Linear Algebra and its Applications 468 (2015) 63–79

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Binary factorizations of the matrix of all ones

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ARTICLE INFO

Article history:
Received 26 September 2013
Accepted 11 January 2014
Available online 20 January 2014
Submitted by P. Psarrakos

MSC:
15A23
15A24

Keywords:
Matrix equation
Standard factorization
Binary matrix
De Bruijn matrix
Isomorphism

ABSTRACT

In this paper, we consider the problem of factorizing the $n \times n$ matrix J_n of all ones into the $n \times n$ binary matrices. We show that under some conditions on the factors, these are isomorphic to a row permutation of a De Bruijn matrix. Moreover, we consider in particular the binary roots of J_n, i.e. the binary solutions to $A^m = J_n$. On the one hand, we prove that any binary root with minimum rank is isomorphic to a row permutation of a De Bruijn matrix whose row permutation is represented by a block diagonal matrix. On the other hand, we partially solve Hoffman's open problem of characterizing the binary solutions to $A^2 = J_n$ by providing a characterization of the binary solutions to $A^2 = J_n$ with minimum rank. Finally, we provide a class of roots which are isomorphic to a De Bruijn matrix.

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0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2014.01.011

1. Introduction

In this paper, we consider the problem of factorizing the $n \times n$ matrix

$$J_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

into the binary matrices. Namely, we restrict ourselves to square $n \times n$ factors A_i that have all their elements in $\{0, 1\}$, i.e. that are adjacency matrices of graphs with n nodes. We are thus looking for the solutions of

$$\prod_{i=1}^{m} A_i = A_1 A_2 \dots A_m = J_n$$

and in particular, the case when all the factors are identical, i.e. we investigate the binary solutions to the equation

$$A^m = J_n.$$

The g-circulant binary solutions to $A^m = J_n$ were studied through a convenient representation by Hall polynomials [1,3,2,4]. Remind that a matrix is called g-circulant if each row is obtained from the previous one by shifting all its elements of g positions to the right. In particular, it has been proved [1] that some g-circulant solutions are isomorphic to a De Bruijn matrix, originally defined in [5]. Nowadays, there are very few results [6] about the general binary solutions to $A^m = J_n$. However, these general solutions are of interest in many problems. Indeed, a solution of $A^m = J_n$ is the adjacency matrix of a directed graph for which given any two nodes u and v, there is a unique directed path of length m from u to v. In [7], it has been shown that these graphs allow to construct a class of algebras. Moreover, in the framework of the finite-time average consensus problem, the binary solutions to $A^m = J_n$ represent all the communication topologies whose interaction strengths are all equal to $1/\sqrt[m]{n}$ and that reach the consensus at time m. In particular, the De Bruijn matrices are of this type and have been shown [8] to be one of the quickest strategies to reach the average consensus. In the present paper, we show how the binary roots of J_n with minimum rank are related to the De Bruijn matrices.

The outline of the paper is as follows: in Section 2 we state some properties on the binary roots of J_n and on the De Bruijn matrices, which are well known roots of J_n. In Section 3, we study the commuting factors of the matrix with all ones. We prove that under some conditions on the commuting factors, these are isomorphic to a row permutation of a De Bruijn matrix. In Section 4, we prove that any binary root with minimum rank is isomorphic to a row permutation of a De Bruijn matrix, whose row permutation is represented by a block diagonal matrix. In Section 5, we provide a characterization of the binary solutions to $A^2 = J_n$ with minimum rank, which partially solves Hoffman's open problem of characterizing any binary solution to $A^2 = J_n$. Finally, in Section 6, we provide a class of roots, not necessarily g-circulant, which are isomorphic to a De Bruijn matrix.

For convenience, the rows and columns of a matrix of dimension n will be indexed from 0 to $n-1$ and $e_{i,n}$ $(0 \leqslant i \leqslant n-1)$ denotes the $n \times 1$ unit vector with 1 in its i-th position. The vector $\mathbf{1}_n$ denotes the $n \times 1$ vector of all 1's.

The Kronecker product of two matrices A, B is denoted $A \otimes B$.

When there is no ambiguity, the square matrix of dimension n with all ones will be denoted by J instead of J_n.

Any $n \times n$ binary matrix A is seen as the adjacency matrix of a graph G with n nodes. One then says that the matrix A represents the graph G.

Any graph is said to be p-regular if it has out and in-degree p.

