproduction probabilities remain unchanged. Hence, only the ratio b/c effectively plays a role up to a rescaling of the intensity of selection. Since the point at which the evolution is neutral (i.e. ρ equals the initial proportional frequency), is independent of β, this point will only depend on the ratio b/c. So, we normalized the cost $c = 1$. To verify this, we also ran additional simulations with different costs, which indeed gave the same results.

We stop the simulation whenever one of the types becomes fixated in the population. With *fixation* we mean that all other types have gone extinct, and only a single type remains. If no type has become fixated after 1,000 generations, we terminate the simulation and count as winner the most frequent type. This almost never happens, and the simulation usually stops after a relatively small number of generations.

In total, we repeat this process 1,000 times for the results in the main text, and for the low ($\beta = 0.5$) and high ($\beta = 5$) selective pressure 100 times. This means that we run the evolutionary dynamics until one of the types has become fixated, and we record which type has "won". After that, we again start from the first generation, and run until fixation, and repeat this. Finally, we calculate how many rounds a type has "won" compared to the total number of rounds, which yields the fixation probability ρ.

Supplementary Information

I. PRELIMINARIES

We investigate matrix differential equations of the form $\dot{X} = F(X, X^T)$, where X is a real $n \times n$ matrix, and F is a one of two specific, smooth functions. These functions are such that it turns out to be advantageous to consider the dynamics of the symmetric and skew-symmetric parts of X. Recall that $\mathbb{R}^{n\times n} = \mathcal{S} \oplus \mathcal{A}$, where $\mathcal{S}$ is the linear subspace of real symmetric matrices, and $\mathcal{A}$ the linear subspace of skew-symmetric matrices. Thus, given any $X \in \mathbb{R}^{n\times n}$, we can find unique symmetric $S \in \mathcal{S}$ and skew-symmetric $A \in \mathcal{A}$ such that $X = S + A$. More explicitly, $S = (X + X^T)/2$ and $A = (X - X^T)/2$. Moreover, using the inner product $\langle X, Y\rangle = \mathrm{tr}(XY^T)$, there holds that

$$\mathcal{A}^\perp = \mathcal{S}. \tag{S1}$$

The norm induced by this inner product is the Frobenius norm $|X|_F = (\mathrm{tr}(XX^T))^{\frac{1}{2}}$. Recall that the Frobenius norm is unitarily invariant, i.e. if U is orthogonal (i.e. $UU^T = I_n$), then

$$|UXU^T|_F = |X|_F. \tag{S2}$$

We denote by I_n the $n \times n$ identity matrix, and by J_n a specific skew symmetric matrix:

$$J_n = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}, n \text{ even}. \tag{S3}$$

For all other linear algebra related terminology and properties we refer to [48].

We briefly review two key ingredients of Heider's (static) theory on social balance, namely those of a *balanced triangle* and a *balanced network*:

Definition 1. *A triangle of (not necessarily distinct) agents i, j and k is called balanced if*

$$X_{ij}X_{ik}X_{kj} > 0. \tag{S4}$$

A network is said to be balanced if all triangles of agents in the network are balanced.

It turns out that a balanced network takes on a specific structure, in that at most 2 factions emerge, where members within each faction have positive opinions about each other, but members in different factions have negative opinions about each other. This result is known as the Structure Theorem [3, 49]:

Theorem 1 (Structure Theorem in [3, 49]). *Let X represent a balanced network. Then up to a permutation of agents, the matrix X has the following sign structure:*

$$(+) \;\; or \;\; \begin{pmatrix} + & - \\ - & + \end{pmatrix}.$$

Conversely, if, up to permutation, X has one of these structures, then it represents a balanced network.

Notice that the same theorem holds irrespective of any permutation of i,j and k in definition 1.

II. EQUATION $\dot{X} = X^2$

Consider the model studied numerically in [11] and analysed for symmetric initial conditions in [12]:

$$\dot{X} = X^2, X(0) = X_0, \tag{S5}$$

where each X_{ij} is real-valued and denotes the opinion agent i has about agent j. Positive values mean that agent i thinks favourably about j, whereas negative values mean that i thinks unfavourably about j. More explicitly, model S5 can also be written entrywise:

$$\dot{X}_{ij} = \sum_k X_{ik}X_{kj}. \tag{S6}$$

The basic question in this context is whether or not the solutions of S5 evolve towards a state which corresponds to a balanced network. A minor technical issue is that the solution $X(t)$ of S5 often blows up in finite time $\bar{t}$ as we shall see later. To resolve this problem we investigate the sign pattern of the matrix limit $\lim_{t\to\bar{t}} X(t)/|X(t)|_F$ instead, and say that the network evolves to a balanced state, if this matrix limit is balanced.

Normal initial condition

We start by defining

$$\mathcal{N} = \{X \in \mathbb{R}^{n\times n} | XX^T = X^TX\},$$

the set of real, normal matrices. Notice that if X belongs to $\mathcal{N}$ then so does X^2, hence the set $\mathcal{N}$ is invariant for $\dot{X} = X^2$.

Recall that normal matrices are (block)-diagonalizable with blocks of size at most 2 by an orthogonal transformation: if $X_0 \in \mathcal{N}$, then

$$U^TX_0U = \Lambda_0, \tag{S7}$$

where Λ_0 consists of real 1×1 scalar blocks A_i and real 2×2 blocks $B_j = \alpha_j I_2 + \beta_j J_2$ with $\beta_j \neq 0$.

Note that if $\Lambda(t)$ is the solution to the initial value problem $\dot{\Lambda} = \Lambda^2$, $\Lambda(0) = \Lambda_0$, then $X(t) := U\Lambda(t)U^T$ is the solution to Eq. S5. This shows it is sufficient to solve system S5 in case of scalar X or in case of a specific, 2×2, normal matrix X. The scalar case is easy to solve: the solution of $\dot{x} = x^2$, $x(0) = x_0$, is

$$x(t) = \frac{x_0}{1 - x_0 t}, \tag{S8}$$

which is easily verified, so we turn to the 2×2 case by considering:

$$\dot{X} = X^2,\ X(0) = \alpha I_2 + \beta J_2,\ \text{where } \beta > 0. \tag{S9}$$

Lemma 1. *The forward solution $X(t)$ of S9 is defined for all $t \in [0, +\infty)$, and*

$$\lim_{t\to+\infty} X(t) = 0 \text{ and } \lim_{t\to+\infty} \frac{X(t)}{|X(t)|_F} = -\frac{\sqrt{2}}{2} I_2.$$

Proof. Let $X_0 = S_0 + A_0$, $S_0 = \alpha I_2$ and $A_0 = \beta J_2$ where J_2 is as defined in Eq. S3. Then the solution $X(t)$ of S9 can be decomposed as $S(t) + A(t)$, where

$$\dot{S} = S^2 + A^2,\quad S(0) = S_0, \tag{S10}$$

$$\dot{A} = AS + SA,\quad A(0) = A_0. \tag{S11}$$

Note that S10 is a matrix Riccati differential equation with the property that the set $\mathcal{L} := \{sI_2 + aJ_2 | s,\ a \in \mathbb{R}\}$, is an invariant set under the flow. Therefore it suffices to solve the scalar Riccati differential equation corresponding to the dynamics of the scalar coefficients s and a:

$$\dot{s} = s^2 - a^2,\ s(0) = \alpha, \tag{S12}$$

$$\dot{a} = 2as,\ a(0) = \beta, \tag{S13}$$

whose solution is given implicitly by:

$$s^2 + \left(a - \frac{1}{2c}\right)^2 = \left(\frac{1}{2c}\right)^2 \text{ if } c \neq 0,$$

where c is an integration constant. So, the orbits form circles which are centered at $(0, 1/2c)$ and pass through $(0,0)$, and by $a = 0$ if $c = 0$. The phase portrait of system S12-S13 is illustrated in Fig. S3.

