The Generalized Eigenstructure Problem in Linear System Theory

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Abstract—**The algebraic theory of linear time-invariant systems has been studied in large detail during the past few decades and numerous computational algorithms have been developed to solve problems arising in this context. In this paper the numerical aspects of a certain class of such algorithms—dealing with what the author calls generalized eigenstructure problems—are discussed. Some new and/or modified algorithms are presented. Both the numerical stability of the algorithms and the conditioning of the problems they solve are analyzed using numerical criteria.**

I. Preliminaries

THROUGHOUT this paper we use the following notation and conventions. Uppercase is used for matrices and lowercase for vectors and both are defined over the field of complex numbers $\mathbb{C}$ unless otherwise stated. We use A^T and A^* for the transpose and conjugate transpose of A and $\text{diag}\{A_1, A_2, \cdots, A_k\}$ denotes a block diagonal matrix whose blocks A_i are not necessarily square. A constant polynomial or rational matrix is called *regular* when it is square and has a nonzero determinant. It is called *singular* otherwise. The matrix norm $\|\cdot\|$ stands for both the Frobenius norm $\|\cdot\|_F$ and the spectral norm $\|\cdot\|_2$, which are both invariant under unitary transformations, i.e., $\|UAV\| = \|A\|$ if $U^*U = I$ and $V^*V = I$. The results stated in the text hold for both norms if not specifically mentioned. Sans Serif is used for vectorspaces. $\text{im}\, A$ and $\ker A$ denote, respectively, the image and the kernel of A; $A\mathsf{X}$ is the image of X under A. $\mathsf{X}+\mathsf{Y}$ and $\mathsf{X}\oplus\mathsf{Y}$ are the sum and direct sum, respectively, of the spaces X and Y. The orthogonal complement of X based on the natural inner product in a given coordinate system, is denoted by $\mathsf{X}^\perp$. An *invariant subspace* X of A satisfies $A\mathsf{X} \subset \mathsf{X}$ where $\subset$ denotes the inclusion; a *deflating subspace* X of a regular pencil $\lambda B - A$ satisfies $\dim(A\mathsf{X}+B\mathsf{X}) = \dim\mathsf{X}$ where dim stands for "dimension of." It is easy to check that deflating subspaces generalize the concept of invariant subspaces to arbitrary regular pencils. System models of the type

$$\begin{cases} T(\lambda)x(t) = U(\lambda)u(t) \\ y(t) = V(\lambda)x(t) + W(\lambda)u(t) \end{cases} \tag{1}$$

where $T(\lambda)$, $U(\lambda)$, $V(\lambda)$, and $W(\lambda)$ are polynomial matrices in λ and where $T(\lambda)$ is regular (i.e., $\det T(\lambda) \not\equiv 0$), are briefly denoted as $\{T(\lambda), U(\lambda), V(\lambda), W(\lambda)\}$. The vectors $u(t)$, $y(t)$, and $x(t)$ are, respectively, the input, output, and state of the *polynomial system* (1); λ can be the differential operator d/dt or the advance operator z, depending on the type of equations described by (1). The *transfer function* and *system matrix* of the system (1) are defined, respectively, as (we use $\triangleq$ when defining something in an equation)

$$R(\lambda) \triangleq V(\lambda)T^{-1}(\lambda)U(\lambda) + W(\lambda);$$

$$P(\lambda) \triangleq \left[\begin{array}{c|c} T(\lambda) & U(\lambda) \\ \hline -V(\lambda) & W(\lambda) \end{array}\right].$$

Additional notation is explained in the text.

In this paper we discuss numerical algorithms that compute some structural elements of some specific system models (state-space model, generalized state-space model, etc.) of a linear time-invariant system. These structural elements are of basic importance in most of the problems encountered in linear system theory and their computation can be viewed as a "generalized eignenstructure problem" [1]. This leads us to the analysis of these system theoretic problems and of the algorithms that solve them, using methods and criteria that are familiar from the "classical eigenvalue problem." The numerical background we will use is briefly surveyed in Section II and is more extensively treated in [2]–[4].

