A Novel Scheme for Positive Real Balanced Truncation

Kari Unneland, Paul Van Dooren and Olav Egeland

***Abstract*— Here, model reduction, based on balanced truncation, of stable and passive systems will be considered. An overview over some of the already existing techniques will be given; Lyapunov balancing, Riccati balancing and Stochastic balancing. Subsequently a novel scheme for positive real balanced truncation will be proposed. This new method is a combination of the already existing Lyapunov balancing and Riccati balancing. Using Riccati balancing, the solution of two Riccati equations are needed to obtain passive reduced order systems. For the suggested method, only one Lyapunov equation and one Riccati equation are solved in order to obtain passive reduced order systems, which is less computationally demanding. A numerical example is given at the end to compare the approximation error of the different schemes.**

I. INTRODUCTION

Recently there has been an increasing interest in passivity preserving model reduction, and several methods have been suggested, e.g. [12], [3], [13], [4], [6]. Here the focus will be on balanced truncation methods. Compared to reduction algorithms based on Krylov-iterations or other parameter-matching schemes, the balanced truncation algorithms are suited for small systems up until an order around 1000. In the next sections, an overview over some of the most commonly used balanced truncation schemes are given; Lyapunov balancing, Riccati balancing and Stochastic balancing. Further a new algorithm is proposed; mixed gramian balancing, with the aim of keeping the passivity properties of the original system. At the end a numerical example is given to compare the different reduction schemes and to verify the efficiency of the proposed method.

II. REDUCTION BY BALANCED TRUNCATION

Given a n'th-order minimal LTI system in state space form

$$\dot{x} = Ax + Bu \quad (1)$$
$$y = Cx + Du, \quad (2)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$. The associated transfer function is

$$G(s) = C(sI - A)^{-1}B + D. \quad (3)$$

Model reduction deals with finding a reduced order system, with order $r \ll n$ which captures the main features of the original system, i.e. dynamics, system stability and passivity or structural properties. The reduced model of order $r \ll n$ is denoted as:

$$\dot{x_r} = A_r x_r + B_r u \quad (4)$$
$$y = C_r x_r + D_r u. \quad (5)$$

Many different coordinate systems can be used to describe the dynamical system in (1)-(2). Let $T \in \mathbb{R}^{n\times n}$ be a nonsingular matrix, and let the system undergo a similarity transformation;

$$\bar{x} = Tx \quad (6)$$
$$\dot{\bar{x}} = TAT^{-1}\bar{x} + TBu \quad (7)$$
$$y = CT^{-1} + Du. \quad (8)$$

These equations have the same dynamics for any nonsingular matrix T. The idea behind the balanced truncation is to transform the system in (1)-(2), by choosing T in terms of some physical measure, and discarding the parts of the dynamics which are less important in terms of this measure. The most commonly used method is Lyapunov balancing where the system is balanced in terms of controllability and observability.

III. LYAPUNOV BALANCING

Lyapunov balanced truncation was introduced to the systems and control society by [11]. The Lyapunov balancing procedure is based on information from two Lyapunov equations giving the controllability gramian, P, and the observability gramian, Q;

$$AP + PA^T + BB^T = 0 \quad (9)$$
$$A^TQ + QA + C^TC = 0. \quad (10)$$

Notice that the gramians are positive definite if the system is minimal. The idea behind the Lyapunov balancing is to transform the mathematical model to a basis where the states which are difficult to control are also difficult to observe, and the reduced model is obtained by discarding the states which have this property. In Table III the algorithm for finding the balancing transformations T and T^{-1}, from (6)-(8), based on the Lyapunov equations, is written out sequentially [2].