All solutions $(s(t), a(t))$ of system S12-S13, not starting on the s-axis, converge to zero as $t \to +\infty$, and approach the origin in the second quadrant for solutions in the upper-half-plane, and in the third quadrant for solutions in the lower-half-plane. Moreover, since the s-axis is the tangent line to every circular orbit at the origin, the slopes $a(t)/s(t)$ converge to 0 along every solution $\lim_{t\to+\infty} a(t)/s(t) = 0$. Consequently, the forward solution $X(t)$ of S9 satisfies:

$$\lim_{t\to+\infty} X(t) = \lim_{t\to+\infty} s(t)I_2 + a(t)J_2 = 0,$$

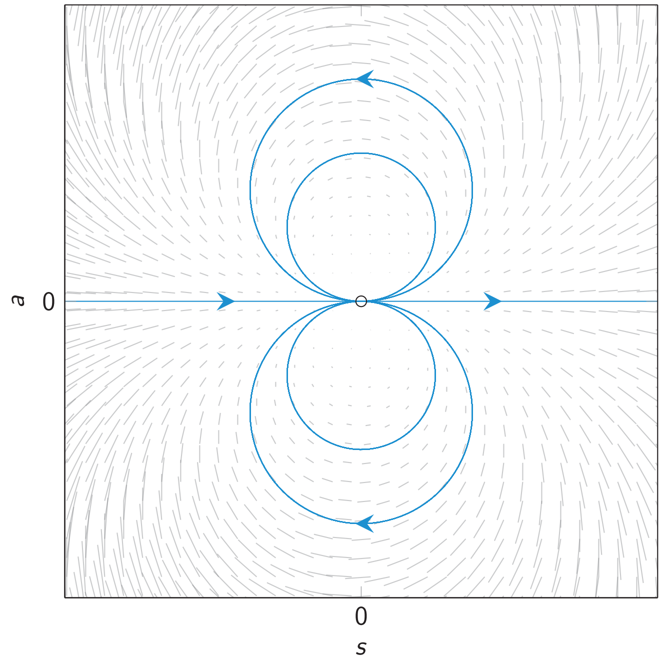


FIG. S3 Phase portrait of system S12-S13. Circular orbits in the upper half plane ($a > 0$) are traversed counter clockwise, whereas circular orbits in the lower half plane ($a < 0$) are traversed clockwise.

and

$$\lim_{t\to+\infty} \frac{X(t)}{|X(t)|_F} = -\frac{\sqrt{2}}{2} I_2.$$

□

Combining the solution for the scalar and 2×2 case yields our main result in the normal case:

Theorem 2. *Let $X_0 \in \mathcal{N}$, and let (U, Λ_0) be as in Eq. S7. Define*

$$\bar{t}_i = \begin{cases} 1/a_i & \text{if } a_i > 0 \\ +\infty & \text{if } a_i \leq 0 \end{cases} \qquad \text{for all } i = 1, \ldots, k,$$

and let $\bar{t} = \min_i \bar{t}_i$. Then the forward solution $X(t)$ of S5 is defined for $[0, \bar{t})$.

If there is a unique $i^* \in \{1, \ldots, k\}$ such that $\bar{t} = \bar{t}_{i^*}$ is finite, then

$$\lim_{t\to\bar{t}_{i^*}-} \frac{X(t)}{|X(t)|_F} = U_{i^*} U_{i^*}^T,$$

where U_{i^*} is the i^*th column of U, an eigenvector corresponding to eigenvalue a_{i^*} of X_0.

Proof. Consider the initial value problem:

$$\dot{\Lambda} = \Lambda^2,\ \Lambda(0) = \Lambda_0.$$

Its solution is given by

$$\Lambda(t)=\begin{pmatrix} \frac{a_1}{1-a_1t} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \frac{a_k}{1-a_kt} & 0 & \dots & 0 \\ 0 & \dots & 0 & X_1(t) & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & X_l(t) \end{pmatrix},$$

where for all $j=1,\dots,l$, $X_j(t)$ is the forward solution of S9, which is defined for all t in $[0,+\infty)$, and converges to 0 as $t\to+\infty$ by Lemma 1.

This clearly shows that $\Lambda(t)$ is defined in forward time for t in $[0,\bar{t})$. Since the solution of S5 is given by $X(t)=U\Lambda(t)U^T$, $X(t)$ is also defined in forward time for t in $[0,\bar{t})$. It follows from S2 that

$$\frac{X(t)}{|X(t)|_F}=U\frac{\Lambda(t)}{|\Lambda(t)|_F}U^T.$$

If $i^*\in\{1,\dots,k\}$ is the unique value such that $\bar{t}=\bar{t}_{i^*}$, then using S2:

$$\lim_{t\to\bar{t}_i^*}\frac{X(t)}{|X(t)|_F}=U\lim_{t\to\bar{t}_i^*}\frac{\Lambda(t)}{|\Lambda(t)|_F}U^T=Ue_{i^*}e_{i^*}^TU^T=U_{i^*}U_{i^*}^T,$$

where e_{i^*} denotes the i^*th standard unit basis vector of $\mathbb{R}^n$. □

Theorem 2 provides a sufficient condition guaranteeing that social balance in the sense of definition 1 is achieved. If X_0 has a simple, positive, real eigenvalue a_{i^*}, and if no entry of the eigenvector U_{i^*} is zero, then the network becomes balanced. Indeed, there holds that, up to a permutation of its entries, the sign pattern of the eigenvector U_{i^*} is either:

$$U_{i^*}=(+)\ \text{or}\ (-)\implies U_{i^*}U_{i^*}^T=(+),$$

or

$$U_{i^*}=\begin{pmatrix}+\\-\end{pmatrix}\implies U_{i^*}U_{i^*}^T=\begin{pmatrix}+&-\\-&+\end{pmatrix}.$$

In either case, Theorem 1 implies that the normalized state of the system becomes balanced in finite time.

Generic initial condition

Although Theorem 2 provides a sufficient condition for the emergence of social balance, it requires that the initial condition X_0 is normal. But the set $\mathcal{N}$ of normal matrices has measure zero in the set of all real $n\times n$ matrices, and thus the question arises if social balance will arise for non-normal initial conditions as well. We investigate this issue here, and will see that generically, social balance is not achieved.

If X_0 is a general real $n\times n$ matrix, we can put it in real Jordan canonical form by means of a similarity transformation:

$$X(0)=T\Lambda_0T^{-1},\ \ TT^{-1}=I_n, \tag{S14}$$

with $\Lambda_0=\operatorname{diag}(A_1,\dots,A_k,B_1,\dots,B_l)$, where A_i are real Jordan blocks and

$$B_j=\begin{pmatrix} C_i & I_2 & \dots & 0\\ 0 & C_i & \ddots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & C_i\end{pmatrix}, C_j=\alpha_jI_2+\beta_jJ_2, \tag{S15}$$

with $\beta_j\neq 0$.

We again observe that if $\Lambda(t)$ is the solution to the initial value problem $\dot{\Lambda}=\Lambda^2$, $\Lambda(0)=\Lambda_0$, then $X(t):=T\Lambda(t)T^{-1}$, is the solution to Eq. S5. Again, it is sufficient to solve system S5 in case of specific block-triangular X of the form A_i or B_j as in S15. To deal with the first form A_i, we first we consider more general, triangular Toeplitz initial conditions:

$$X(0)=\begin{pmatrix} x_1(0) & x_2(0) & \cdots & x_n(0)\\ 0 & x_1(0) & \ddots & x_{n-1}(0)\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & x_1(0)\end{pmatrix}, \tag{S16}$$

with $x_i(0)$ reals, and denote $\mathcal{TT}=\{X\ |\ X \text{ is of the form } S16\}$. It turns out that this is an invariant set for the system, which can be easily verified by noting that if X belongs to $\mathcal{TT}$, then so does X^2.