In the next three sections we discuss the computation of the structural elements connected to state-space and generalized state-space models. Each of these structural elements is given by the Kronecker canonical decomposition of an appropriate pencil $\lambda B - A$:

$$S(\lambda B - A)T = \text{diag}\left\{L_{l_1}, \cdots, L_{l_s}, L_{r_1}^T, \cdots, L_{r_t}^T, I - \lambda N, \lambda I - J\right\} \tag{2}$$

where i) S and T are constant invertible row and column transformations; ii) L_k is the $(k+1)\times k$ bidiagonal pencil

$$L_k \triangleq \underbrace{\begin{bmatrix} \lambda & & \\ -1 & \ddots & \\ & \ddots & \lambda \\ & & -1 \end{bmatrix}}_{k} \Bigg\} k+1;$$

and iii) N is nilpotent and both N and J are in Jordan canonical form.

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Using this canonical form, we call the elementary divisors of $\lambda I-J$ the *finite elementary divisors* of $\lambda B-A$, the elementary divisors of $\mu I-N$ the *infinite elementary divisors* of $\lambda B-A$, and the index sets $\{l_1,\cdots,l_s\}$ and $\{r_1,\cdots,r_t\}$ the *left and right Kronecker indexes* of $\lambda B-A$. The Kronecker canonical from (2) can be considered as the generalization of the Jordan canonical form of a pencil $\lambda I-A$ (see [5] for an extensive discussion).

From a numerical point of view [6] the computation of the canonical form (2) is not recommended because of the possible bad conditioning of the transformations S and T, whence numerical stability cannot be ensured. Instead, using unitary transformations U and V one can always reduce an arbitrary pencil $\lambda B-A$ to the form

$$U(\lambda B-A)V \triangleq \begin{bmatrix} \lambda B_l-A_l & 0 & 0 & 0 \\ \times & \lambda B_f-A_f & 0 & 0 \\ \times & \times & \lambda B_i-A_i & 0 \\ \times & \times & \times & \lambda B_r-A_r \end{bmatrix} \tag{3}$$

where i) λB_l-A_l and λB_r-A_r are nonsquare pencils with only *left* and *right* null spaces, respectively, and containing the corresponding *Kronecker indexes* of $\lambda B-A$; and ii) λB_i-A_i and λB_f-A_f are regular pencils which have only *infinite* and *finite elementary divisors*, respectively, which are those of $\lambda B-A$.

Furthermore, the form (3) can be obtained with a backward stable algorithm [6] which at the same time determines the Kronecker indexes and the infinite elementary divisors of $\lambda B-A$ through the fine structure obtained in λB_l-A_l, λB_r-A_r, and λB_i-A_i. This algorithm, described in [6], will be referred to as the *pencil algorithm*. The eigenstructure of λB_f-A_f (generalized eigenvalues and eigenvectors) can be computed in a stable way using the *QZ algorithm* [7] which constructs unitary transformations Q and Z that reduce a regular pencil—in this case λB_f-A_f—to a lower triangular form (see [9], [61] for more details)

$$Q(\lambda B_f-A_f)Z \triangleq \lambda \begin{bmatrix} b_{11} & & 0 \\ \times & \ddots & \\ & & b_{kk} \end{bmatrix} - \begin{bmatrix} a_{11} & & 0 \\ \times & \ddots & \\ & & a_{kk} \end{bmatrix} \tag{4}$$

The ratios $\lambda_i = a_{ii}/b_{ii}$ are called the *generalized eigenvalues* of λB_f-A_f and the vectors x_i satisfying $(\lambda_i B_f - A_f)x_i = 0$ are the corresponding *generalized eigenvectors*.