2. Matrix roots of J_n and De Bruijn matrices

In this section, we prove some properties of the binary roots of the square matrix J_n of all ones. Moreover, we remind some properties on the De Bruijn matrices which are a class of solutions to the equation $A^m = J_n$.

Lemma 2.1. *Let $A^m = kJ_n$, where $k \neq 0$ and $A \in \{0,1\}^{n\times n}$, then A represents a p-regular graph, the trace of A is p, p and k are positive integers and $p^m = kn$.*

Proof. Clearly, the spectrum of kJ_n is given by

$$\Lambda(kJ_n) = \{kn, 0, \ldots, 0\}.$$

The spectrum of any m-th root A must therefore be equal to

$$\Lambda(A) = \{p, 0, \ldots, 0\}, \quad p^m = kn,$$

but since A is a binary matrix its trace, which is then equal to p, must be a nonnegative integer. Moreover, the elements of A^m must be equal to k, which is therefore also a non-negative integer. Since we ruled out $k = 0$, both p and k must be positive integers. Since p is then the only strictly positive eigenvalue of A, its left and right Perron vectors must again be proportional to $\mathbf{1}_n$:

$$A\mathbf{1}_n = p\mathbf{1}_n, \qquad A^T\mathbf{1}_n = p\mathbf{1}_n,$$

which implies that A is p-regular. $\square$

The De Bruijn matrices are well known to meet these properties, especially when $k = 1$.

Definition 2.2. The De Bruijn matrix of order p and dimension n is an $n \times n$ matrix defined as:

$$D(p,n) := \mathbf{1}_p \otimes I_{n/p} \otimes \mathbf{1}_p^T,$$

where $I_{n/p}$ is the identity matrix of dimension n/p and $\mathbf{1}_p$ is the $p \times 1$ vector with all ones. Moreover, $n = p^m$ for some integer m.

Example 2.3. The De Bruijn matrix $D(2,8)$ of order 2 and dimension 8 is

$$D(2,8) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Below we remind some well known properties of the De Bruijn matrices.

Lemma 2.4. *The i-th power of the De Bruijn matrix $D(p,n)$ (with $n = p^m$) is equal to*

$$D(p,n)^i = D\big(p^i, n\big) = \mathbf{1}_{p^i} \otimes I_{p^{m-i}} \otimes \mathbf{1}_{p^i}^T$$

for $i \leqslant m$ and equal to

$$D(p,n)^i = p^{i-m} J_n$$

for $i \geqslant m$.

A direct consequence of this lemma is the following.

Corollary 2.5. *The De Bruijn matrices $D(p,n)$ are such that*

$$D(p,n)^m = J_n, \quad \forall n = p^m,$$
$$D(p,n)^m = kJ_n, \quad \forall kn = p^m.$$

The De Bruijn matrices $D(p,n)$ for which $n = p^m$ are thus m-th roots of J_n. In the next section, we will show in particular that under a certain condition on the rank of the m-th roots of J_n, these are isomorphic to a row permutation of a De Bruijn matrix.

3. Factorizations into commuting factors

In this section we describe the factorization problem into two commuting factors over the $n \times n$ binary matrices:

$$AB = BA = J_n, \quad A, B \in \{0,1\}^{n\times n}. \tag{1}$$

We assume that the factors A and B represent regular graphs, say with in- and out-degree p for A and in- and out-degree l for B. We then have Theorem 3.2. In the proof, we use the following terminology.

Let P_1 and P_2 be two permutation matrices. Saying the permutations of P_1 are absorbed in P_2 means that we pose $P_2 := P_1P_2$.

The notation $A[a : b, c : d]$ refers to the submatrix of A with the rows of A indexed from a to b and the columns of A indexed from c to d.

Definition 3.1. Two matrices A and B are said to be isomorphic if they are permutation similar, that is, if there is a permutation matrix P such that $PAP^T = B$. We write $A \cong B$.