Lemma 2. *Let $X(0)\in\mathcal{TT}$ with*

$$x_i(0)=\begin{cases} a\neq 0 \ \textit{if}\ i=1\\ 1\ \textit{if}\ i=2\\ 0\ \textit{otherwise}\end{cases}.$$

Then the forward solution $X(t)$ of S5 is defined on $[0,t^*)$ where $t^*=1/a$ if $a>0$ and on $t^*=\infty$ if $a\leq 0$, belongs to $\mathcal{TT}$, and satisfies

$$x_i(t)=p_i\left(\frac{1}{1-at}\right),\ \ t\in[0,t^*),$$

where each $p_i(z)$ is a polynomial of degree i:

$$p_i(z)=\begin{cases} az\ \textit{if}\ i=1\\ \frac{1}{a^{i-2}}z^i+\dots+c_iz^2\ \textit{otherwise}\end{cases}, \tag{S17}$$

where c_i is some real constant, so that $p_i(z)$ has no constant or first order terms when $i>1$.

Proof. First note that system S5 can be solved recursively, starting with $x_1(t)$, followed by $x_2(t), x_3(t), \ldots$. Only the first equation for x_1 is nonlinear, whereas the equations for $x_2, x_3, \ldots$ are linear. To see this, we write these equations:

$$\dot{x}_i = \begin{cases} x_1^2, & \text{if } i = 1 \\ (2x_1(t))x_2, & \text{if } i = 2 \\ (2x_1(t))x_i + \sum_{k=2}^{i-1} x_k(t)x_{i-(k-1)}(t), & \text{if } i > 2 \end{cases},$$

with $x_1(0) = a$, $x_2(0) = 1$ and $x_i(0) = 0$ for $i > 2$. The forward solution for x_1 is $x_1(t) = \frac{a}{1-at}$, for $t \in [0, t^*)$, which establishes the result if $i = 1$. The forward solution for x_2 is: $x_2(t) = \frac{1}{(1-at)^2}$, for $t \in [0, t^*)$, which establishes the result if $i = 2$. If $i > 2$, we obtain the proof by induction on n. Assume the result holds for $i = 1, \ldots, n$, for some $n \geq 2$, and consider the equation for x_{n+1}. Using that $x_n(0) = 0$ for $n \geq 2$, the solution is given by:

$$x_{n+1}(t) = \mathrm{e}^{\int_0^t 2x_1(s)ds}\Bigg[0 + \int_0^t \left(\sum_{k=2}^{n} x_k(s)x_{n-k+2}(s)\right)\mathrm{e}^{\int_0^s -2x_1(\tau)d\tau}\, ds\Bigg].$$

Since $\mathrm{e}^{\int_0^t 2x_1(s)ds} = x_2(t)$ and thus $\mathrm{e}^{\int_0^s -2x_1(\tau)d\tau} = 1/x_2(s)$, it follows that:

$$x_{n+1}(t) = \frac{1}{(1-at)^2}\Bigg[\int_0^t \Big(\sum_{k=2}^{n} p_k(1/(1-as)) p_{n-k+2}(1/(1-as))\Big)(1-as)^2 ds\Bigg].$$

Since the polynomials appearing in the integral take the form of Eq. $S17$, they are all missing first order and constant terms, and thus there follows that

$$x_{n+1}(t) = \frac{1}{(1-at)^2}\Bigg[\int_0^t \Big(\sum_{k=2}^{n} \frac{1}{a^{n-2}}\frac{1}{(1-as)^{n+2}} + \cdots + c_k c_{n-k+2}\frac{1}{(1-as)^4}\Big)(1-as)^2 ds\Bigg]$$

and so that

$$x_{n+1}(t) = \frac{1}{a^{n-1}}\frac{1}{(1-at)^{n+1}} + \cdots + \frac{c_{n+1}}{(1-at)^2}, \quad t \in [0, t^*),$$

where K_{n+1} and c_{n+1} are certain constants (which are related in some way which is irrelevant for what follows). This shows that $x_{n+1}(t)$ is indeed of the form $p_{n+1}(1/(1-at))$ with $p_{n+1}(z)$ as in S17. □

Next we consider equation S5 in case $X(0)$ is a block triangular Toeplitz initial condition:

$$X(0) = \begin{pmatrix} B_1(0) & B_2(0) & \cdots & B_n(0) \\ 0 & B_1(0) & \ddots & B_{n-1}(0) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_1(0) \end{pmatrix}, \tag{S18}$$

with $B_i(0) = \alpha_i I_2 + \beta_i J_2$ with $\alpha_i, \beta_i \in \mathbb{R}$, and denote $\mathcal{BTT} = \{X \mid X \text{ is of the form } S18\}$. Again the set $\mathcal{BTT}$ is invariant for system S5. We use this to solve equation S5 in case $X(0)$ is a real Jordan block corresponding to a pair of eigenvalues $\alpha \pm j\beta$.

Lemma 3. *Let $X(0) \in \mathcal{BTT}$ with*

$$B_i(0) = \begin{cases} \alpha I_2 + \beta J_2 & \text{if } i = 1 \\ I_2 & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}.$$

Then the forward solution $X(t)$ of S5 is defined on $[0, +\infty)$, and it belongs to $\mathcal{BTT}$.

Proof. Just like in the proof of Proposition 2, we note that system S5 can be solved recursively, starting with $X_1(t)$, followed by $X_2(t), X_3(t), \ldots$. Only the first equation for X_1 is nonlinear, whereas the equations for $X_2, X_3, \ldots$ are linear. To see this, we write these equations:

$$\dot{X}_i = \begin{cases} X_1^2, & \text{if } i = 1 \\ (2X_1(t))X_2, & \text{if } i = 2 \\ (2X_1(t))X_i + \sum_{k=2}^{i-1} X_k(t)X_{i-(k-1)}(t), & \text{if } i > 2 \end{cases}.$$

with $X_1(0) = \alpha I_2 + \beta J_2$, $X_2(0) = I_2$ and $X_i(0) = 0$ for $i > 2$. Here we have used the fact that $X_1X_i + X_iX_1 = 2X_1X_i$, since any two matrices of the form $pI_2 + qJ_2$ commute and the matrices $X_i(t)$ are of this form.

By Lemma 1, the forward solution for $X_1(t)$ is defined for all t in $[0, +\infty)$ (and in fact, converges to zero as $t \to +\infty$).

Since the $X_1(t)$ commute for every pair of t's, the forward solution for $X_2(t)$ is given by [50] $X_2(t) = \mathrm{e}^{\int_0^t 2X_1(s)ds}$, for $t \in [0, +\infty)$, where this solution exists for all forward times t because $X_1(t)$ is bounded and continuous. Similarly, the forward solution for $X_i(t)$ when $i > 2$, is given by the variation of constants formula:

$$X_i(t) = X_2(t)\left[\int_0^t X_2^{-1}(s)\left(\sum_{k=2}^{i-1} X_k(s)X_{i-(k-1)}(s)\right)ds\right],$$

for $t \in [0, +\infty)$ when $i > 2$, where these solutions are recursively defined for all forward times because the formula only involves integrals of continuous functions. □

Combining both results, puts us in a position to state and prove our main result.

