Paul Van Dooren

Gramian Based Model Reduction of Large-Scale Dynamical Systems

Abstract

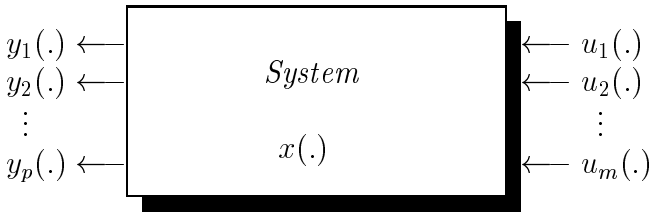
We describe model reduction techniques for large scale dynamical systems, modeled via systems of equations of the type

$$\begin{cases} F(\dot{x}(t), x(t), u(t)) = 0 \\ y(t) = H(x(t), u(t)), \end{cases}$$

as encountered in the study of control systems with input $u(t) \in \Re^m$, state $x(t) \in \Re^N$ and output $y(t) \in \Re^p$. These models arise from the discretization of continuum problem and correspond to sparse systems of equations $F(.\ ,.\ ,.)$ and $H(.\ ,.)$. The state dimension N is typically very large, while m and p are usually reasonably small. Although the numerical simulation of such systems may still be viable for large state dimensions N, most control problems of such systems are of such high complexity that they require model reduction techniques, i.e. techniques that construct a lower order model via a projection on a state space of lower dimension. We survey such techniques and put emphasis on the case where $F(.\ ,.\ ,.)$ and $H(.\ ,.)$ are linear time-invariant or linear time-varying.

1 Linear, time-varying and nonlinear models

Model reduction can have a lot of meanings. In this paper we refer to the problem of representing a complex dynamical system by a much simpler one. More precisely, we focus on dynamical systems describing a relation between a vector $u(.) \in \Re^m$ of input functions of time and a vector $y(.) \in \Re^p$ of output functions of time. The relation between inputs and outputs is a "system" $\mathcal{S}$ which is often represented as a "black box":



and uses an internal "state" $x(.) \in \Re^N$ to describe the relation between inputs and outputs.

Even though the original physical model typically has an infinite dimensional state-space to start with we assume that the model has already been discretized in space by e.g. using a finite element model. This however leads to a state-space dimension N that is typically very large if one wants to obtain a sufficient approximation of the true physical phenomenon. On the other hand, the resulting models are then often sparse and when simulating such systems using ODE techniques, this can be exploited to yield a reasonable complexity.

For control applications one has often to solve associated problems that are not sparse anymore and hence have a much higher complexity. Typical examples of this are optimal control or Kalman filtering over a finite horizon. These both require the solution of an adjoint problem without sparsity.

The models that are used in control systems design can be both continuous-time and discrete-time and are nonlinear in their most general form. For computational reasons these are typically linearized around their trajectory. The resulting linear time-varying systems can then subsequently be approximated by a time invariant one over a period of time where the model does not change too much. So we differentiate between six types of models that are used in this area. Notice that a discrete-time model will e.g. result from using an ODE solvers for the simulation of a corresponding continuous-time model. Moreover, we simplified the models by choosing *explicit* state equations, although the ideas presented in later sections generalize to a large extent to *implicit* state equations.

continuous-time	$\Longrightarrow$	discrete-time
$\left\{ \begin{array}{l} \dot{x}(t) = G(x(t), u(t)) \\ y(t) = H(x(t), u(t)) \end{array} \right.$		$\left\{ \begin{array}{l} x(k+1) = G(x(k), u(k)) \\ y(k) = H(x(k), u(k)) \end{array} \right.$
$\Downarrow$	linearize	$\Downarrow$
$\left\{ \begin{array}{l} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{array} \right.$		$\left\{ \begin{array}{l} x(k+1) = A(k)x(k) + B(k)u(k) \\ y(k) = C(k)x(k) + D(k)u(k) \end{array} \right.$
$\Downarrow$	freeze time	$\Downarrow$
$\left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{array} \right.$		$\left\{ \begin{array}{l} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{array} \right.$

In order to reduce the complexity of the solution of a control problem for any of these models one can use reduced order models. These are essentially projectors $P(.)$ of the state-space vector $x(.)$ to a vector $\hat{x}(.) \doteq P(.)x(.)$ of much lower dimension. The control problem is then solved in the lower dimensional setting, after which the solution can be "lifted back" to the original coordinate system. This of course does not yield the correct solution but an approximation whose quality depends on several factors. We now describe the different types of projection techniques that can be used for each of these models, starting with the simplest case, the linear time-invariant case. Throughout the paper we assume all signals and models are real. When we need an infinite dimensional setting we assume that we are working in a Hilbert space with the usual inner product leading to the ℓ_2 norm. We also assume that vectors (i.e. signals)

have bounded ℓ_2 norm.