Theorem 3.2. *Let A and B be two regular graphs satisfying (1), then $pl = n$. Moreover,* $\operatorname{rank}(A) = n/p$ *(resp.* $\operatorname{rank}(B) = n/l$*) if and only if there is a permutation matrix P such that*

$$A \cong PD(p,n) \quad \bigl(\textit{resp. } B \cong PD(l,n)\bigr).$$

Proof. First of all, notice that we can suppose without loss of generality that the entries a_{00} and b_{00} of A and B respectively both equal 1. Indeed, since $BA = J$, there exist i, j such that $b_{ij}.a_{ji} = 1$. Hence, there are permutation matrices P_i, P_j such that the matrices $\tilde{B} = P_iBP_j^T$ and $\tilde{A} = P_jAP_i^T$ have their element $(0,0)$ equal to 1. Moreover,

$$\tilde{A}\tilde{B} = \tilde{B}\tilde{A} = J,$$

and if $\tilde{A}$ (resp. $\tilde{B}$) is isomorphic to a matrix of the form $PD(p,n)$ (resp. $PD(l,n)$), then so it is for A (resp. B). Consider therefore that $a_{00} = b_{00} = 1$. Since each column of B has l nonzero elements, there exists a row permutation P_2 such that $P_2Be_{0,n} = [\mathbf{1}_l^T, 0, \dots, 0]^T$. Therefore, the block A_1 of the first l columns of AP_2^T satisfies

$$A_1\mathbf{1}_l = \mathbf{1}_n, \qquad \mathbf{1}_n^T A_1 = p\mathbf{1}_l^T.$$

For such matrices there is a row permutation P_1 such that

$$P_1A_1 = \mathbf{1}_p \otimes I_l.$$

This implies that $pl = n$. Furthermore, since $a_{00} = b_{00} = 1$ we may assume that $e_{0,n}^T P_1 = e_{0,n}^T P_2 = e_{0,n}^T$. We thus have now

$$\big(P_1AP_2^T\big)(e_{0,p}\otimes I_l)=P_1A_1=\mathbf{1}_p\otimes I_l,\qquad \big(P_2BP_1^T\big)e_{0,n}=e_{0,p}\otimes \mathbf{1}_l$$

and obviously we also have

$$\big(P_1AP_2^T\big)\big(P_2BP_1^T\big)=\big(P_2BP_1^T\big)\big(P_1AP_2^T\big)=J.$$

It is then straightforward to see that in $P_2BP_1^T$ all elements (i,j) with $0\leqslant i\leqslant l-1$ and $j>0$ such that $j\equiv 0 \bmod l$ are zero. As a consequence, there is a permutation matrix P which permutes the $n-l$ last rows of $P_2BP_1^T$ in such a way that

$$\big(PP_2BP_1^T\big)e_{0,n}=e_{0,p}\otimes \mathbf{1}_l,\qquad \big(PP_2BP_1^T\big)e_{l,n}=e_{1,p}\otimes \mathbf{1}_l$$

and

$$\big(P_1AP_2^TP^T\big)(e_{0,p}\otimes I_l)=\mathbf{1}_p\otimes I_l.$$

Further in $PP_2BP_1^T$ all the entries (i,j) with $l\leqslant i\leqslant 2l-1$ and $j\neq l$ such that $j\equiv 0 \bmod l$ are zero. Repeating this process on the last $n-2l$ rows of $PP_2BP_1^T$ and absorbing all row permutations in P_2 we have permutation matrices P_1, P_2 such that

$$\big(P_1AP_2^T\big)(e_{0,p}\otimes I_l)=\mathbf{1}_p\otimes I_l$$

and for any $i\equiv 0 \bmod l$,

$$\big(P_2BP_1^T\big)e_{i,n}=e_{i/l,p}\otimes \mathbf{1}_l.$$

In addition, since $(P_2BP_1^T)(P_1AP_2^T)=J$, there is a permutation matrix P such that,

$$e_{0,n}^T\big(P_2BP_1^TP^T\big)=e_{0,p}^T\otimes \mathbf{1}_l^T\quad\text{and}\quad \big(PP_1AP_2^T\big)(e_{0,p}\otimes I_l)=\mathbf{1}_p\otimes I_l.$$

Absorbing P in P_1, since $(P_1AP_2^T)(P_2BP_1^T)=(P_2BP_1^T)(P_1AP_2^T)=J$, we notice that every block $P_1AP_2^T[0:l-1,i:i+l-1]$ ($i\equiv 0 \bmod l$) has exactly a 1 in each row and each column. Consequently, there is a permutation matrix P such that

$$\big(P_1AP_2^TP^T\big)(e_{0,p}\otimes I_l)=\mathbf{1}_p\otimes I_l,\qquad \big(e_{0,p}^T\otimes I_l\big)\big(P_1AP_2^TP^T\big)=\mathbf{1}_p^T\otimes I_l.$$

Let us absorb P in P_2. We thus have

$$P_1AP_2^T=\left[\begin{array}{c|c|c|c} I_l & I_l & \cdots & I_l\\ \hline I_l & & & \\ \hline \vdots & & & \\ \hline I_l & & & \end{array}\right],$$

where every block is of size l.