Theorem 3. *Let $X(0) \in \mathbb{R}^{n\times n}$ and (T, Λ_0) as in S14 with S15. Let $a_1 > a_2 \geq \cdots \geq a_k$ with $a_1 > 0$ a simple eigenvalue with corresponding right and left-eigenvectors U_1 and V_1^T respectively:*

$$X(0)U_1 = a_1 U_1 \text{ and } V_1^T X(0) = a_1 V_1^T.$$

Then the forward solution $X(t)$ of S5 is defined for $[0, 1/a_1)$, and

$$\lim_{t\to 1/a_1} \frac{X(t)}{|X(t)|_F} = \frac{U_1 V_1^T}{|U_1 V_1^T|_F}.$$

Proof. Consider the initial value problem $\dot{\Lambda} = \Lambda^2$ with $\Lambda(0) = \Lambda_0$, whose solution is given by

$$\Lambda(t) = \mathrm{diag}(A_1(t), \ldots, A_k(t), B_1(t), \ldots, B_l(t)),$$

where for all $i = 1, \ldots, k$, $A_i(t)$ is the forward solution of S5 with $A_i(0)$ of the form A_i in S15, which by Lemma 2 is defined for all $t \in [0, 1/a_i)$. Since $a_1 > a_2 \geq \cdots \geq a_k$, $A_1(t)$ blows up first when $t \to 1/a_1$. The matrices $B_j(t)$, $j = 1, \ldots, l$, are the forward solution of S5 with $B_j(0)$ of the form B_j in S15, and by Lemma 3, they are defined for all t in $[0, +\infty)$.

This clearly shows that $\Lambda(t)$ is defined in forward time for t in $[0, 1/a_1)$. Since the solution of S5 is given by $X(t) = T\Lambda(t)T^{-1}$, $X(t)$ is also defined in forward time for t in $[0, 1/a_1)$, and it follows that

$$\lim_{t\to 1/a_1} \frac{X(t)}{|X(t)|_F} = \lim_{t\to 1/a_1} \frac{T\Lambda(t)T^{-1}}{|X(t)|_F} = \frac{Te_1 e_1^T T^{-1}}{|Te_1 e_1^T T^{-1}|_F} = \frac{U_1 V_1^T}{|U_1 V_1^T|_F},$$

where e_1 denotes the first standard unit basis vector of $\mathbb{R}^n$. □

Theorem 3 implies that social balance is usually not achieved when $X(0)$ is an arbitrary real initial condition. Indeed, if X_0 has a simple, positive, real eigenvalue a_1, and if we assume that no entry of the right and left eigenvectors U_1 and V_1^T are zero (an assumption which is generically satisfied), then in general, up to a permutation of its entries, the sign patterns of U_1 and V_1^T are:

$$U_1 = \begin{pmatrix} + \\ + \\ \hline - \\ - \end{pmatrix} \text{ and } V_1^T = \begin{pmatrix} + & - & | & + & - \end{pmatrix}$$

implies that

$$U_1 V_1^T = \left(\begin{array}{cc|cc} + & - & + & - \\ + & - & + & - \\ \hline - & + & - & + \\ - & + & - & + \end{array}\right).$$

Then Theorem 1 implies that the normalized state of the system does not become balanced in finite time.

This shows that in general, unless X_0 is normal (so that Theorem 2 is applicable), we cannot expect that social balance will emerge for system S5.

III. EQUATION $\dot{X} = XX^T$

We now consider

$$\dot{X} = XX^T, X(0) = X_0, \tag{S19}$$

where again, each X_{ij} denotes the real-valued opinion agent i has about agent j. As before, for $i = j$, the value of X_{ii} is interpreted as a measure of self-esteem of agent i. We can also write the equations entrywise:

$$\dot{X}_{ij} = \sum_k X_{ik} X_{jk}. \tag{S20}$$

As in the case of model $\dot{X} = X^2$, we split up the analysis in two parts. First we consider system S19 with normal initial condition X_0, and we shall see that not all initial conditions lead to the emergence of a balanced network in this case, in contrast to the behavior of S5. Secondly, we will see that for non-normal, generic initial conditions X_0, we typically do get the emergence of social balance, also contrasting the behavior of S5.

Normal initial condition

As for the model $\dot{X} = X^2$ the set $\mathcal{N}$ is invariant for system S19. By using the same diagonalization as in Eq. S7, if $\Lambda(t)$ is the solution to the initial value problem $\dot{\Lambda} = \Lambda\Lambda^T$, $\Lambda(0) = \Lambda_0$, then $X(t) := U\Lambda(t)U^T$, is the solution to Eq. S19. This shows it is sufficient to solve system S19 in case of scalar X or in case of a specific 2×2 normal matrix X. The scalar case is easy to solve and follows Eq. S8, so we turn to the 2×2 case by considering

$$\dot{X} = XX^T, \; X(0) = \alpha I_2 + \beta J_2, \text{ where } \beta \neq 0. \tag{S21}$$

We define the angle ϕ as

$$\phi = \arctan\left(\frac{\alpha}{\beta}\right), \; \phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{S22}$$

Lemma 4. *Define $\bar{t}$ as*

$$\bar{t} = \frac{\pi}{2\beta} - \frac{\phi}{\beta}. \tag{S23}$$

Then the forward solution $X(t)$ of S21 is:

$$X(t) = \beta\tan(\beta t + \phi)I_2 + \beta J_2, \;\; t \in [0, \bar{t}). \tag{S24}$$

Moreover,

$$\lim_{t\to\bar{t}-} X(t) = +\infty I_2 + \beta J_2 \text{ and } \lim_{t\to\bar{t}-} \frac{X(t)}{|X(t)|_F} = \frac{\sqrt{2}}{2} I_2.$$

Proof. Let $X_0 = S_0 + A_0$, $S_0 = \alpha I_2$, and $A_0 = \beta J_2$. Then the solution $X(t)$ of S21 can be decomposed as $S(t) + A(t)$, where

$$\dot{S} = (S + A)(S - A), \;\; S(0) = S_0, \tag{S25}$$

$$\dot{A} = 0, \;\; A(0) = A_0, \tag{S26}$$

so $A(t) = A_0$, and reduces to

$$\dot{S} = (S + A_0)(S - A_0), \quad S(0) = S_0 \tag{S27}$$

Note that S27 is a matrix Riccati differential equation with the property that the line $\mathcal{L} = \{\alpha I_2 | \alpha \in \mathbb{R}\}$, is an invariant set under the flow. Therefore it suffices to solve the scalar Riccati differential equation corresponding to the dynamics of the diagonal entries of S: $\dot{s} = s^2 + \beta^2$, $s(0) = \alpha$, whose forward solution is: $s(t) = \beta \tan(\beta t + \phi)$, for $t \in (0, \bar{t})$, where $\bar{t}$ is given by S23. Consequently, the forward solution $X(t)$ of S21 is given by: $X(t) = S(t) + A_0 = \beta \tan(\beta t + \phi) I_2 + \beta J_2$, for $t \in (0, \bar{t})$, and thus $\lim_{t \to \bar{t}-} X(t) = +\infty I_2 + \beta J_2$ and

$$\lim_{t \to \bar{t}-} \frac{X(t)}{|X(t)|_F} = \frac{X(t)}{\sqrt{2}|\beta \sec(\beta t + \phi)|} = \frac{\sqrt{2}}{2} I_2.$$

□

Combining the solution for the 1×1 scalar case in Eq. S8 and Lemma 4 yields our main result:

Theorem 4. *Let $X_0 \in \mathcal{N}$, and let (U, Λ_0) be as in Lemma S7. Define*

$$\bar{t}_i = \begin{cases} 1/a_i & \text{if } a_i > 0 \\ +\infty & \text{if } a_i \leq 0 \end{cases} \quad \text{for all } i = 1, \ldots, k,$$

and

$$\bar{t}_j = \frac{\pi}{2\beta_j} - \frac{\phi_j}{\beta_j} \text{ for all } j = 1, \ldots, l,$$

where $\phi_j = \arctan\left(\frac{\alpha_j}{\beta_j}\right)$ and let $\bar{t} = \min_{i,j}\{\bar{t}_i, \bar{t}_j\}$. Then the forward solution $X(t)$ of S19 is defined for $[0, \bar{t})$.

If there is a unique $i^* \in \{1, \ldots, k\}$ such that $\bar{t} = \bar{t}_{i^*}$ is finite, then

$$\lim_{t \to \bar{t}_{i^*}-} \frac{X(t)}{|X(t)|_F} = U_{i^*} U_{i^*}^T,$$

where U_{i^*} is the i^*th column of U, an eigenvector corresponding to eigenvalue a_{i^*} of X_0.