2 Linear Time Invariant Systems

This is the simplest case of model reduction and it is also the one that has been studied the most. We formulate the approximation problem in continuous-time but everything extends to discrete-time systems as well. The original system is given by the state space equations

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{2.1}$$

where $u(t) \in \Re^m$, $y(t) \in \Re^p$ and $x(t) \in \mathcal{R}^N$ are the vectors of input variables, output variables and state variables, respectively. The input dimension m and output dimension p are assumed much smaller than the state dimension N. A reduced-order approximation of (2.1) takes the corresponding form

$$\begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \\ \hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}u(t), \end{cases} \tag{2.2}$$

where $\hat{y}(t) \in \Re^p$ and $\hat{x}(t) \in \mathcal{R}^n$. Notice that both systems are driven by the same input $u(t)$ and that their outputs $y(t)$ and $\hat{y}(t)$ are of the same dimension, and hence can directly be compared (this does not hold for $x(t)$ and $\hat{x}(t)$). The idea of model reduction is to find a smaller model (2.2) whose output $\hat{y}(t)$ is close to the original output $y(t)$. In other words, we are trying to construct a dynamical system of much lower complexity that nevertheless approximates well the behavior of the original system. How can we assess the quality of the reduced order model and how can we compute at low cost good approximations ? These are the questions to be addressed here.

2.1 Transfer functions and norms

Linear time invariant systems have a transfer function which (for continuous-time systems) is obtained under Laplace transform of the differential equations [8]. This yields

$$T(s) = C(sI_N - A)^{-1}B + D, \quad \hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D},$$

respectively, for the full and reduced order models. These transfer functions play an important role since the approximation problem can now be phrased in the frequency domain. Let

$$u_f(\omega) = \mathcal{F}u(t), \quad y_f(\omega) = \mathcal{F}y(t), \quad \hat{y}_f(\omega) = \mathcal{F}\hat{y}(t)$$

be the Fourier transforms of the corresponding time domain signals defined for $t \in (-\infty, +\infty)$ (we assume these signals to be of bounded ℓ_2 norm). If both transfer functions $T(s)$ and $\hat{T}(s)$ correspond to *stable* dynamical systems (i.e. their poles must lie in the open left half plane), then the inputs and outputs of these systems are related

at each frequency ω by the so-called frequency response of the systems defined as the transfer function evaluated at $s = j\omega$ [8] :

$$y_f(\omega) = T(j\omega)u_f(\omega), \quad \hat{y}_f(\omega) = \hat{T}(j\omega)u_f(\omega).$$

As a consequence we also have that the error $e(t) \doteq [y(t) - \hat{y}(t)]$ has a Fourier transform

$$\mathcal{F}e(t) = e_f(\omega) = [T(j\omega) - \hat{T}(j\omega)]u_f(\omega).$$

Since the Fourier transform is a linear transformation preserving the energy of a signal (namely $\|x_f(\omega)\|_2 = \sqrt{2\pi}\|x(t)\|_2$), the energy $\|e(t)\|_2$ of the error signal $e(t)$ is minimized for all unit norm inputs $u(t)$ by minimizing $\|e_f(\omega)\|_2$ for all $\|u_f(\omega)\|_2 = 1$. The so-called H_∞-norm, denoted by $\| \, . \, \|_\infty$, is defined as follows [14] :

$$\|T(j\omega)\|_\infty = \sup_{u_f \neq 0} \|T(j\omega)u_f(\omega)\|_2 / \|u_f(\omega)\|_2.$$

This norm is thus defined as the largest possible energy increase between inputs and outputs of a dynamical system and since Fourier transforms are linear, this holds as well in the time domain as in the frequency domain. For a stable transfer function, one shows that [14] :

$$\|T(.)\|_\infty \doteq \max_\omega \|T(j\omega)\|_2.$$

Applying this to the difference between the original and approximate system, we obtain

$$\|T(.) - \hat{T}(.)\|_\infty \doteq \max_\omega \|T(j\omega) - \hat{T}(j\omega)\|_2,$$

so that we will minimize the worst case error $\|e_f(\omega)\|_2$ by minimizing $\|T(.) - \hat{T}(.)\|_\infty$.