It follows from $\operatorname{rank}(A) = l = n/p$ that all blocks in $P_1AP_2^T$ must be I_l. Therefore, we can update P_1 and P_2 so that

$$P_1AP_2^T = D(p, n).$$

Since A and B are commuting factors, we have the same result for B. This concludes the proof. □

From the proof of Theorem 3.2, notice that in any case $\operatorname{rank}(A) \geqslant n/p$ and $\operatorname{rank}(B) \geqslant n/l$.

Remark 3.3. The commuting factors may not have a minimum rank. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

It is a solution to $A^2 = J_9$. However, $\operatorname{rank}(A) = 4 > 9/3$.

Remark 3.4. From Theorem 3.2, we could wonder whether the factors with minimum rank are in particular isomorphic to a De Bruijn matrix. The matrix

$$A = \operatorname{diag}(I_9, Q_2, Q_3)D(3, 27),$$

with

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

is a solution to $A^3 = J_{27}$ with a rank equal to 27/3. However, since $\operatorname{rank}(A^2) = 4 \neq 3 = \operatorname{rank}(D(3, 27)^2)$, A is not isomorphic to $D(3, 27)$.

Theorem 3.2 can be generalized to the case of any number of commuting factors of J_n.

Corollary 3.5. *Let $\{A_i\}_{i\in I}$ be a finite set of $n\times n$ binary matrices, each of them p_i-regular such that all their products commute, i.e. for any permutations σ_1, σ_2, $\prod_i A_{\sigma_1(i)} = \prod_i A_{\sigma_2(i)}$, and satisfy*

$$\prod_{i\in I} A_i = J_n.$$

Then, $\prod_{i\in I} p_i = n$. Moreover, for any $i \in I$, $\operatorname{rank}(A_i) = n/p_i$ if and only if there is a permutation matrix P such that

$$A_i \cong PD(p_i, n).$$

Proof. First of all, we prove that $\prod_{i\geqslant 2} A_i$ is a binary matrix. Indeed, it is clear that all the entries of $\prod_{i\geqslant 2} A_i$ are nonnegative integers. Suppose that entry (i, j) of $\prod_{i\geqslant 2} A_i$ is greater than one. Then, since A_1 is p_1-regular, column i of A_1 has at least one nonzero element, say $a_{ki} = 1$. Therefore, entry (k, j) of $A_1(\prod_{i\geqslant 2} A_i)$ is greater than one, which is a contradiction since $\prod_{i\in I} A_i = J_n$.

Moreover, $\prod_{i\geqslant 2} A_i$ is $(\prod_{i\geqslant 2} p_i)$-regular. Indeed,

$$\Bigl(\prod_{i\geqslant 2} A_i\Bigr)\mathbf{1}_n = p_2\Bigl(\prod_{i\geqslant 3} A_i\Bigr)\mathbf{1}_n = \cdots = \Bigl(\prod_{i\geqslant 2} p_i\Bigr)\mathbf{1}_n.$$

We identically show that

$$\mathbf{1}_n^T\Bigl(\prod_{i\geqslant 2} A_i\Bigr) = \Bigl(\prod_{i\geqslant 2} p_i\Bigr)\mathbf{1}_n^T.$$

Theorem 3.2 shows then the result for A_1. The same argument repeated on each factor completes the proof. $\square$

Theorem 3.2 can also be applied to the particular case of the binary roots of J_n. We thus have the following result.

Corollary 3.6. *Let A be a binary matrix satisfying*

$$A^m = J_n.$$

Then A is p-regular. Moreover, $\operatorname{rank}(A) = n/p$ if and only if there is a permutation matrix P such that

$$A \cong PD(p, n).$$

Proof. From Lemma 2.1, we know that A is p-regular. With the same argument as in the proof of Corollary 3.5, we can show that A^{m-1} is a binary matrix. Further, since $A\mathbf{1}_n = p\mathbf{1}_n$ and $\mathbf{1}_n^T A = p\mathbf{1}_n^T$, we deduce that A^{m-1} is p^{m-1}-regular. Theorem 3.2 completes the proof. $\square$

From the proof of Theorem 3.2, we deduce that even in the case of an m-th root A of J_n, $\operatorname{rank}(A) \geqslant n/p$. Remark 3.3 shows that the rank of A may be greater than n/p and Remark 3.4 shows that A may be nonisomorphic to the De Bruijn matrix even though A has a rank equal to n/p.