If there is a unique $j^* \in \{1, \ldots, l\}$ such that $\bar{t} = \bar{t}_{j^*}$, then

$$\lim_{t \to \bar{t}_{j^*}-} \frac{X(t)}{|X(t)|_F} = \frac{\sqrt{2}}{2} U_{j^*} U_{j^*}^T,$$

where U_{j^*} is an $n \times 2$ matrix consisting of the two consecutive columns of U which correspond to the columns of the 2×2 block B_{j^*} in Λ_0.

Proof. Consider the initial value problem:

$$\dot{\Lambda} = \Lambda \Lambda^T, \ \Lambda(0) = \Lambda_0.$$

By Lemma 4 its solution is given by

$$\Lambda(t) = \begin{pmatrix} \frac{a_1}{1-a_1 t} & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \frac{a_k}{1-a_k t} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & X_1(t) & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & X_l(t) \end{pmatrix},$$

where for all $j = 1, \ldots, l$, $X_j(t)$ is given by the 2×2 matrix in S24 with β, ϕ and $\bar{t}$ replaced by β_j, ϕ_j and $\bar{t}_j$ respectively. This clearly shows that $\Lambda(t)$ is defined in forward time for t in $[0, \bar{t})$. Since the solution of S19 is given by $X(t) = U\Lambda(t)U^T$, $X(t)$ is also defined in forward time for t in $[0, \bar{t})$. It follows from S2 that

$$\frac{X(t)}{|X(t)|_F} = U \frac{\Lambda(t)}{|\Lambda(t)|_F} U^T.$$

If $i^* \in \{1, \ldots, k\}$ is the unique value such that $\bar{t} = \bar{t}_{i^*}$, then

$$\begin{aligned} \lim_{t \to \bar{t}_i^*} \frac{X(t)}{|X(t)|_F} &= U \lim_{t \to \bar{t}_i^*} \frac{\Lambda(t)}{|\Lambda(t)|_F} U^T \\ &= U e_{i^*} e_{i^*}^T U^T = U_{i^*} U_{i^*}^T, \end{aligned}$$

where e_{i^*} denotes the i^*th standard unit basis vector of $\mathbb{R}^n$.

If $j^* \in \{1, \ldots, l\}$ is the unique value such that $\bar{t} = \bar{t}_{j^*}$, then by Lemma 4:

$$\begin{aligned} \lim_{t \to \bar{t}_j^*} \frac{X(t)}{|X(t)|_F} &= U \lim_{t \to \bar{t}_j^*} \frac{\Lambda(t)}{|\Lambda(t)|_F} U^T \\ &= \frac{\sqrt{2}}{2} U E_{j^*} U^T = \frac{\sqrt{2}}{2} U_{j^*} U_{j^*}^T, \end{aligned}$$

where E_{j^*} has exactly two non-zero entries equal to 1 on the diagonal positions corresponding to the block B_{j^*} in Λ_0. □

A particular consequence of Theorem 4 is that if X_0 has a complex pair of eigenvalues, the solution of S19 always blows up in finite time, even if all real eigenvalues of X_0 are non-positive. Recall that the solution of S5 blows up in finite time, if and only if X_0 has a positive, real eigenvalue. Another implication of Theorem 4 is that if blow-up occurs, it may be due to a real eigenvalue of X_0, or to a complex eigenvalue. In contrast, if the solution of S5 blows up in finite time, it is necessarily due to a positive, real eigenvalue, and never to a complex eigenvalue. When the solution of S19 blows up because of a positive, real eigenvalue of X_0, the system will achieve balance, just as in the case of system S5. If on the other hand, finite time blow up of S19 is caused by a complex eigenvalue of X_0, we show that in general one cannot expect to achieve a balanced network. Assume there is a unique j^* such that:

$$\lim_{t\to\bar{t}^*_j-}\frac{X(t)}{|X(t)|_F}=\frac{\sqrt{2}}{2}U_{j^*}U_{j^*}^T,$$

Assuming that no entry of U_{j^*} is zero, the sign pattern of $U_{j^*}U_{j^*}^T$, with

$$U_j^*=\begin{pmatrix} p_1 & q_1\\ p_2 & -q_2\\ -p_3 & q_3\\ -p_4 & -q_4\end{pmatrix}$$

is given by:

$$\begin{pmatrix} + & ? & ? & -\\ ? & + & - & ?\\ ? & - & + & ?\\ - & ? & ? & +\end{pmatrix},$$

up to a suitable permutation, where all p_i and q_i, $i=1,\dots,4$, are entrywise positive vectors, and where

$$\langle p_1,q_1\rangle+\langle p_4,q_4\rangle=\langle p_2,q_2\rangle+\langle p_3,q_3\rangle,$$

because U is an orthogonal matrix. The ? are not entirely arbitrary because $U_{j^*}U_{j^*}^T$ is a symmetric matrix, but besides that their signs can be arbitrary.

Generic initial condition

Consider

$$\dot{X}=XX^T, X(0)=X_0, \tag{S28}$$

where X is a real $n\times n$ matrix, which is not necessarily normal.

We first decompose the flow S28 into flows for the symmetric and skew-symmetric parts of X. Let $X=S+A, X_0=S_0+A_0$, where $S,S_0\in\mathcal{S}$ and $A,A_0\in\mathcal{A}$ are the unique symmetric and skew-symmetric parts of X and X_0 respectively. If $X(t)$ satisfies S28, then it can be verified that $S(t)$ and $A(t)$ satisfy the system:

$$\dot{S}=(S+A)(S-A),\ S(0)=0, \tag{S29}$$

$$\dot{A}=0,\ A(0)=A_0, \tag{S30}$$

Consequently, $A(t)=A_0$ for all t, and thus the skew-symmetric part of the solution $X(t)$ of S28 remains constant and equal to A_0. Throughout this subsection we assume that $A_0\neq 0$, for otherwise $X(0)$ is symmetric, hence normal, and the results from the previous subsection apply. It follows that we only need to understand the dynamics of the symmetric part. Then the solution $X(t)$ to S28 is given by $X(t)=S(t)+A_0$, where $S(t)$ solves S29, and in view of S1, there follows by Pythagoras' Theorem that: $|X(t)|_F^2=|S(t)|_F^2+|A_0|_F^2$, and thus

$$\frac{X(t)}{|X(t)|_F}=\frac{S(t)+A_0}{(|S(t)|_F^2+|A_0|_F^2)^{\frac{1}{2}}}. \tag{S31}$$

Next we shall derive an explicit expression for the solution $S(t)$ of S29. We start by performing a change of variables:

$$\hat{S}(t)=\mathrm{e}^{-tA_0}\,S(t)\,\mathrm{e}^{tA_0}. \tag{S32}$$

This yields the equation

$$\dot{\hat{S}}=\hat{S}^2-A_0^2,\ \ \hat{S}(0)=S_0. \tag{S33}$$

We perform a further transformation which diagonalizes $-A_0^2$: Let V be an orthogonal matrix such that $-V^TA_0^2V=D^2$, where $D:=\mathrm{diag}(0,\omega_1I_2,\dots,\omega_kI_k)$ where $k\geq 1$ (because $A_0\neq 0$) and all $\omega_j>0$ without loss of generality. Setting

$$\tilde{S}=V^T\hat{S}V, \tag{S34}$$

and multiplying equation S33 by V on the left, and by V^T on the right, we find that:

$$\dot{\tilde{S}}=\tilde{S}^2+D^2,\ \ \tilde{S}(0)=\tilde{S}_0:=V^TS_0V. \tag{S35}$$

Notice that this is a matrix Riccati differential equation, a class of equations with specific properties which are briefly reviewed next.