The physical interpretation of the Lyapunov balancing can be related to the L_2-norm of the input and the output of the system. The controllability gramian P is connected to the solution of the minimum L_2-norm problem [8]:

$$\min_{u\in L_2(-\infty,0)} \left\{ \int_{-\infty}^{0} u(t)^T u(t) dt \text{ s.t. } x(0) = x_0 \right\} = x_0^T P^{-1} x_0. \quad (11)$$

Kari Unneland is with the Department of Engineering Cybernetics (ITK), Norwegian University of Science and Technology (NTNU), 7491 Trondheim, Norway. unneland@itk.ntnu.no

Paul Van Dooren is with the Department of Mathematical Engineering (INMA), Universite Catholique de Louvain (UCL), 1348 Louvain la Neuve, Belgium. vdooren@csam.ucl.ac.be

Olav Egeland is with the Department of Engineering Cybernetics (ITK), Norwegian University of Science and Technology (NTNU), 7491 Trondheim, Norway. olav.egeland@ntnu.no

TABLE I

BALANCED TRUNCATION ALGORITHM

1) Choose a pair of positive definite matrices; P, Q.
2) Solve out Cholesky factor U of P; $P = UU^T$.
3) Do the eigenvalue decomposition of U^TQU; $U^TQU = K\Sigma^2K^T$.
4) Get the balancing transformations; $T=\Sigma^{1/2}K^TU^{-1}$, $T^{-1}=UK\Sigma^{-1/2}$.
5) Compute the balanced realizations; $\hat{A} = TAT^{-1}$, $\hat{B} = TB$, $\hat{C} = CT^{-1}$.
6) Truncate $\hat{A},\hat{B},\hat{C}$ to form the reduced order system A_r,B_r,C_r.

In this setting, the sizes of the eigenvalues of P describe how much energy, in terms of the L_2-norm, is needed to control the associated state eigenvector. The observability gramian, Q, is related to the L_2-norm of the output; if the system is released at $x(0) = x_0$ with $u(t) = 0, \forall t \geq 0$ the following equality holds:

$$\int_0^\infty y(t)^T y(t) dt = x_0^T Q x_0. \tag{12}$$

In this setting, the sizes of the eigenvalues of Q describe how much energy, in terms of the L_2-norm of the output, is produced when the associated state eigenvector is in free evolution.

When applied to asymptotically stable systems, the Lyapunov balancing preserves the stability of the system, but a property like passivity might not be preserved. In the next section, Riccati balancing, which keeps the passivity properties of a system will be presented.

IV. RICCATI BALANCING

Riccati balancing [5] is used to reduce positive real systems, and will keep the positive real properties of the system. If a system (A,B,C,D) is positive real, it will satisfy the positive real (PR) equations [9];

$$AR + RA^T = -B_l B_l^T \tag{13}$$
$$RC^T - B = -B_l D_l^T \tag{14}$$
$$-D - D^T = -D_l D_l^T \tag{15}$$

Here $R = R^T > 0$, B_l and D_l are to be solved from the equations (13)-(14). A dual pair of positive real equations can be obtained by pre- and postmultiplying (13) by R^{-1}, and premultiplying (14) by R^{-1}. By defining;

$$O \equiv R^{-1} \tag{16}$$
$$C_r \equiv -J^T B_l^T R^{-1} \tag{17}$$
$$D_r \equiv J^T D_l^T, \tag{18}$$

where J is an arbitrary matrix, and $JJ^T = I$, the dual positive real equations can be obtained;

$$A^T O + OA = -C_r^T C_r \tag{19}$$
$$OB - C^T = -C_r^T D_r \tag{20}$$
$$-D - D^T = -D_r^T D_r \tag{21}$$

These equations shows that the dual system $G^T(-s) = B^T(sI - A^T)^{-1}C^T + D^T$ of $G(s)$ is positive real. In Riccati balanced truncation the minimal solutions to (13)-(15) and (19)-(21) are used. These can be obtained by rewriting (13)-(15) and (19)-(21) as a dual pair of Riccati equations, and then solve for R and O;

$$AR + RA^T + (RC^T - B)(D + D^T)^{-1}(CR - B^T) = 0 \tag{22}$$
$$A^T O + OA + (OB - C^T)(D + D^T)^{-1}(B^T O - C) = 0. \tag{23}$$

Riccati balancing can now be achieved by substituting (R,O) with (P,Q) in the balanced truncation algorithm in Table I.