Notice that the previous corollary does not provide a full characterization of the roots with minimum rank since not any row permutation of the De Bruijn matrix is a root of J_n. In the following section, we complete the result of Corollary 3.6 by showing that P can always be chosen as being a block diagonal matrix.

4. Roots of J_n with minimum rank and De Bruijn matrices

From the proof of Theorem 3.2, we have deduced that any binary solution A to the equation $A^m = J_n$ (remind that A is then p-regular) has a rank of at least n/p. Moreover, we have shown in the previous section that any binary root with minimum rank is isomorphic to a matrix of the form $PD(p,n)$, where P is a permutation matrix. In this section, we show that P can always be chosen as being a block diagonal matrix.

Lemma 4.1. *Let $A \in \{0,1\}^{n\times n}$ such that $A^m = J_n$ and A p-regular. If $\operatorname{rank}(A) = n/p$, then A is isomorphic to a matrix of the form*

$$B \otimes \mathbf{1}_p^T,$$

where B is of the form

$$\begin{bmatrix} I_{n/p} \\ Q_2 \\ \vdots \\ Q_p \end{bmatrix},$$

with any $Q_i \in \{0,1\}^{(n/p)\times(n/p)}$ a permutation matrix.

Proof. From Lemma 2.1, we know that $\operatorname{trace}(A) = p$. So, we can assume without loss of generality that $a_{00} \neq 0$. Indeed, if it is not the case, since there exists a nonzero diagonal entry (i,i), then there is a permutation matrix P_i such that A is isomorphic to

$$P_i A P_i^T$$

with entry $(0,0)$ which is nonzero.

Since A is p-regular and since A^2 is a binary matrix (this can be proved with the same argument as in the proof of Corollary 3.5), there is a permutation matrix P such that

$$PAP^T = \begin{bmatrix} p\left\{ \overbrace{\begin{matrix} \mathbf{1}_p^T & 0 & \cdots & 0 \\ 0 & \star & \cdots & \star \\ \vdots & \vdots & & \vdots \\ & \star & \cdots & \star \end{matrix}}^{n} \right. \\ \vdots \end{bmatrix}.$$

With the same argument we can update P such that:

$$PAP^T = \begin{bmatrix} I_p \otimes \mathbf{1}_p^T & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Since the first row of A^2 is then $[\mathbf{1}_{p^2}^T \ 0 \ \cdots \ 0]$, we can update P such that

$$PAP^T = \begin{bmatrix} p \text{ blocks of size } p \left\{ \begin{matrix} I_p \otimes \mathbf{1}_p^T & 0 & \cdots & \cdots & 0 \\ 0 & I_p \otimes \mathbf{1}_p^T & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & I_p \otimes 1_p^T & 0 \end{matrix} \right. \\ \vdots \end{bmatrix}.$$

Moreover, since for any $1 \leqslant k \leqslant m-1$, the first row of A^k is $[\mathbf{1}_{p^k}^T \ 0 \ \cdots \ 0]$, by updating P, we see that A is isomorphic to

$$PAP^T = \begin{bmatrix} n/p \left\{ \begin{matrix} 1 & \cdots & 1 & & & & & & \\ & & & 1 & \cdots & 1 & & & \\ & & & & & & \ddots & & \\ & & & & & & 1 & \cdots & 1 \end{matrix} \right. \\ \vdots \end{bmatrix}.$$

Since A is p-regular with rank n/p, the rest of the matrix PAP^T is made of rows chosen among the first n/p ones. This matrix, isomorphic to A, has p blocks $B_1, \ldots, B_p$ with n/p rows each. Up to a row permutation, all these blocks are identical. Indeed, if it was not the case, a block B_i would have two identical rows. Hence, there would be a column such that in B_i, the sum of the elements in that column is greater than 1. However,

$$(PAP^T)^{m-1} = \begin{bmatrix} I_p \\ \vdots \end{bmatrix} \otimes \mathbf{1}_{n/p}^T.$$

Consequently, $(PAP^T)^m$ would not be a binary matrix, which is a contradiction.