Consider a general matrix Riccati differential equation:

$$\dot{S}=SMS-SL-L^TS+N, \tag{S36}$$

where $M=M^T$,$N=N^T$ and L arbitrary, defined on $\mathcal{S}$. Associated to this equation is a linear system

$$\begin{pmatrix}\dot{P}\\ \dot{Q}\end{pmatrix}=H\begin{pmatrix}P\\ Q\end{pmatrix},\ \ H:=\begin{pmatrix}L & -M\\ N & -L^T\end{pmatrix}, \tag{S37}$$

where H is a Hamiltonian matrix, i.e. $J_{2n}H=(J_{2n}H)^T$ holds, where J_{2n} is as defined in Eq. S3. The following fact is well-known.

Lemma 5. *Let $\begin{pmatrix}P(t)\\ Q(t)\end{pmatrix}$ be a solution of S37. Then, provided that $P(t)$ is non-singular,*

$$S(t)=Q(t)P(t)^{-1}, \tag{S38}$$

is a solution of S36. Conversely, if $S(t)$ is a solution of S36, then there exists a solution $\begin{pmatrix}P(t)\\ Q(t)\end{pmatrix}$ of S37 such that S38 holds, provided that $P(t)$ is non-singular.

Proof. Taking derivatives in $S(t)P(t)=Q(t)$ yields that $\dot{S}=(\dot{Q}-S\dot{P})P^{-1}$, and using S37,

$$\dot{S}=(NP-L^TQ-S(LP-MQ))P^{-1}=N-L^TS-SL+SMS,$$

showing that $S(t)$ solves S36. For the converse, let $S(t)$ be a solution of S36. Let $\begin{pmatrix}P(t)\\ Q(t)\end{pmatrix}$ with $\begin{pmatrix}P(0)\\ Q(0)\end{pmatrix}=\begin{pmatrix}I_n\\ S(0)\end{pmatrix}$ be the solution of S37. Then

$$\begin{aligned}&\frac{d}{dt}\left(Q(t)P^{-1}(t)\right)\\ =&\dot{Q}P^{-1}-QP^{-1}\dot{P}P^{-1}\\ =&(NP-L^TQ)P^{-1}-QP^{-1}(LP-MQ)P^{-1}\\ =&(QP^{-1})M(QP^{-1})-(QP^{-1})L-L^T(QP^{-1})+N,\end{aligned}$$

which implies that QP^{-1} is a solution to S36. Since $S(0) = Q(0)P^{-1}(0)$, it follows from uniqueness of solutions that $S(t) = Q(t)P^{-1}(t)$. $\square$

In other words, in principle we can solve the nonlinear equation S36 by first solving the linear system S37, and then use formula S38 to determine the solution of S36.

We carry this out for our particular Riccati equation S35 which is of the form S36 if $M = I_n$, $L = 0$, $N = D^2$. The corresponding Hamiltonian is $H = \begin{pmatrix} 0 & -I_n \\ D^2 & 0 \end{pmatrix}$. We partition D in singular and non-singular parts:

$$D = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{D} \end{pmatrix}, \text{ where } \tilde{D} := \begin{pmatrix} \omega_1 I_2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_k I_2 \end{pmatrix},$$

where $\tilde{D}$ is positive definite since all $\omega_j > 0$. Partitioning H correspondingly:

$$H = \left(\begin{array}{cc|cc} 0 & 0 & -I_{n-2k} & 0 \\ 0 & 0 & 0 & -I_{2k} \\ \hline 0 & 0 & 0 & 0 \\ 0 & \tilde{D}^2 & 0 & 0 \end{array}\right). \tag{S39}$$

This matrix is then exponentiated to solve system S37:

$$\begin{pmatrix} P(t) \\ \hline Q(t) \end{pmatrix} = \left(\begin{array}{cc|cc} I_{n-2k} & 0 & -tI_{n-2k} & 0 \\ 0 & c & 0 & -\tilde{D}^{-1}s \\ \hline 0 & 0 & I_{n-2k} & 0 \\ 0 & \tilde{D}s & 0 & c \end{array}\right) \begin{pmatrix} P(0) \\ \hline Q(0) \end{pmatrix},$$

where we have introduced the following notation:

$$s(t) := \operatorname{diag}(\sin(\omega_1 t)I_2, \dots, \sin(\omega_k t)I_2) = \sin(\tilde{D}t),$$

and similarly $c(t) = \cos(\tilde{D}t)$. By setting $P(0) = I_n$, and $Q(0) = \tilde{S}_0$, and using Lemma 5, it follows that the solution of the initial value problem S35 is given by $\tilde{S}(t) = Q(t)P(t)^{-1}$,

$$\begin{pmatrix} P(t) \\ \hline Q(t) \end{pmatrix} = \left(\begin{array}{c} \begin{pmatrix} (I_{n-2k} - t)\tilde{S}_0 & 0 \\ 0 & c(t) - \tilde{D}^{-1}s(t)\tilde{S}_0 \end{pmatrix} \\ \hline \begin{pmatrix} I_{n-2k}\tilde{S}_0 & 0 \\ 0 & \tilde{D}s(t) + c(t)\tilde{S}_0 \end{pmatrix} \end{array}\right), \tag{S40}$$

for all t for which $P(t)$ is non-singular. We now make the following assumption:

Assumption A. *The matrix $P(t)$ is non-singular for all t in $[0, \bar{t})$, where $\bar{t}$ is finite and such that $s(t)$ is non-singular for all t in $(0, \bar{t})$. Moreover, $P(\bar{t})$ has rank $n-1$, or equivalently, has a simple eigenvalue at zero.*

Later we will show that this assumption is generically satisfied, and also that

$$t^* = \bar{t}, \tag{S41}$$

where $[0, t^*)$ is the maximal forward interval of existence of the solution $\tilde{S}(t)$ of the initial value problem S35. Consequently, the theory of ODE's implies that $\lim_{t\to\bar{t}} |\tilde{S}(t)|_F = +\infty$, i.e. that $\bar{t}$ is the blow-up time for the solution $\tilde{S}(t)$.

Assuming for the moment that assumption A is satisfied, back-transformation using S32 and S34, yields that the solution $S(t)$ of S29 is $S(t) = \mathrm{e}^{tA_0}\, V\tilde{S}(t)V^T\, \mathrm{e}^{-tA_0}$, which is defined for all t in $[0, \bar{t})$, because $\mathrm{e}^{tA_0}\, V$ is bounded for all t (as it is an orthogonal matrix). It follows from S2 that

$$\lim_{t\to\bar{t}} \frac{S(t)}{|S(t)|_F} = \mathrm{e}^{\bar{t}A_0}\, V \left(\lim_{t\to\bar{t}} \frac{\tilde{S}(t)}{|\tilde{S}(t)|_F}\right) V^T\, \mathrm{e}^{-\bar{t}A_0}, \tag{S42}$$

provided that at least one of the two limits exists. Partitioning $\tilde{S}_0$ in S40 as follows:

$$\tilde{S}_0 = \begin{pmatrix} (\tilde{S}_0)_{11} & (\tilde{S}_0)_{12} \\ (\tilde{S}_0)_{12}^T & (\tilde{S}_0)_{22} \end{pmatrix}, \text{ with } \begin{array}{c} (\tilde{S}_0)_{11} = (\tilde{S}_0)_{11}^T \\ (\tilde{S}_0)_{22} = (\tilde{S}_0)_{22}^T \end{array},$$

we can rewrite $P(t)$ and $Q(t)$ on the time interval $(0, \bar{t})$ as: $P(t) = \Delta(t)M(t)$ with,

$$\Delta(t) = \begin{pmatrix} tI_{n-2k} & 0 \\ 0 & \tilde{D}^{-1}s(t) \end{pmatrix},$$

and

$$M(t) = \begin{pmatrix} 1/t - (\tilde{S}_0)_{11} & -(\tilde{S}_0)_{12} \\ -(\tilde{S}_0)_{12}^T & \tilde{D}c(t)s^{-1}(t) - (\tilde{S}_0)_{22} \end{pmatrix} = M^T(t),$$

and

$$Q(t) = \begin{pmatrix} (\tilde{S}_0)_{11} & (\tilde{S}_0)_{12} \\ c(t)(\tilde{S}_0)_{12}^T & \tilde{D}s(t) + c(t)(\tilde{S}_0)_{22} \end{pmatrix}.$$