Another way of checking if a system is passive, is in terms of Lyapunov theory and the use of storage functions [15]. In these terms a system is said to be passive if there exists a storage function, $V(x) > 0$, such that the following inequality holds;

$$V(x) \leq V(x(0)) + \int_0^t s(u(t),y(t)) dt. \tag{24}$$

Here $s(u(t),y(t))$ is called the supply function, and describes the rate at which power is supplied to the system. Two quantities can be defined from the notion of a storage function [15]; the required supply, V_r, and the available storage, V_a. The required supply, V_r, is defined as;

$$0 \leq V_r(x_0) = \inf_{u(t)|x(0)=x_0} \left[\int_{-\infty}^0 s(u(t),y(t)) dt \right], \tag{25}$$

and it is the minimum amount of energy that must be injected into the system, in order to control the system to state x_0 at time 0. The solution of (22) is related to the required supply [12];

$$x_0^T R^{-1} x_0 = V_r(x_0). \tag{26}$$

In this setting, the sizes of the eigenvalues of R describe how much energy is needed to control the associated state eigenvector. Small eigenvalues of R implies that a large amount of energy is needed to reach the associated mode. R can be regarded as an input energy gramian; and here it is denoted as the required supply gramian.

The available storage is defined as;

$$0 \leq V_a(x_0) = \sup_{x(0)=x_0} - \left[\int_0^\infty s(u(t),y(t)) dt \right], \tag{27}$$

it is the maximum amount of energy which can be extracted from the system in free evolution. The solution of (23) is related to the available storage;

$$x_0^T O x_0 = V_a(x_0). \tag{28}$$

Here, the sizes of the eigenvalues of O describe how much energy can be extracted from the system in free evolution. Small eigenvalues of O implies that a small amount of energy can be extracted from the associated mode. O can be interpreted as an output energy gramian; and here it is denoted as the available storage gramian.

Another interpretation can also be given to the gramian pair (R,O) in terms of spectral factors and power spectrums [1]. Letting Φ be the power spectrum of the positive real minimal degree transfer function $Z(s)$, then we have the following relation;

$$\Phi = Z(s) + Z^T(-s) = V(s)V^T(-s) = W^T(-s)W(s). \tag{29}$$

Here $Z(s)$ is denoted as the phase system, $V(s)$ as the left spectral factor of $Z(s)$, and $W(s)$ as the right spectral factor of $Z(s)$. In this section, let us assume that the original system $G(s)$ equals the phase system $Z(s)$ in (29), then;

$$\Phi = G(s) + G^T(-s) = V(s)V^T(-s) = W^T(-s)W(s). \quad (30)$$

Subsequently, let (R, B_l, D_l) be the solution to the PR equations (13)-(15), then the left spectral factor associated with (R, B_l, D_l) is

$$V(s) = C(sI - A)^{-1}B_l + D_l. \quad (31)$$

By looking at the PR equations in (13)-(15), one can see that solving for the required supply gramian, R, of $G(s)$ is the same as solving for the controllability gramian, R, of the left spectral factor, $V(s)$;

$$AR + RA^T = -B_l B_l^T. \quad (32)$$

A similar result is also true for the available storage gramian. Let (O, C_r, D_r) be the solution to the DPR equations in (19)-(21), then the right spectral factor associated with (O, C_r, D_r) is

$$W(s) = C_r(sI - A)^{-1}B + D_r. \quad (33)$$

By looking at the DPR equations in (19)-(20), one can see that solving for the available storage gramian, O, of $G(s)$, is the same as solving for the controllability gramian, O, of the associated right spectral factor $W(s)$;

$$A^T O + OA = -C_r^T C_r \quad (34)$$

When applied to positive real systems, the Riccati balancing preserves the positive real properties of the system. In the next section stochastic balancing, a method closely related to the Riccati balancing will be presented. It can only preserve the stability of the system, but has some interesting phase matching properties which is worth investigating in relation with passive systems.