So the problem of model reduction of a stable linear time invariant systems can be stated as follows : *find the best stable approximation* $\hat{T}(.)$ *of a given degree for* $T(.)$ *where the error is measured in the* H_∞ *norm.*

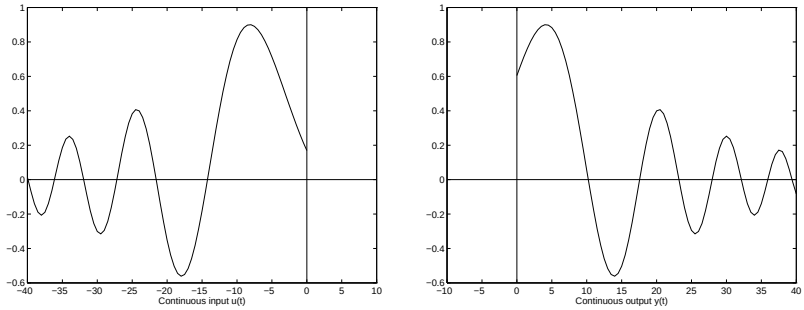
Unfortunately, it is not easy to find good (stable) approximations $\hat{T}(.)$ of a given (stable) transfer functions $T(.)$ using this norm. Another norm that is quite close to the H_∞-norm is the Hankel norm, which is still the largest possible energy increase between inputs and outputs, but where we restrict inputs to be nonzero in the interval $(-\infty, 0)$ (i.e. the "past") and outputs to be nonzero in the interval $[0, \infty)$ (i.e. the "future"). It turns out that the input/output map is then easy to describe and its norm easy to compute. For the continuous-time case we have

$$y(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)d\tau = Ce^{At} \cdot \int_0^\infty e^{A\tau}Bu(-\tau)d\tau, \quad t \in [0, \infty).$$

This map obviously factorizes in two sub-maps :

$$y(t) = Ce^{At}x(0), \quad x(0) = \int_0^\infty e^{A\tau}Bu(-\tau)d\tau.$$

The infinite dimensional "Hankel" operator $\mathcal{H}$ mapping $u(t)$, $t \in (-\infty, 0)$ to $y(t)$, $t \in [0, \infty)$ is then of rank at most N since $x(0) \in \Re^N$.



$$u(t),\ t \in (-\infty, 0) \Longrightarrow x(0) \Longrightarrow y(t),\ t \in [0, \infty)$$

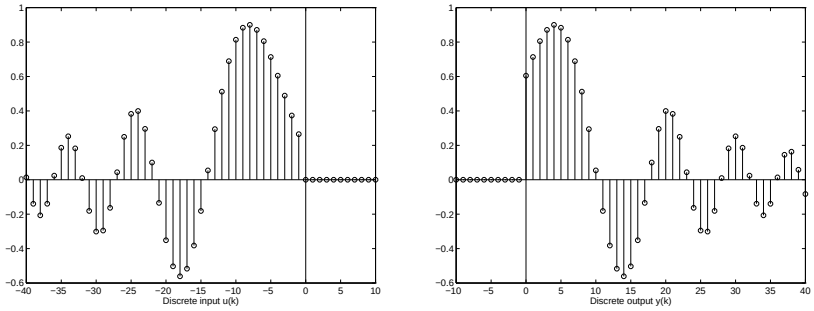
The same holds for discrete-time systems where the input/output relation becomes

$$y(k) = \sum_{-\infty}^{0} CA^{(k-j)}Bu(j) = CA^k \cdot \sum_{0}^{\infty} A^j Bu(-j), \quad k \in [0, \infty).$$

This map obviously factorizes in two sub-maps since

$$y(k) = CA^k x(0), \quad x(0) = \sum_{0}^{\infty} A^j Bu(-j).$$

The infinite dimensional "Hankel" operator $\mathcal{H}$ mapping $u(k),\ k \in (-\infty, 0)$ to $y(k),\ k \in [0, \infty)$ is then of rank at most N since $x(0) \in \Re^N$.



$$u(k),\ k \in (-\infty, 0) \Longrightarrow x(0) \Longrightarrow y(k),\ k \in [0, \infty)$$

Since the Hankel operator has finite rank N, it has only a finite number of nonzero singular values and these can easily be computed. For this, we represent the two linear maps as follows :

$$y([0, \infty)) = \mathcal{O}x(0), \quad x(0) = \mathcal{C}u((-\infty, 0)).$$

and define the dual maps as

$$\mathcal{O}^* : y([0, \infty)) \mapsto x(0), \quad \mathcal{C}^* : x(0) \mapsto u((-\infty, 0)).$$

2.2 Gramians

The products $G_o \doteq \mathcal{O}^*\mathcal{O}$ and $G_c \doteq \mathcal{C}\mathcal{C}^*$ are $N \times N$ matrices and are called the Gramians of the system.