Therefore, there is a permutation matrix P such that

$$PAP^T = \begin{bmatrix} I_{n/p} \\ Q_2 \\ \vdots \\ Q_p \end{bmatrix} \otimes \mathbf{1}_p^T,$$

where any $Q_i \in \{0,1\}^{(n/p)\times(n/p)}$ is a permutation matrix. $\square$

Theorem 4.2. *Let $A \in \{0,1\}^{n\times n}$ be p-regular. If $A^m = J_n$ and* $\operatorname{rank}(A) = n/p$, *then A is isomorphic to a matrix*

$$PD(p,n),$$

where $P = \operatorname{diag}(Q_1, \dots, Q_p)$ and any $Q_i \in \{0,1\}^{(n/p)\times(n/p)}$ is a permutation matrix.

Proof. We have seen in the previous lemma that A is isomorphic to a matrix

$$\begin{bmatrix} I_{n/p} \\ Q_2 \\ \vdots \\ Q_p \end{bmatrix} \otimes \mathbf{1}_p^T$$

which can be written as

$$P(\mathbf{1}_p \otimes I_{n/p}) \otimes \mathbf{1}_p^T,$$

with $P = \operatorname{diag}(I_{n/p}, Q_2, \dots, Q_p)$. Hence, A is isomorphic to

$$PD(p,n). \qquad \square$$

Of course, not all the matrices of the form $PD(p,n)$ like in the previous theorem are solutions to $A^m = J_n$. Indeed, let us have a look at the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the previous result is not a full characterization of the solutions with minimum rank. However, in the next section, we prove that, if $m = 2$, the previous result is actually a full characterization of the solutions with minimum rank.

5. A characterization of the binary solutions to $A^2 = J_n$ with minimum rank

In 1967, Hoffman [9] was interested in a characterization of the binary solutions to the equation $A^2 = J_n$. This is still an open problem and to our best knowledge, none subclass of solutions has been characterized. In this section, we provide a characterization of the solutions to $A^2 = J_n$ with minimum rank.

The goal of this section is to prove Corollary 5.6. To do so, we need the following definitions.

Let Q be an $n \times n/p$ matrix with $n = p^m$. Q is made of n/p p-row blocks $C_1, \ldots, C_{n/p}$. C_1 contains the first p rows of Q, C_2 the p next ones, etc.

Definition 5.1. Let Q be an $n \times n/p$ binary matrix with $n = p^m$. Its reduced p-form Q_r^0 is an $n/p \times n/p$ matrix whose i-th row is the sum of the p rows in the i-th p-row block C_i of Q.

Example 5.2. Q_r^0 is the reduced 2-form of Q

$$Q = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \qquad Q_r^0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix}.$$

Remind that $I_{n/p}$ denotes the identity matrix of dimension n/p.

Notice that the reduced p-form of Q can be written as $Q_r^0 = (I_{n/p} \otimes \mathbf{1}_p^T)Q$.

Let $Q_1, \ldots, Q_p$ be matrices of dimension n/p. The notation $Q := [Q_1; \ldots; Q_p]$ refers to the $n \times n/p$ matrix

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_p \end{bmatrix}.$$

Lemma 5.3. *Let A, B be two matrices of the form $A := [Q_1; \ldots; Q_p] \otimes \mathbf{1}_p^T$ and $B := [R_1; \ldots; R_p] \otimes \mathbf{1}_p^T$, where any Q_i and any R_i is a matrix of dimension n/p. Then,*

$$AB = (QR_r^0) \otimes \mathbf{1}_p^T,$$

where $Q := [Q_1; \ldots; Q_p]$ *and* $R := [R_1; \ldots; R_p]$.

Proof. Notice that any matrix of the form $M(I_{n/p} \otimes \mathbf{1}_p^T)$ can be written as

$$M(I_{n/p} \otimes \mathbf{1}_p^T) = (M \otimes 1)(I_{n/p} \otimes \mathbf{1}_p^T) = M \otimes \mathbf{1}_p^T.$$

We can then write $A = Q(I_{n/p} \otimes \mathbf{1}_p^T)$ and $B = R(I_{n/p} \otimes \mathbf{1}_p^T)$.

Therefore,

$$\begin{aligned} AB &= Q(I_{n/p} \otimes \mathbf{1}_p^T)R(I_{n/p} \otimes \mathbf{1}_p^T) \\ &= QR_r^0(I_{n/p} \otimes \mathbf{1}_p^T) \\ &= (QR_r^0) \otimes \mathbf{1}_p^T. \qquad \square \end{aligned}$$

Definition 5.4. Let $Q_1, \ldots, Q_p$ be matrices of dimension n/p. Let Q be the $n \times n/p$ matrix defined as $Q := [Q_1; \ldots; Q_p]$. The sequence $\{Q_r^i\}$ is such that $Q_r^i := ((I_{n/p} \otimes \mathbf{1}_p^T)Q)^{i+1}$.