V. STOCHASTIC BALANCING

Stochastic balancing was introduced by [5] with the purpose of balancing stochastic systems. A more generalized approach was later presented in [9] together with an error bound. As for the Riccati balancing, the method is related to results from spectral factorization. In this section, let us assume that the original system $G(s)$ equals the left spectral factor $V(s)$ in (29);

$$\Phi = G(s)G^T(-s) = Z(s) + Z^T(-s) = W^T(-s)W(s) \quad (35)$$
$$G(s) = C(sI - A)^{-1}B_l + D_l. \quad (36)$$

Let R denote the controllability gramian of the system $G(s)$;

$$AR + RA^T + B_l B_l^T = 0. \quad (37)$$

Solving for the controllability gramian of $G(s)$ is the same as solving for the required supply gramian of the phase system $Z(s)$. When R is found, since (A, B_l, C, D_l) is given, the B in (14) can be solved for,

$$B = RC^T + B_l D_l^T. \quad (38)$$

Also having the relations (15) and (21),

$$D_l D_l^T = D + D^T = D_r^T D_r, \quad (39)$$

the available storage gramian of $Z(s)$ can be solved for,

$$A^T O + OA + (OB - C^T)(D_l D_l^T)^{-1}(B^T O - C) = 0. \quad (40)$$

Stochastic balancing uses the solution of (R, O) in the balancing scheme. Due to this, the stochastic balancing is based on balancing one gramian from a Lyapunov equation (37) and one gramian from a Riccati equation (40).

There is no similar interpretation of the stochastic balancing procedure, due to energy, as for the Lyapunov and Riccati balancing. But in [10] it is pointed out that stochastic balancing gives reduced order systems with closeness in phase to the original system, and from this closeness in magnitude. The stochastic balancing can be seen as a phase matching reduction technique. For SISO systems, one can check if the systems are passive by looking at their phase, which should be between $\mp 90^\circ$. This gives an indication that the algorithm might often give passive reduced order systems, since their phase will be well matched. In the next section a novel algorithm for positive real balanced truncation will be given, based on the results presented in the first sections.

VI. MIXED GRAMIAN BALANCING

In this section a new algorithm for achieving positive real reduced order systems is proposed. So far, the only algorithm presented which will guarantee positive real reduced order systems is the Riccati balancing. When Riccati balancing is used, the system is balanced based on the solution of two Riccati equations. Since the balanced system satisfy the PR equations, this gives positive real reduced order systems. The solution of two Riccati equations are computationally demanding. The idea behind the new approach is to solve one Riccati equation and one Lyapunov equation. As long as the PR equations are satisfied, this also holds for the balanced system and for the reduced system.

In this approach we balance the solution of one Riccati equation and one Lyapunov equation. Letting $G(s) = C(sI - A)^{-1}B + D$ denote the original system. By taking the controllability gramian, P in (9);

$$AP + PA^T + BB^T = 0 \quad (41)$$

and balancing it with the available storage gramian, O, from (23)

$$A^T O + OA + (OB - C^T)(D + D^T)^{-1}(B^T O - C) = 0, \quad (42)$$

positive real reduced systems are achieved. The same yields if one takes the observability gramian, Q, in (10),