For a continuous-time system (2.1) the observability and controllability Gramians are equal to [8] :

$$G_o = \int_0^{+\infty} (Ce^{At})^T (Ce^{At}) dt, \tag{2.3}$$

$$G_c = \int_0^{+\infty} (e^{At}B)(e^{At}B)^T dt, \tag{2.4}$$

which by Parseval's theorem are also equal to

$$G_o = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (-j\omega I - A^T)^{-1} C^T C (j\omega I - A)^{-1} d\omega, \tag{2.5}$$

$$G_c = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (j\omega I - A)^{-1} B B^T (-j\omega I - A^T)^{-1} d\omega. \tag{2.6}$$

These Gramians can be computed as the solution of the Lyapunov equations

$$A^T G_o + G_o A + C^T C = 0 \quad \text{and} \quad A G_c + G_c A^T + B B^T = 0. \tag{2.7}$$

For a discrete-time system the corresponding time domain definitions are :

$$G_o = \sum_{k=0}^{+\infty} (CA^k)^T (CA^k), \tag{2.8}$$

$$G_c = \sum_{k=0}^{+\infty} (A^k B)(A^k B)^T, \tag{2.9}$$

which by Parseval's theorem can be transformed to the frequency domain :

$$G_o = \frac{1}{2\pi} \int_0^{2\pi} (e^{-j\omega} I - A^T)^{-1} C^T C (e^{j\omega} I - A)^{-1} d\omega \tag{2.10}$$

$$G_c = \frac{1}{2\pi} \int_0^{2\pi} (e^{j\omega} I - A)^{-1} B B^T (e^{-j\omega} I - A^T)^{-1} d\omega. \tag{2.11}$$

These Gramians can again be computed as the solution of the Stein equations

$$A^T G_o A - G_o + C^T C = 0 \quad \text{and} \quad A G_c A^T - G_c + B B^T = 0. \tag{2.12}$$

How can one compute the norm of the Hankel map

$$\mathcal{H} = \mathcal{O}\mathcal{C} \ ?$$

One shows that for any two positive definite matrices there always exists a (so-called contragradient) transformation

$$\hat{G}_o \doteq T^T G_o T, \quad \hat{G}_c \doteq T^{-1} G_c T^{-T},$$

such that both new Gramians are equal and diagonal [9] :

$$\hat{G}_o = \hat{G}_c = \Lambda \doteq \text{diag } \{\lambda_1, \ldots, \lambda_N\}. \tag{2.13}$$

The transformed Gramians $\hat{G}_o$ and $\hat{G}_c$ are easily seen to be the new Gramians of the transformed system

$$\left\{\hat{A},\hat{B},\hat{C},\hat{D}\right\} \doteq \left\{T^{-1}AT, T^{-1}B, CT, D\right\} \overset{T}{\Longleftarrow} \{A,B,C,D\}\,.$$

Note that the corresponding state-to-outputs and inputs-to-state maps of the transformed system equal

$$\hat{\mathcal{O}} = \mathcal{O}T, \quad \hat{\mathcal{C}} = T^{-1}\mathcal{C}.$$

In this new coordinate system we thus have

$$\hat{\mathcal{O}}^*\hat{\mathcal{O}} = \Lambda = \hat{\mathcal{C}}\hat{\mathcal{C}}^*. \tag{2.14}$$

The transformation T is in fact a similarity that diagonalizes the product G_cG_o since

$$T^{-1}(G_cG_o)T = \hat{G}_c\hat{G}_o = \Lambda^2.$$

Moreover, the λ_i's are the singular values of the Hankel map since the maps

$$\mathcal{H}^*\mathcal{H} = \mathcal{C}^*\mathcal{O}^*\mathcal{O}\mathcal{C}, \quad \text{and} \quad \mathcal{C}\mathcal{C}^*\mathcal{O}^*\mathcal{O} = T\Lambda^2T^{-1}$$

have the same nonzero eigenvalues. Finally (2.14) also implies that Λ contains the square of the singular values of the transformed maps $\hat{\mathcal{O}}$ and $\hat{\mathcal{C}}$.

2.3 Approximation via projection methods

How is this now used for model reduction ? It is well known that the best approximation M_n of a given rank $n \ll N$ to a linear map M of rank N is obtained from the singular value decomposition of M. Notice that this decomposition always exists if M is bounded and has finite rank N to start with. Let us partition the eigenvalue matrix Λ and the system matrices $\{\hat{A},\hat{B},\hat{C}\}$ conformably such that the $(1,1)$ blocks are $n \times n$:

$$\Lambda \doteq \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix}, \hat{A} \doteq \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \hat{B} \doteq \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \hat{C} \doteq \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix}. \tag{2.15}$$

Define also

$$\hat{Y}^T \doteq \begin{bmatrix} I_n & 0 \end{bmatrix}, \quad \hat{X} \doteq \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad \hat{P} \doteq \hat{X}\hat{Y}^T, \tag{2.16}$$

then $\hat{P}$ is a projector since $\hat{Y}^T\hat{X} = I_n$ and $\hat{P}^2 = \hat{P}$. It follows from (2.14) that there exist orthogonal transformations U and V which we partition conformably, such that

$$\hat{\mathcal{O}} = \begin{bmatrix} U_1\Lambda_{11}^{\frac{1}{2}} & U_2\Lambda_{22}^{\frac{1}{2}} \end{bmatrix}, \hat{\mathcal{C}} = \begin{bmatrix} \Lambda_{11}^{\frac{1}{2}}V_1^T \\ \Lambda_{22}^{\frac{1}{2}}V_2^T \end{bmatrix}, \|\mathcal{H} - \hat{\mathcal{O}}\hat{P}\hat{\mathcal{C}}\|_2 = \|U_2\Lambda_{22}V_2^T\|_2 = \lambda_{n+1}.$$