Theorem 5.5. *Let* $A \in \{0,1\}^{n\times n}$ *be a* p*-regular matrix.* A *is an* m*-th root of* J_n *with minimum rank if and only if* A *is isomorphic to a matrix of the form* $\operatorname{diag}(Q_1, \ldots, Q_p)D(p,n)$, *where any* Q_i *is a permutation matrix of dimension* n/p, *with* $Q_r^{m-2} = J_{n/p}$.

Proof. Let A be a matrix of the form $\operatorname{diag}(Q_1, \ldots, Q_p)D(p,n) = [Q_1; \ldots; Q_p] \otimes \mathbf{1}_p^T$, where any Q_i is a permutation matrix of dimension n/p. As usually, pose $Q := [Q_1; \ldots; Q_p]$. By applying repeatedly Lemma 5.3, we notice that $A^m = (QQ_r^{m-2}) \otimes \mathbf{1}_p^T$. So, since Q is made of permutation matrices, $A^m = J_n$ if and only if $Q_r^{m-2} = J_{n/p}$. Theorem 4.2 concludes the proof. $\square$

Corollary 5.6. *Let* $A \in \{0,1\}^{n\times n}$ *be a* p*-regular matrix.* A *is a binary solution to* $A^2 = J_n$ *with minimum rank if and only if* A *is isomorphic to a matrix of the form*

$$PD(p,n),$$

where $P = \operatorname{diag}(Q_1, \ldots, Q_p)$ *and any* $Q_i \in \{0,1\}^{p\times p}$ *is a permutation matrix.*

Proof. Since any Q_i is a $p \times p$ permutation matrix, it is clear that $Q_r^0 = J_p$. $\square$

6. A class of roots of J_n isomorphic to a De Bruijn matrix

In [1], it is proved that some g-circulant binary roots of J_n are isomorphic to a De Bruijn matrix. More specifically, the following result is shown.

Proposition 6.1. *Let A be a g-circulant binary solution to $A^m = J_n$ and A p-regular. If $g^m = 0$ $(mod\ n)$, then A is isomorphic to the De Bruijn matrix $D(p,n)$.*

In this section, we extend the results of [1] by identifying another class of binary solutions to $A^m = J_n$ isomorphic to a De Bruijn matrix.

Definition 6.2. A *nice permutation matrix* is built as follows: start with a $p \times p$ permutation matrix. Then, replace all the zeros by a $p \times p$ zero matrix and each one by a $p \times p$ permutation matrix. Repeat this m times. Then, you obtain a permutation matrix of dimension p^m. Such a matrix is called a nice permutation matrix.

An interpretation of a nice permutation matrix: an $n \times n$ matrix A such that $n = p^m$ for some integers p and m is made of p n/p-row blocks; the first block contains the first n/p rows of A, etc. In the same way, each of these blocks is made of p n/p^2-row blocks, and so on until we have 1-row blocks. So, we have a cascading block structure. Multiplying A to the left by a nice permutation matrix performs block permutations inside each set of p blocks with n/p^i rows included in a SAME block of n/p^{i-1} rows.

Definition 6.3. A nice permutation of the De Bruijn matrix $D(p,n)$ is a matrix of the form $PD(p,n)$, where P is a nice permutation matrix.

Definition 6.4. A nice permutation of level i $(1 \leqslant i \leqslant m)$ permutes blocks of p^{i-1} rows included in a same block of p^i rows.

Definition 6.5. A nice permutation matrix of level i is a nice permutation matrix performing only nice permutations of level i.

Notice that multiplying to the right a nice permutation $\tilde{D}(p,n)$ of the De Bruijn matrix $D(p,n)$ by P_i^T, where P_i is a nice permutation matrix of level i, is equivalent to performing nice permutations of level less than i on the rows of $\tilde{D}(p,n)$. This is illustrated in the following example.

Example 6.6. Consider the following nice permutation matrix of the De Bruijn matrix $D(2,8)$:

$$\tilde{D}(2,8) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

If P_3 is the nice permutation matrix of level 3 which permutes the two blocks of 4 rows, we have:

$$\tilde{D}(2,8)P_3^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that if we take these nice permutation matrices P_1 and P_2 of level 1 and 2 respectively

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

then $P_2P_1\tilde{D}(2,8) = \tilde{D}(2,8)P_3^T$.