$$A^T Q + QA + C^T C = 0 \quad (43)$$

and combines it with the required storage gramian, R, from (22),

$$AR + RA^T + (RC^T - B)(D + D^T)^{-1}(CR - B^T) = 0. \quad (44)$$

TABLE II

OVERVIEW OVER THE DIFFERENT BALANCING SCHEMES AND ASSOCIATED EQUATIONS TO BE SOLVED

	Left Spectral Factor $V(s)=H(sI-F)^{-1}G_l+J_l$	Phase System $Z(s)=H(sI-F)^{-1}G+J$	Right Spectral Factor $W(s)=H_r(sI-F)^{-1}G+J_r$
		$Z(s)=G(s)$ $G(s)=(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ given	
Lyapunov balancing; balance P and Q. Solve for P and Q.		Controllability gramian, P: $\mathbf{A}P+P\mathbf{A}^T+\mathbf{B}\mathbf{B}^T=0$ $\times$ Observability gramian, Q: $\mathbf{A}^{\mathbf{T}}Q+Q\mathbf{A}+\mathbf{C}^T\mathbf{C}=0$	
Riccati balancing; balance R and O. Solve for R and O.	Controllability gramian, R: $\Leftarrow$ $\mathbf{A}R+R\mathbf{A}^{\mathbf{T}}+G_lG_l^T=0$	$\Rightarrow$Required supply gramian, R: $\mathbf{A}R+R\mathbf{A}^T=-G_lG_l^T$ $R\mathbf{C}^T-\mathbf{B}=-G_lJ_l^T$ $-\mathbf{D}-\mathbf{D}^T=-J_lJ_l^T$ $\Updownarrow$ $\mathbf{A}R+R\mathbf{A}^T+(R\mathbf{C}^T-\mathbf{B})(\mathbf{D}+\mathbf{D}^T)^{-1}(\mathbf{C}R-\mathbf{B}^T)=0$ $\times$ Available storage gramian, O:$\Leftarrow$ $\mathbf{A}^TO+O\mathbf{A}=-H_r^TH_r$ $O\mathbf{B}-\mathbf{C}^T=-H_r^TJ_r$ $-\mathbf{D}-\mathbf{D}^T=-J_r^TJ_r$ $\Updownarrow$ $\mathbf{A}^TO+O\mathbf{A}+(O\mathbf{B}-\mathbf{C}^T)(\mathbf{D}+\mathbf{D}^T)^{-1}(\mathbf{B}^TO-\mathbf{C})=0$	$\Rightarrow$Observability gramian, O: $\mathbf{A}^TO+O\mathbf{A}+H_r^TH_r$
Mixed balancing; balance P and O, or balance R and Q. Solve for P and O, or solve for R and Q.	Controllability gramian, R:$\Leftarrow$ $\mathbf{A}R+R\mathbf{A}^T+G_lG_l^T=0$	Controllability gramian, P: $\mathbf{A}P+P\mathbf{A}^T+\mathbf{B}\mathbf{B}^T=0$ $\times$ Available storage gramian, O:$\Leftarrow$ $\mathbf{A}^TO+O\mathbf{A}+(O\mathbf{B}-\mathbf{C}^T)(\mathbf{D}+\mathbf{D}^T)^{-1}(\mathbf{B}^TO-\mathbf{C})=0$ or $\Rightarrow$Required supply gramian, R: $\mathbf{A}R+R\mathbf{A}^T+(R\mathbf{C}^T-\mathbf{B})(\mathbf{D}+\mathbf{D}^T)^{-1}(\mathbf{C}R-\mathbf{B}^T)=0$ $\times$ Observability gramian, Q: $\mathbf{A}^TQ+Q\mathbf{A}+\mathbf{C}^T\mathbf{C}=0$	$\Rightarrow$Observability gramian, O: $\mathbf{A}^TO+O\mathbf{A}+H_r^TH_r$
	$V(s)=G(s)$ $G(s)=(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ given		
Stochastic balancing; balance R and O. Solve for R, G and O.	Controllability gramian, R:$\Leftarrow$ $\mathbf{A}R+R\mathbf{A}^T+\mathbf{B}\mathbf{B}^T=0$	$\Rightarrow$Required supply gramian, R: $\mathbf{A}R+R\mathbf{A}^T=-\mathbf{B}\mathbf{B}^T$ $G=R\mathbf{C}^T+\mathbf{B}\mathbf{D}^T$ $-J-J^T=-\mathrm{DD}^T$ $\times$ Available storage gramian, O:$\Leftarrow$ $\mathbf{A}^TO+O\mathbf{A}+(OG-\mathbf{C}^T)(\mathbf{D}\mathbf{D}^T)^{-1}(G^TO-\mathbf{C})=0$	$\Rightarrow$Observability gramian, O $\mathbf{A}^TO+O\mathbf{A}+H_r^TH_r$

Now, the following definition of a mixed gramian balanced system can be given,

Definition 1: The positive real minimal system $G(s)$ is called mixed gramian balanced if the system is balanced based on the gramian pair (P,O) from (9) and (23) or on the gramian pair (Q,R) from (10) and (22).

$$P=O=\Sigma=diag(\sigma_1 I_{m1},\ldots,\sigma_q I_{mq}) \tag{45}$$

$$\text{or} \tag{46}$$

$$R=Q=\Sigma=diag(\sigma_1 I_{m1},\ldots,\sigma_q I_{mq}), \tag{47}$$

where $\sigma_1>\sigma_2>\ldots>\sigma_q>0$, m_i, $i=1,\ldots,q$ are the multiplicities of σ_i and $m_1+\ldots+m_q=n$.

The following theorem can now be stated.