The mapping $\hat{\mathcal{O}}\hat{P}\hat{\mathcal{C}}$ is therefore an optimal rank n approximation to $\mathcal{H}$ but it does not necessarily have a Hankel structure, in which case it does not correspond to a time invariant system [14]. It turns out that if the gap $\lambda_n - \lambda_{n+1}$ is large then the projected system

$$\left\{\hat{A}_{11}, \hat{B}_1, \hat{C}_1, D\right\} = \left\{\hat{Y}^T\hat{A}\hat{X}, \hat{Y}^T\hat{B}, \hat{C}\hat{X}, D\right\}$$

has a corresponding Hankel map that is very close to $\hat{\mathcal{O}}\hat{P}\hat{\mathcal{C}}$. For obvious reasons, this technique of producing reduced order models is called "balanced truncation". One shows [9] that balanced truncation of stable systems always produces stable reduced order models as well. This technique has the additional advantage to yield simultaneously *all* reduced order models of degree 1 up to $N-1$ and one can choose e.g. the order n of the approximate model based on the error estimate λ_n between both Hankel maps. A more involved model reduction technique that is derived from the balanced coordinate system is the so-called optimal Hankel norm approximation [4]. Its construction involves more work but yields a stable and optimal Hankel norm approximation of a given order n. When the gap $\lambda_n - \lambda_{n-1}$ is large, both techniques yield very close models [14].

2.4 Large-scale models

We pointed out that balanced truncation is based on the construction of two restrictions X and Y satisfying $Y^TX = I_n$ and hence of a projector $P = XY^T$. This projector $\hat{P}$ is orthogonal in the coordinate system of the balanced realization (2.16), but the corresponding projector P is not orthogonal in the original coordinate system. Defining

$$X = T\hat{X}, \quad Y^T = \hat{Y}^TT^{-1},$$

it follows that $Y^TX = \hat{Y}^TT^{-1}T\hat{X} = I_n$ and that the balanced truncation amounts to

$$\left\{\hat{A}_{11}, \hat{B}_1, \hat{C}_1, D\right\} = \left\{Y^TAX, Y^TB, CX, D\right\}$$

in the original coordinate system. The construction of the projector in the previous section was based on an eigenvalue decomposition of the product of two Gramians, which is a dense matrix even if the original system $\{A, B, C, D\}$ is sparse.

So how can one produce an "approximation" of this projector P without having to perform eigenvalue or singular value decompositions of dense $N \times N$ matrices ? For this it is informative to look at the Hankel map of the discrete-time case :

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = \mathcal{OC} \begin{bmatrix} u_{-1} \\ u_{-2} \\ u_{-3} \\ \vdots \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \dots \end{bmatrix} \begin{bmatrix} u_{-1} \\ u_{-2} \\ u_{-3} \\ \vdots \end{bmatrix}.$$

From this one expects Krylov sequences to play an important role in approximating both factors $\mathcal{O}$ and $\mathcal{C}$. A Krylov subspace is defined as follows :

$$\mathcal{K}_j(M, R) = Im\left\{R, MR, M^2R, \dots, M^{j-1}R\right\} \tag{2.17}$$

where the matrices M and R are $N \times N$ and $N \times k$, respectively. The construction of bases of Krylov spaces of a particular order have been studied for many years in the numerical linear algebra literature (especially for sparse M and R). The link with the present problem is obviously the eigenvalue problem since we would like to approximate the dominant eigenspace of the product of the Gramians $G_o G_c$. One wishes in fact to construct an n dimensional basis with $n \ll N$ and yet capture the dominant features of both Gramians.

For linear time invariant systems, one can exploit the frequency domain identities (2.6,2.11) very efficiently. They suggest that the Gramians are well approximated if the transfer function is interpolated in points where the expressions (2.6,2.11) are large. The transfer function $T(s)$ can be expanded in a Taylor series around any point σ that is not a pole of $T(s)$:

$$T(s) = T_0 + T_1(s-\sigma)^1 + T_2(s-\sigma)^2 + \cdots, \tag{2.18}$$

where the coefficients T_i – called moments – are equal to :

$$T_0 \doteq T(\sigma) = D - C(A-\sigma I)^{-1}B, \quad T_i \doteq \frac{1}{i!}\frac{\partial^i T(s)}{\partial s^i}\bigg|_{s=\sigma} = -C(A-\sigma I)^{-(i+1)}B, \ i > 0.$$

In general, one can produce a reduced-order model that interpolates the frequency response and its derivatives at multiple points $\{\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(K)}\}$. A model meeting these constraints is denoted a multipoint Padé approximation or a rational interpolant. Recently, a general theory for such interpolations, based on Krylov subspace computations was presented [5]. It is shown in [6] that these approximations also satisfy certain Galerkin conditions for the Lyapunov equations (2.7,2.12). Moreover, it is shown there that these techniques also apply to implicit systems of differential equations. Another approach is to use interpolation ideas to accelerate particular iterative methods for computing the solution of Lyapunov and Stein equations directly [10].