The following lemma will be useful to prove that any nice permutation of the De Bruijn matrix $D(p,n)$ is isomorphic to $D(p,n)$.

Lemma 6.7. *Let $\tilde{D}(p,n)$ be a nice permutation of the De Bruijn matrix $D(p,n)$ (with $n = p^m$) and P_i be a nice permutation matrix of level i. Then, $P_i\tilde{D}(p,n)$ is isomorphic to $\tilde{D}(p,n)$.*

Proof. By induction on level i.

- If $i = 1$, it is clear that $P_i\tilde{D}(p,n) = P_i\tilde{D}(p,n)P_i^T$.
- Multiplying to the right $P_i\tilde{D}(p,n)$ by P_i^T is equivalent to performing nice permutations of level less than i on the rows of $P_i\tilde{D}(p,n)$.
  Hence, there is a nice permutation matrix $\tilde{P}$ performing only permutations of level less than i such that

$$\tilde{P}P_i\tilde{D}(p,n) = P_i\tilde{D}(p,n)P_i^T.$$

Therefore, $P_i\tilde{D}(p,n)P_i^T$ is a nice permutation of $D(p,n)$. Since $\tilde{P}^T$ is a product of nice permutation matrices of level less than i, by induction, we know that $\tilde{P}^T P_i\tilde{D}(p,n)P_i^T$ is isomorphic to $P_i\tilde{D}(p,n)P_i^T$ and therefore isomorphic to $\tilde{D}(p,n)$.
As a consequence, since $P_i\tilde{D}(p,n) = \tilde{P}^T P_i\tilde{D}(p,n)P_i^T$, $P_i\tilde{D}(p,n)$ is isomorphic to $\tilde{D}(p,n)$. □

Proposition 6.8. *Any nice permutation of the De Bruijn matrix $D(p,n)$ is isomorphic to $D(p,n)$.*

Proof. Any nice permutation of $D(p,n)$ can be written as $P_m \dots P_2 P_1 D(p,n)$, where any P_i is a nice permutation matrix of level i.

So, from the previous lemma, it follows that such a matrix is isomorphic to $D(p,n)$. □

Corollary 6.9. *Any nice permutation of the De Bruijn matrix $D(p,n)$ $(n = p^m)$ is an m-th root of J_n, isomorphic to $D(p,n)$.*

The example in Remark 3.4 shows that not any root of J_n with minimum rank is a nice permutation of a De Bruijn matrix.

7. An open problem

In this paper, we have shown that any m-th root of J_n with minimum rank is isomorphic to a row permutation of a De Bruijn matrix, whose row permutation is represented by a block diagonal matrix (see Theorem 4.2). We have also shown that if $m \neq 2$, then the opposite is false.

In the future, it would be interesting to have a full characterization of all the binary roots of J_n with minimum rank.

Acknowledgements

The authors acknowledge support from the Belgian Programme of Interuniversity Attraction Poles and an Action de Recherche Concertée (ARC) of the Wallonia–Brussels Federation.

References

[1] Y. Wu, R. Jia, Q. Li, g-Circulant solutions to the $(0,1)$ matrix equation $A^m = J_n$, Linear Algebra Appl. 345 (2002) 195–224.
[2] S.L. Ma, W.C. Waterhouse, The g-circulant solutions of $A^m = \lambda J$, Linear Algebra Appl. 85 (1987) 211–220.
[3] F. King, K. Wang, On the g-circulant solutions to the matrix equation $A^m = \lambda J$, J. Combin. Theory Ser. A 38 (1985) 182–186.
[4] K. Wang, On the g-circulant solutions to the matrix equation $A^m = \lambda J$, J. Combin. Theory Ser. A 33 (1982) 287–296.
[5] N.G. de Bruijn, A combinatorial problem, Nederl. Akad. Wetensch. Proc. 49 (1946) 758–764.

[6] M. Malyshev, V.E. Tarakanov, Generalized De Bruijn graphs, Math. Notes 62 (4) (1997) 449–456.
[7] N.S. Mendelsohn, An application of matrix theory to a problem in universal algebra, Linear Algebra Appl. 1 (1968) 471–478.
[8] J.-C. Delvenne, R. Carli, S. Zampieri, Optimal strategies in the average consensus problem, Systems Control Lett. 58 (2009) 759–765.
[9] A.J. Hoffman, Research problems 2–11, J. Combin. Theory 2 (1967) 393.