Theorem 1: Let the positive real and minimal system $G(s)$ have the mixed gramian balanced realization,

$$G(s)=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]=\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right], \tag{48}$$

with $P=O=\Sigma=diag(\Sigma_1,\Sigma_2)$ or $R=Q=\Sigma=diag(\Sigma_1,\Sigma_2)$ where $\Sigma_1=diag(\sigma_1 I_{m_1},\ldots,\sigma_k I_{m_k})$ and $\Sigma_2=diag(\sigma_{m_{k+1}} I_{m_{k+1}},\ldots,\sigma_q I_{m_q})$. Then the reduced order model,

$$G_r(s)=\begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix} \tag{49}$$

obtained by truncation is positive real.

Proof: Since (A,B,C,D) is balanced, the two gramians, (P,O) are equal, $P=O=\Sigma$, and satisfy one Lyapunov equation (9) and one Riccati equation (23),

$$A\Sigma+\Sigma A^T+BB^T=0 \tag{50}$$

$$A^T\Sigma+\Sigma A+(\Sigma B-C^T)(D+D^T)^{-1}(B^T\Sigma-C)=0. \tag{51}$$

Writing out the second equation in terms of its partitioned matrices gives the following $(1,1)$ block;

$$A_{11}^T\Sigma_1+\Sigma_1 A_{11}+(\Sigma_1 B_1-C_1^T)(D+D^T)^{-1}(B_1^T\Sigma_1-C_1)=0. \tag{52}$$

Since $\Sigma_1>0$ the positive realness of the reduced order system (A_{11},B_1,C_1,D) can be concluded.

The same can be shown for the pair (R,Q). Since (A,B,C,D) is balanced, the two gramians, (R,Q) are equal, $R=Q=\Sigma$, and satisfy one Lyapunov equation (9) and one Riccati equation (23),

$$A\Sigma+\Sigma A^T+(\Sigma C^T-B)(D+D^T)^{-1}(C\Sigma-B^T)=0 \quad (53)$$
$$A^T\Sigma+\Sigma A+C^TC=0 \quad (54)$$

Writing out the first equation in terms of its partitioned matrices gives the following $(1,1)$ block;

$$A_{11}\Sigma_1+\Sigma_1 A_{11}^T+(\Sigma_1 C_1^T-B_1)(D+D^T)^{-1}(C_1\Sigma_1-B_1^T)=0 \quad (55)$$

Since $\Sigma_1>0$ the positive realness of the reduced order system (A_{11},B_1,C_1,D) can be concluded. ■

Further it can be shown that the reduced order system (A_{11},B_1,C_1,D) will be the same, either if the gramian pair (P,O) or the gramian pair (R,Q) is used as a basis for the mixed balanced truncation algorithm. The dual system of $G(s)$ in (3) can be written as:

$$G^T(-s)=\left[\begin{array}{cc} A^T & C^T \\ B^T & D^T \end{array}\right]. \quad (56)$$

Substituting the system $G(s)$ with the dual system $G^T(-s)$ in the equations (41)-(42), we get the following equations;

$$A^TP+PA+C^TC=0 \quad (57)$$
$$AO+OA^T+(OC^T-B)(D^T+D)^{-1}(CO-B^T)=0. \quad (58)$$

Solving for these equations, where the original system, $G(s)$, has been substituted with its dual, $G^T(-s)$, is the same as solving for the required supply gramian (43) and controllability gramian (44) of the original system, $G(s)$.

Subsequently, the system $G(s)$ is replaced with the dual system $G^T(-s)$ in the equations (43)-(44):

$$A^TR+RA+(RB-C^T)(D^T+D)^{-1}(B^TR-C)=0 \quad (59)$$
$$AQ+QA^T+BB^T=0 \quad (60)$$

Solving for these equations, is the same as solving for the controllability gramian (41) and the available storage gramian (42) of the original system $G(s)$.

Due to this duality, which comes from the duality of the Lyapunov equations (9)-(10) and the Riccati equations (22)-(23), the balanced system (A,B,C,D) from either using (P,O) or (R,Q) as a basis for the balanced truncation will be equal. Hence the reduced order system (A_{11},B_1,C_1,D) will also be equal.