3 Time-varying systems

Time varying systems are described by systems of differential or difference equations. Assume that we are given a large-scale time-varying system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases} \quad \begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases} \tag{3.1}$$

which we would like to approximate by a lower order model

$$\begin{cases} \dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t) + \hat{B}(t)u(t) \\ \hat{y}(t) = \hat{C}(t)\hat{x}(t) + \hat{D}(t)u(t) \end{cases} \quad \begin{cases} \hat{x}(k+1) = \hat{A}(k)\hat{x}(k) + \hat{B}(k)u(k) \\ \hat{y}(k) = \hat{C}(k)\hat{x}(k) + \hat{D}(k)u(k). \end{cases} \tag{3.2}$$

It is obvious that projectors have to be time-varying as well in order to capture the dynamics of the system at each time instant. Since there is no transfer function

anymore, one should use instead the Gramians of the state-to-outputs and inputs-to-state maps.

Gramians of time-varying systems are based on the fundamental solution matrix $\Phi(.,.)$ of the corresponding homogeneous system, which yields the state at a particular time in terms of the state at a previous time [8] :

$$\begin{array}{cc} \dot{x}(t) = A(t)x(t), & x(k+1) = A(k)x(k), \\ x(T) = \Phi(T,t)x(t), & x(K) = \Phi(K,k)x(k). \end{array}$$

For a continuous-time system the observability and controllability Gramians can be defined as follows for a finite time window $t \in [t_i, t_f]$:

$$G_o(t) = \int_t^{t_f} (C(\tau)\Phi(\tau,t))^T (C(\tau)\Phi(\tau,t)) d\tau, \tag{3.3}$$

$$G_c(t) = \int_{t_i}^{t} (\Phi(t,\tau)B(\tau))(\Phi(t,\tau)B(\tau))^T d\tau. \tag{3.4}$$

Using $\frac{d}{d(-t)}\Phi(T,t) = \Phi(T,t)A(t)$ and $\frac{d}{dT}\Phi(T,t) = A(T)\Phi(T,t)$ we observe that the Gramians can be computed as the solution of the Lyapunov differential equations

$$\begin{array}{cc} \frac{d}{d(-t)}G_o(t) = A(t)^T G_o(t) + G_o(t)A(t) + C(t)^T C(t), & G_o(t_f) = 0, \\ \frac{d}{dt}G_c(t) = A(t)G_c(t) + G_c(t)A(t)^T + B(t)B(t)^T, & G_c(t_i) = 0. \end{array}$$

Notice that the first equation goes "backward" in time, while the second goes forward in time. For a discrete-time system the corresponding Gramians are defined for $k \in [k_i, k_f]$ as follows :

$$G_o(k) = \sum_{j=k}^{k_f} (C(j)\Phi(j,k))^T (C(j)\Phi(j,k)), \tag{3.5}$$

$$G_c(k) = \sum_{j=k_i}^{k} (\Phi(k,j)B(j))(\Phi(k,j)B(j))^T. \tag{3.6}$$

Using $\Phi(K,k) = \Phi(K,k+1)A(k)$ and $\Phi(K+1,k) = A(K)\Phi(K,k)$ we observe that the Gramians can be computed as the solution of the Lyapunov difference equations

$$\begin{array}{cc} G_o(k) = A(k)^T G_o(k+1)A(k) + C(k)^T C(k), & G_o(k_f+1) = 0, \\ G_c(k+1) = A(k)G_c(k)A(k)^T + B(k)B(k)^T, & G_c(k_i-1) = 0. \end{array}$$

Again we point out that both recurrences evolve differently with time. In the literature (see e.g. [11, 12]) one typically assumes $t_i = k_i = -\infty$ and $t_f = k_f = \infty$ but in order to be able to compute the Gramians we consider here a finite interval of time.

The continuous-time and discrete-time problems are very similar but since continuous-time systems need to be discretized anyway, we focus here on discrete-time systems.