Note that the factorization of $P(t)$ is well-defined on $(0, \bar{t})$ because by assumption A, the matrix $s(t)$ is non-singular in the interval $(0, \bar{t})$. Moreover, assumption A also implies there exists a nonzero vector u corresponding to the zero eigenvalue of $M(\bar{t})$, i.e. $M(\bar{t})u = 0$, and that u is uniquely defined up to scalar multiplication because the zero eigenvalue is simple. More explicitly, partitioning u as $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, there holds that

$$\begin{pmatrix} 1/\bar{t} - (\tilde{S}_0)_{11} & -(\tilde{S}_0)_{12} \\ -(\tilde{S}_0)_{12}^T & \tilde{D}c(\bar{t})s^{-1}(\bar{t}) - (\tilde{S}_0)_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0. \tag{S43}$$

Notice that $M(t)$ is at least real-analytic on the interval $(0, \bar{t})$. Hence, it follows from [51] (see also [52, 53]), that there is an orthogonal matrix $U(t)$, and a diagonal matrix $\Lambda(t)$, both real-analytic on $(0, \bar{t})$, such that: $M(t) = U(t)\Lambda(t)U^T(t)$, for $t \in (0, \bar{t})$, and thus $M^{-1}(t) = U(t)\Lambda^{-1}(t)U^T(t)$, for $t \in (0, \bar{t})$. Returning to S42, we obtain that:

$$\begin{aligned}
&\lim_{t\to\bar{t}} \frac{S(t)}{|S(t)|_F} \\
&= e^{\bar{t}A_0}\, V \lim_{t\to\bar{t}} \frac{Q(t)U(t)\Lambda^{-1}(t)U^T(t)\Delta^{-1}(t)}{|Q(t)U(t)\Lambda^{-1}(t)U^T(t)\Delta^{-1}(t)|_F} V^T\, e^{-\bar{t}A_0} \\
&= e^{\bar{t}A_0}\, V \frac{Q(\bar{t})uu^T\Delta^{-1}(t)}{|Q(\bar{t})uu^T\Delta^{-1}(t)|_F} V^T\, e^{-\bar{t}A_0}\,.
\end{aligned}$$

Here, we have used the fact that $M^{-1}(t)$ is positive definite on the interval $(0,\bar{t})$, so that its largest eigenvalue (which is simple for all $t<\bar{t}$ sufficiently close to $\bar{t}$, because of assumption A approaches $+\infty$ -and not $-\infty$- as $t\to\bar{t}$. To see this, note that from its definition follows that $M(t)$ is positive definite for all sufficiently small $t>0$, because $\tilde{D}$ is positive definite. Moreover, $M(t)$ is non-singular on $(0,\bar{t})$ since by assumption (**A**), $P(t)$ is non-singular on $(0,\bar{t})$, and because $M(t)=\Delta^{-1}(t)P(t)$ (it is clear from its definition and assumption A that $\Delta(t)$ is non-singular on $(0,\bar{t})$ as well). Consequently, the smallest eigenvalue of $M(t)$ remains positive in $(0,\bar{t})$, as it approaches zero as $t\to\bar{t}$. This implies that the largest eigenvalue of $M^{-1}(t)$ is positive on $(0,\bar{t})$, and approaches $+\infty$ as $t\to\bar{t}$, as claimed.

Note that:

$$\begin{aligned}
Q(\bar{t})u &= \begin{pmatrix} (\tilde{S}_0)_{11} & (\tilde{S}_0)_{12} \\ c(\bar{t})(\tilde{S}_0)_{12}^T & \tilde{D}s(\bar{t})+c(\bar{t})(\tilde{S}_0)_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
&= \begin{pmatrix} (1/\bar{t})u_1 \\ \tilde{D}s^{-1}(\bar{t})u_2 \end{pmatrix} = \Delta^{-1}(\bar{t})u,
\end{aligned}$$

where in the second equality, we used the second row of S43 , multiplied by $c(\bar{t})$. From this follows that

$$\begin{aligned}
\lim_{t\to\bar{t}} \frac{S(t)}{|S(t)|_F} &= e^{\bar{t}A_0}\, V \frac{\Delta^{-1}(\bar{t})uu^T\Delta^{-1}(\bar{t})}{|\Delta^{-1}(\bar{t})uu^T\Delta^{-1}(\bar{t})|_F} V^T\, e^{-\bar{t}A_0} \\
&= \frac{ww^T}{|ww^T|_F},
\end{aligned}$$

where $w := e^{\bar{t}A_0}V\Delta^{-1}(\bar{t})u$.

Taking limits for $t\to\bar{t}$ in S31, and using the above equality, we finally arrive at the following result, which implies that system S28 evolves to a socially balanced state (in normalized sense) when $t\to\bar{t}$:

Proposition 1. *Suppose that assumption A holds and $A_0\neq 0$. Then the solution $X(t)$ of S28 satisfies:*

$$\lim_{t\to\bar{t}} \frac{X(t)}{|X(t)|_F} = \frac{ww^T}{|ww^T|_F}.$$

with $w = e^{\bar{t}A_0}V\Delta^{-1}(\bar{t})u$.

Genericity

Generically, assumption A holds, and S41 holds as well. There are two aspects to assumption A:

1. The matrix $P(t)$ is nonsingular in the interval $[0,\bar{t})$, but singular at some finite $\bar{t}$ such that:

$$\bar{t} < \min_{j=1,\ldots,k} \frac{\pi}{\omega_j}. \tag{S44}$$

2. $P(\bar{t})$ has a simple zero eigenvalue.

To deal with the first item, suppose that the solution $\tilde{S}(t)$ of S35 is defined for all $t\in[0,t^*)$ for some finite positive t^*. By Lemma 5, there exist $P(t)$ and $Q(t)$ such that $\tilde{S}(t)=Q(t)P^{-1}(t)$, where $P(t)$ and $Q(t)$ are components of the solution of system S37 with H defined in S39. Then necessarily $\bar{t}\leq t^*$. Thus, if we can show that $t^*<\min_j \pi/\omega_j$, then S44 holds. To show that $t^*<\min_j \pi/\omega_j$, we rely on a particular property of matrix Riccati differential equations S36: their solutions preserve the order generated by the cone of non-negative symmetric matrices, see [54]. More precisely, if $S_1(t)$ and $S_2(t)$ are solutions of S36, and if $S_1(0)\preceq S_2(0)$, then $S_1(t)\preceq S_2(t)$, for all t ≥ 0 for which both solutions are defined. The partial order notation $S_1(t)\preceq S_2(t)$ means that the difference $S_2(t)-S_1(t)$ is a positive semi-definite matrix.