In Table II an overview over the different balancing schemes are given, and the relations to the different spectral factors of the system $G(s)$ are given.

In the next section an numerical example will be given, where the different algorithms are applied with the aim of keeping the passivity properties of the model.

VII. NUMERICAL EXAMPLE

The 3 degrees of freedom dynamical positioning equations for a marine vessel can be written as follows [7];

$$M\dot{\nu}+C_{RB}(\nu)\nu+\tau_{pd} = \tau_c+\tau_A+\tau_{FK+diff} \quad (61)$$
$$\dot{x} = Ax+B\nu \quad (62)$$
$$\tau_{pd} = Cx+D\nu. \quad (63)$$

Here $\nu=[u,v,r]$ is the velocity vector consisting of the velocities in surge, sway and yaw. $M\in\mathbb{R}^{6\times 6}$ is the rigid-body inertia matrix, C_{RB} is the rigid-body Coriolis and centripetal matrix, τ_{pd} represents the potential damping of the system, τ_A represents the control forces, τ_c represents the current forces and $\tau_{FK+diff}$ represents the Froude-Krylov and diffraction forces. In this model the surge dynamics are decoupled from the dynamics in sway and yaw. This gives one SISO-system representing the dynamics in surge and one MIMO-system representing the dynamics in sway and yaw.

The subsystem (62)-(63) representing the potential damping forces is usually identified from data from towing tank experiment or hydrodynamical software (Wamit, Veres, Octopus Seaway), and the realizations are of high order. This makes model reduction necessary in order to make the overall model (61)-(63) efficient for simulation or control design. It can be shown that the mapping from $\nu\mapsto\tau_{pd}$ is passive [14], and this property should be kept during the reduction.

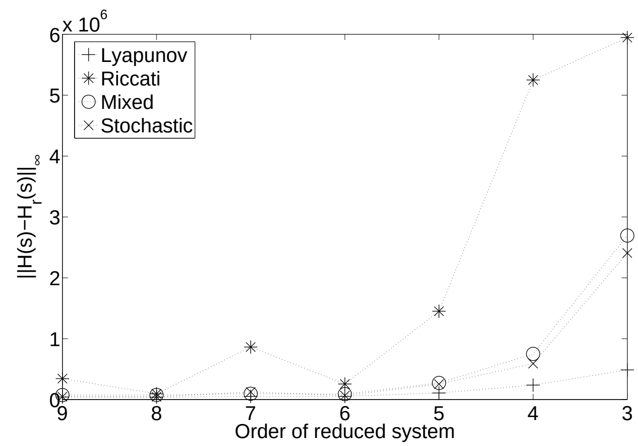

Fig. 1. $\|H(s)-H_r(s)\|_\infty$ for different order of the reduced system, applying the different reduction schemes, where $H(s)$ is of order 10. Notice that Lyapunov balancing does not generate passive systems for a order less than 5.

The vessel used is a $S-175$ tanker, and an $10th$-order model of the dynamics in surge is achieved. This model is reduced to different orders, with the different schemes. The worst case error between the original system and the reduced order system, $\|H(s)-H_r(s)\|_\infty$, is plotted in Figure 1, for different order of the reduced system. The reduced order systems of order lower than 5, obtained with Lyapunov balancing, are no longer passive. For all the other reduction schemes, passive reduced order systems are obtained. As one can see from Figure 1 the stochastic balancing and mixed gramian balancing schemes gives reduced order systems with less $\|\cdot\|_\infty$-error than the Riccati scheme. Stochastic balancing does not ensure passive reduced order systems, but since it has nice properties in terms of phase matching, as mentioned in section V, it seems as if it very often gives passive reduced order systems. This is well reflected in the results which is obtained here.