The state-to-outputs and inputs-to-state maps of the discrete-time case are :

$$\begin{bmatrix} y(k) \\ y(k+1) \\ y(k+2) \\ \vdots \end{bmatrix} = \begin{bmatrix} C(k) \\ C(k+1)A(k) \\ C(k+2)A(k+1)A(k) \\ \vdots \end{bmatrix} x(k),$$

$$x(k) = \begin{bmatrix} B(k-1) & A_(k-1)B(k-2) & A(k-1)A(k-2)B(k-3) & \dots \end{bmatrix} \begin{bmatrix} u(k-1) \\ u(k-2) \\ u(k-3) \\ \vdots \end{bmatrix}.$$

The Gramians clearly reflect the *energy* of both these maps and hence play an important role in their approximation. Provided we compute only a finite window of the above Gramians then one can easily come up with a square root version for the Gramians. Let $R_o^T(k)R_o(k) = G_o(k)$ and $R_c^T(k)R_c(k) = G_c(k)$ be Cholesky factorizations of these Gramians then $R_o(k)$ and $R_c(k+1)$ are the upper triangular factors of the QR factorizations

$$Q_o(k)R_o(k) = \begin{bmatrix} R_o(k+1)A(k) \\ C(k) \end{bmatrix}, \quad Q_c(k+1)R_c(k+1) = \begin{bmatrix} R_c(k)A(k)^T \\ B(k)^T \end{bmatrix}$$

respectively. Rather than computing the exact factors, one can keep a low rank approximation of both Gramians

$$\hat{R}_o^T(k)\hat{R}_o(k) \approx G_o(k), \quad \hat{R}_c^T(k)\hat{R}_c(k) \approx G_c(k). \tag{3.7}$$

The matrices $\hat{R}_o(k+1)$ and $\hat{R}_c(k)$ will have e.g. $n \ll N$ rows, implying that at each step of the QR factorization only the n "dominant" rows of the triangular factors should be kept. The basic idea for this is to keep at each step the leading n row vectors of the singular value decomposition rather than performing the above QR decompositions. How to do this is described in a different context in [13], [7].

It is important to point out here that one can still define the eigenvalues of the product of the Gramians :

$$T^{-1}(k)G_c(k)G_o(k)T(k) = \Lambda(k)^2$$

and these will be positive real if the system is completely controllable and completely observable over the considered time interval [11, 12]. The contragradient transformation then exists and reduced order models can be constructed provided there is a nonzero gap $\lambda_n(k) - \lambda_{n+1}(k) > 0$ at each time instant. The optimal projector would be $P(k) = Y^T(k)X(k)$ where $X(k)$ contains the first n columns of $T(k)$ and $Y^T(k)$ contains the n first rows of $T^{-1}(k)$. It follows from (3.7) that a good approximation is given by

$$\hat{P}(k) \doteq \hat{R}_o^T(k)[\hat{R}_c(k)\hat{R}_o^T(k)]^{-1}\hat{R}_c(k)$$

and its quality will depend on the gap $\lambda_n(k) - \lambda_{n+1}(k) > 0$ at each time step.

For continuous-time problems with Gramians defined over infinite horizon intervals $(-\infty, t]$ and $[t, +\infty)$, it is shown in [11, 12] that if the original system is stable, then so will the reduced order model provided the original system is uniformly completely controllable and observable. If finite intervals are considered, the issue of stability is less crucial of course.

Notice that since we are defining projectors for finite time windows, this could also be applied to linear time invariant systems that are unstable. One can then not show any property of stability for the reduced order system, but the finite horizon Hankel map will at least be well approximated.

3.1 Time-varying linearized problems

The nonlinear problems in dynamical systems are much harder to handle

$$\begin{cases} \dot{x}(t) = G(x(t), u(t)) \\ y(t) = H(x(t), u(t)), \end{cases} \qquad \begin{cases} x(k+1) = G(x(k), u(k)) \\ y(k) = H(x(k), u(k)). \end{cases}$$

One typically linearizes such models along a "nominal" trajectory $(x(t), u(t))$ by computing $A(.), B(.), C(.)$ and $D(.)$ from the Taylor expansion of $G(.,.)$ and $H(.,.)$ around that trajectory. Model reduction techniques for time-varying models could then be applied to this, but the construction of the Gramians becomes too complex, since they evolve in two different time directions.

A simpler idea that has gained popularity in nonlinear systems is to just compute a trajectory $x(.)$ and consider

$$\int_{t_i}^{t_f} x(\tau)x(\tau)^T d\tau, \quad \text{and} \quad \sum_{k=k_i}^{k_f} x(k)x(k)^T,$$

as approximations of Gramians (or "energy functions") for constructing an appropriate projector. This is known as the Proper Orthogonal Decomposition (POD) technique [1].