We apply this to equation S35 with $\tilde{S}_1(0)=\alpha_{\min}I_n$ and $\tilde{S}_2(0)=\tilde{S}(0)$, where we choose $\alpha_{\min}$ as the smallest eigenvalue of $\tilde{S}(0)$ (or equivalently, of $S(0)=S_0$, since $\tilde{S}(0)=V^TS_0V$), so that clearly $\tilde{S}_1(0)\preceq\tilde{S}_2(0)$. Consequently, by the monotonicity property of system S35, it follows that $\tilde{S}_1(t)\preceq\tilde{S}(t)$, as long as both solutions are defined. We can calculate the blow-up time t_1^* of $\tilde{S}_1(t)$ explicitly, and then it follows that $t^*\leq t_1^*$, where t^* is the blow-up time of $\tilde{S}(t)$. Indeed, equations of system S35 decouple for an initial condition of the form $\alpha_{\min}I_n$, and the resulting scalar equations are scalar Riccati equations we have solved before. The blow-up time for $\tilde{S}_1(t)$ is given by:

$$t_1^* = \begin{cases} \min_{j=1,\ldots,k}\left(\frac{\pi}{2\omega_j}-\frac{\phi_j}{\omega_j}\right), & \text{if } \alpha_{\min}\leq 0 \\ \min_{j=1,\ldots,k}\left(\frac{1}{\alpha_{\min}}, \frac{\pi}{2\omega_j}-\frac{\phi_j}{\omega_j}\right), & \text{if } \alpha_{\min}>0 \end{cases}.$$

with $\phi_j := \arctan\left(\frac{\alpha_{\min}}{\omega_j}\right)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Notice that for all $j=1,\ldots,k$, there holds that $\frac{\pi}{2\omega_j}-\frac{\phi_j}{\omega_j}<\frac{\pi}{\omega_j}$, because by definition, $\frac{\phi_j}{\omega_j}\in(-\frac{\pi}{2\omega_j},\frac{\pi}{2\omega_j})$. Consequently,

$$\bar{t}\leq t^*\leq t_1^* < \min_{j=1,\ldots,k}\frac{\pi}{\omega_j},$$

which establishes S44. In other words, we have shown that the first item in assumption A is always satisfied.

The second item in assumption A may fail, but holds for generic initial conditions as we show next. For this we first point out that the derivative of each eigenvalue of $M(t)$ is a strictly decreasing function in the interval $(0,\bar{t})$, independently of the value of the matrix $\tilde{S}_0$. Indeed, the derivative of eigenvalue $\lambda_j(t)$ of $M(t)$ equals (see [51]) :

$$\dot{\lambda}_j(t) = u_j(t)^T \dot{M}(t) u_j(t) = -u_j(t)^T \begin{pmatrix} 1/t^2 & 0 \\ 0 & \tilde{D}^2 s^{-2}(t) \end{pmatrix} u_j(t),$$

where $u_j(t)$ is the normalized eigenvector of $M(t)$ corresponding to $\lambda_j(t)$, and which is analytic in the considered interval. Since $\dot{M}(t)$ is negative definite in that interval, $\dot{\lambda}_j(t)$ is also negative and hence all eigenvalues of $M(t)$ are strictly decreasing functions of t in that interval. Suppose now that $M(t)$ has a multiple eigenvalue 0 at $t = \bar{t}$, then $M(\bar{t})$ is positive semi-definite since $\bar{t}$ is the first singular point of $M(t)$ and the eigenvalues are decreasing function of t. If we now choose a positive semi-definite $\Delta_{\tilde{S}_0}$ of nullity 1, such that $M(\bar{t}) + \Delta_{\tilde{S}_0}$ also has nullity 1, then the perturbed initial condition $(\tilde{S}_0)_p = \tilde{S}_0 - \Delta_{\tilde{S}_0}$ yields the perturbed solution $\tilde{S}_p(t)$ which can be factored as $Q_p(t)P_p^{-1}(t)$, and where $P_p(t) = \Delta(t)M_p(t)$ (note that $\Delta(t)$ remains the same as before the perturbation) for $M_p(t) = M(t) + \Delta_{\tilde{S}_0}$ which now has a single root at the same minimal value $\bar{t}$. To construct such a matrix $\Delta_{\tilde{S}_0}$ is simple since the only condition it needs to satisfy is that $M(\bar{t})$ and $\Delta_{\tilde{S}_0}$ have a common null vector. Those degrees of freedom show that the second item in assumption A is indeed generic.

Now that we have established that A generically holds, we show that S41 is satisfied also. The proof is by contradiction. Earlier, we have shown that $\bar{t} \leq t^*$. Thus, if we suppose that S41 fails, then necessarily $\bar{t} < t^*$. This implies that although $P(\bar{t})$ is singular, the solution $\tilde{S}(t)$ exists for $t = \bar{t}$. Our goal is to show that $\lim_{t\to\bar{t}} |\tilde{S}(t)|_F = +\infty$, which yields the desired contradiction (by the theory of ODE's).

We first claim the following:

$$\text{If } u \neq 0 \text{ and } P(\bar{t})u = 0, \text{ then } Q(\bar{t})u \neq 0. \tag{S45}$$

Indeed, if this were not the case, then there would exist some vector $\bar{u} \neq 0$ such that $P(\bar{t})\bar{u} = 0$ and $Q(\bar{t})\bar{u} = 0$. On the other hand, $P(t)$ and $Q(t)$ are components of the matrix product

$$\begin{pmatrix} P(t) \\ Q(t) \end{pmatrix} = \mathrm{e}^{tH} \begin{pmatrix} I_n \\ \tilde{S}_0 \end{pmatrix},$$

where H is defined in S39. Multiplying the latter in $t = \bar{t}$ by $\bar{u}$, and using the previous expression, it follows from the invertibility of $\mathrm{e}^{\bar{t}H}$ that $\bar{u} = 0$, a contradiction. This establishes S45.

In the previous section, we factored $P(t)$ as $P(t) = \Delta(t)M(t)$. Since $P(t)$ is non-singular on $[0, \bar{t})$, and singular at $\bar{t}$, it follows from S44 and the definition of $\Delta(t)$, that $M(t)$ is non-singular (and, in fact, positive definite as shown in the previous section) on $(0, \bar{t})$, and singular at $\bar{t}$ as well. Therefore, since $M(t)$ is symmetric and real-analytic, it follows from [51] that we can find a positive and real-analytic scalar function $\epsilon(t)$, and a real-analytic unit vector $u(t)$ such that:

$$M(t)u(t) = \epsilon(t)u(t), \ \epsilon(t) > 0$$

on $(0, \bar{t})$, $\epsilon(\bar{t}) = 0$, $|u(t)|_2 = 1$, where $|.|_2$ denotes the Euclidean norm. In particular, $M(\bar{t})u(\bar{t}) = 0$, and since $\Delta(\bar{t})$ is non-singular, it follows that $P(\bar{t})u(\bar{t}) = 0$. Then S45 implies that $Q(\bar{t})u(\bar{t}) \neq 0$. Define the real-analytic unit vector

$$v(t) = \frac{\Delta(t)u(t)}{|\Delta(t)u(t)|_2}, \quad t \in (0, \bar{t}),$$

and calculate

$$\lim_{t\to\bar{t}} |\tilde{S}(t)v(t)|_2 = \lim_{t\to\bar{t}} |Q(t)P^{-1}(t)v(t)|_2 = \frac{|Q(\bar{t})u(\bar{t})|_2}{|\Delta(\bar{t})u(\bar{t})|_2} \lim_{t\to\bar{t}} \frac{1}{\epsilon(t)} = +\infty.$$

Since for any real $n \times n$ matrix A, and for any unit vector x (i.e. $|x|_2 = 1$) holds that $|Ax|_2 \leq |A|_F$, it follows that $\lim_{t\to\bar{t}} |\tilde{S}(t)|_F = +\infty$. This yields the sought-after contradiction.

By combining Proposition 1 and the results in this subsection, we have proved the main result concerning the generic emergence of balance for solutions of system S28.

Theorem 5. *There exists a dense set of initial conditions X_0 in $\mathbb{R}^{n\times n}$ such that the corresponding solution $X(t)$ of S28 satisfies:*

$$\lim_{t\to\bar{t}} \frac{X(t)}{|X(t)|_F} = \frac{ww^T}{|ww^T|_F}.$$

with $w = e^{\bar{t}A_0} V \Delta^{-1}(\bar{t}) u$.

Proof. The set of initial conditions X_0 for which $A_0 \neq 0$ and assumption A holds is dense in $\mathbb{R}^{n\times n}$. ☐