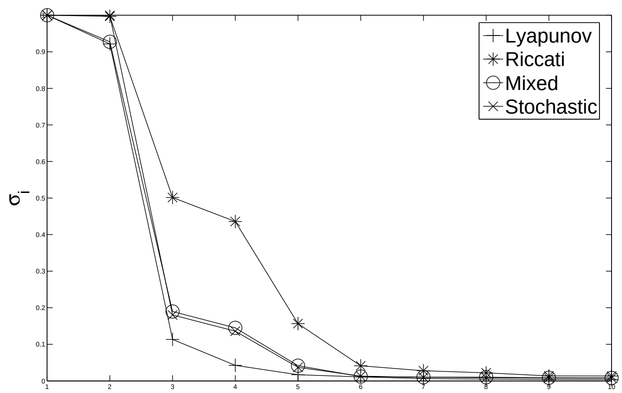

Fig. 2. The normalized singular values obtained for the system, using the different balancing schemes

Looking at the normalized singular value distribution which is plotted in Figure 2, the difference in the $\|\cdot\|_\infty$-error for the different schemes is reflected. As one can see, the singular values obtained from the system where Riccati balancing has been used, decreases slower than for the other schemes. By discarding a state from the Riccati balanced system, much more of the dynamics is canceled, when the distribution is like this.

VIII. CONCLUSIONS AND FUTURE WORK

A novel scheme for obtaining positive real balanced truncated systems has been presented. The new approach, mixed gramian balancing, is a combination of the already existing Lyapunov balancing and Riccati balancing schemes. Compared to Riccati balancing, which ensures passive reduced order systems, it has nice properties. Instead of solving for two Riccati equations, one Lyapunov and one Riccati equation is solved, which is computationally less demanding. A numerical example has also been given, and looking at the $\|\cdot\|_\infty$-error of the original system and reduced order system, the new scheme seems to bee competitive with Ricccati balancing.

Future work will concentrate on finding a computationally bound for the reduction error when using mixed gramian balancing. It is also of interest to check how it works when applied to MIMO-systems.

IX. ACKNOWLEDGMENTS

This project was partially supported from the Center for Ships and Ocean Structures (CeSOS) at the Norwegian University of Science and Technology (NTNU), through the Research Council of Norway.

REFERENCES

[1] B.D.O. Anderson and S. Vongpantilerd. *Network Analysis and Synthesis*. Prentice-Hall, 1973.
[2] A.C. Antoulas. *Approximation of Large-Scale Dynamical Systems*. SIAM, 2005.
[3] A.C. Antoulas. A new result on positive real interpolation and model reduction. *System & Control Letters*, 54:361–374, 2005.
[4] C.I. Byrnes, T.T. Georgiu, and A. Lindquist. A generalized entropy criterion for nevanlinna-pick interpolation with degree constraint. *IEEE Transactions on Automatic Control*, 46:822–839, 2001.
[5] U.B. Desai and D. Pal. A transformation approach to stochastic model reduction. *IEEE Transactions on Automatic Control*, AC-29:1097–1100, 1984.
[6] H. Fassbender and P. Benner. Passivity preserving model reduction via a structured lanczos method. In *Proceeding of the 2006 Intl. Symp. CCA/CACSD/ISIC*, 2006.
[7] T.I. Fossen and Ø.N. Smogeli. Nonlinear time-domain strip theory formulation for low-speed manouvering and station-keeping. *Modelling, Identification and Control*, MIC-25:201–221, 2004.
[8] K. Glover. All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds. *International Journal of Control*, 39:1115–1193, 1984.
[9] M. Green. Balanced Stochastic Realizations. *Linear Algebra and its Applications*, 98:211–247, 1988.
[10] M. Green and B.D.O. Anderson. Model Reduction by Phase Matching. *Mathematich of Control, Signals and Systems*, 2:221–263, 1989.
[11] B.C. Moore. Principal component analysis in linear systems: Controllability, observability, and model reduction. *IEEE Transactions on Automatic Control*, 23:17–32, 1981.
[12] J.R. Phillips, L. Daniel, and L.M. Silveira. Guaranteed passive balancing transformations for model order reduction. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 22:1027–1041, 2003.
[13] D.C. Sorensen. Passivity preserving model reduction via interpolation of spectral zeros. *System & Control Letters*, 54:347–360, 2005.
[14] K. Unneland and O. Egeland. Positive real modeling of ships for dynamic positioning. In *In Proc. of 17th International Symposium on Mathematical Theory of Networks and Systems, July 24-28, Kyoto, Japan*, 2006.
[15] J.C. Willems. Dissipative dynamical systems, Part I: General Theory. *Arch. Rat. Mech. An.*, 45:321–351, 1971.