How does this relate to the time-varying schemes described in the previous section ? If we consider a linear time-varying system :

$$\dot{x}(t) = A(t)x(t), \quad \text{and} \quad x(k+1) = A(k)x(k),$$

with initial conditions $x(t_i)$ and $x(k_i)$, respectively, then

$$x(t) = \Phi(t, t_i)x(t_i), \quad \text{and} \quad x(k) = \Phi(k, k_i)x(k_i).$$

The above expressions then become

$$\int_{t_i}^{t_f} (\Phi(\tau, t_i)x(t_i))(\Phi(\tau, t_i)x(t_i))^T d\tau, \quad \text{and} \quad \sum_{k=k_i}^{k_f} (\Phi(k, k_i)x(k_i))(\Phi(k, k_i)x(k_i))^T,$$

which shows the link with Gramians over a finite time interval. The difference here is that no input matrix $B(.)$ is involved.

4 Concluding remarks

Model reduction of dynamical systems has its roots in many different fields of applied mathematics. The earlier occurrence of such techniques is the approximation of rational functions of high degree by a lower degree one. The first results in that area were formulated in a mathematical setting and included techniques such as Padé approximations, continued fraction expansions and so on [3]. Such results were also used in the context of model reduction techniques or approximation techniques in application areas such as signals and systems and lead to the synthesis of approximating systems by one of a prespecified degree. Examples of algorithmic developments in this area are the Remez algorithm in filter design and the Massey Berlekamp algorithm in convolutional codes. More recent developments in linear systems theory are nicely synthesized in [2].

But these developments are referring mainly to the area of linear time-invariant dynamical systems. In this paper we tried to establish connections between different projection techniques used in the area of systems and control and in particular showed how to extend this to the time varying case and tried to show the connection with a popular technique for nonlinear dynamical systems.

References

[1] P. Holmes, J. Lumley and G. Berkooz, *Turbulence, coherent structures, dynamical systems and symmetry*, Cambridge Univ. Press, Cambridge, 1996.

[2] J. Ball, I. Gohberg and L. Rodman, *Interpolation of rational matrix functions*, Operator Theory : Advances and Applications **45**, Birkhäuser Verlag, Basel, 1990.

[3] C. Brezinski, *Padé-Type Approximation and General Orthogonal Polynomials*, **ISNM 50**, Birkhäuser, Basel, 1980.

[4] K. Glover, All optimal Hankel norm approximations of linear time multivariable systems and their L_∞-error bounds, *Int. J. Contr.* **39**, pp. 1115-1193, 1984.

[5] E. Grimme, K. Gallivan and P. Van Dooren, On Some Recent Developments in Projection-based Model Reduction, in *ENUMATH 97, 2nd European Conference on Numerical Mathematics and Advanced Applications*, H.G. Bock, F. Brezzi, R. Glowinski, G. Kanschat, Yu.A. Kuznetsov, J. Périaux, R. Rannacher (eds.), World Scientific Publishing, Singapore, pp. 98-113, 1998.

[6] E. Grimme, K. Gallivan and P. Van Dooren, Model Reduction of large-scale systems. Rational Krylov versus balancing techniques, in *Error Control and Adaptivity in Scientific Computing*, H. Bulgak and C. Zenger, Eds., Kluwer, pp. 177-190, 1999.

[7] K. Gallivan and P. Van Dooren, Recursive calculation of dominant singular subspaces, Internal Report, CESAME, Univ. catholique de Louvain, Belgium, 1999.

[8] T. Kailath, *Linear systems*, Prentice Hall, Englewood Cliffs, 1980.

[9] B.C. Moore, Principal component analysis in linear systems : controllability, observability and model reduction, *IEEE Trans. Aut. Contr.* **26**, pp. 17-32. 1981.

[10] T. Penzl, *A Cyclic Low Rank Smith Method for Large Sparse Lyapunov Equations with Applications in Model Reduction and Optimal Control*, T.U. Chemnitz, Dept. Mathematics, Preprint SFB393/98-6, 1998.

[11] S. Shokoohi, L. Silverman and P. Van Dooren, Linear time-variable systems: balancing and model reduction, *IEEE Trans. Aut. Contr.*, **28**, pp. 810-822, 1983.

[12] S. Shokoohi, L. Silverman and P. Van Dooren, Stable approximation of time-variable systems, *Automatica*, **20**, pp. 59-67, 1984.

[13] M. Verlaan and A. Heemink, Tidal flow forecasting using reduced rank square root filters, *Stochastic Hydrology and Hydraulics*, **11**, pp. 349-368, 1997.

[14] K. Zhou, J.C. Doyle and K. Glover, *Robust and optimal control*, Upper Saddle River, Prentice Hall, 1996.

Acknowledgement

This work was supported by the National Science Foundation under grant CCR-9796315. Parts of this paper presents research results of the Belgian Programme on Inter-university Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with its authors.
Discussions with K. Gallivan and N. Nichols regarding this paper are also gratefully acknowledged.

Paul M. Van Dooren
CESAME
Université catholique de Louvain
B-1348 Louvain-la-Neuve, Belgium
vdooren@csam.ucl.ac.be
http://www.auto.ucl.ac.be/~vdooren/