In the Sections III, IV, and V we show how to use modifications of these algorithms in order to compute several structural elements of state-space and generalized state-space models. In Section VI we take a look at linear systems represented by polynomial or rational matrices. We discuss some classical algorithms from a numerical point of view and give alternative algorithms for computing the structural information contained in the Smith and Smith–McMillan canonical forms. In the last section we give a brief historical review of the algorithms that lead to or are related to the ones mentioned in this paper. We also give some general comments and conclusions.

Most of the algorithms in this paper rely on a system theoretical background that is not completely elaborated here but adequate references are given where needed. Instead, we spend more time in discussing the numerical implications and difficulties since they are relatively unknown to researchers in this area.

II. Numerical Background

We briefly review concepts such as stability and conditioning by discussing a classical matrix problem that we will often use in the sequel. We intentionally delete tedious details and refer to the literature for a more rigorous discussion when needed. Instead we focus on the ill-posed nature of the problem because of the important role this plays in later sections.

Let A be an arbitrary $m\times n$ matrix. There always exist unitary transformations U and V such that

$$U^*A \triangleq \begin{bmatrix} A_r \\ \hline 0 \end{bmatrix} \begin{matrix} \}\rho \\ \}\mu \end{matrix} \qquad A\cdot V \triangleq \underbrace{[A_c}_{\rho} \mid \underbrace{0]}_{\nu} \tag{5}$$

where A_r and A_c have, respectively, ρ independent rows and columns (ρ is then clearly the rank of A). We call such transformations a *row* and *column compression* of the matrix A, respectively. A_r and A_c are said to have *full row rank*, respectively, *full column rank*. These decompositions can, e.g., be computed with the singular value decomposition (SVD) of the matrix A [8]:

$$A \triangleq U\cdot\Sigma\cdot V^* \tag{6a}$$

where i) U and V are, respectively, $m\times m$ and $n\times n$ unitary matrices and ii) Σ is an $m\times n$ matrix of the form

$$\Sigma \triangleq \left[\begin{array}{c|c} \Sigma_\rho & 0 \\ \hline 0 & 0 \end{array}\right] \quad \text{and} \quad \Sigma_\rho \triangleq \operatorname{diag}\{\sigma_1,\cdots,\sigma_\rho\} \tag{6b}$$

with σ_i being positive and satisfying $\sigma_1 \geq \cdots \geq \sigma_\rho > 0$.

It may be readily verified that U^*A and AV yield, respectively, a row compression and a column compression of A. The computation of such a decomposition is, of course, subject to rounding errors. Denoting computed quantities by an overbar, we generally have, for some error matrix E_A, that

$$\bar{A} \triangleq A + E_A = \bar{U}\bar{\Sigma}\bar{V}^*. \tag{7}$$

Hence, the computed decomposition does not correspond exactly to the given matrix A but rather to a "perturbed" version $A+E_A$. When using the SVD algorithm available in the literature [8], [9], this perturbation E_A can be bounded by

$$\|E_A\| \leq \epsilon_A \triangleq \Pi_A\cdot\epsilon\cdot\|A\| \tag{8}$$

where ϵ is the machine accuracy and Π_A is some polynomial expression in m and n [8]. Very often, this is a rough upper bound and one may prefer to replace Π_A by some statistical estimate $\hat{\Pi}_A$, usually "fairly" close to 1

(see, e.g., [2], [8]). The error E_A induced by this algorithm —called "*backward error*" because it is interpreted as an error on the data—has thus roughly the same norm as the input error E_i performed when reading in the data A in the computer. When such a bound exists for the perturbation E_A induced by a numerical algorithm, it is called *backward stable* [2].

Notice that backward stability does *not* warrant any bounds on the errors in the *result* $\bar{U}$, $\bar{\Sigma}$, and $\bar{V}$. This depends indeed on how perturbations on the data (namely, E_A) effect the resulting decomposition (or the differences $E_U \triangleq \bar{U}-U$, $E_\Sigma \triangleq \bar{\Sigma}-\Sigma$, and $E_V \triangleq \bar{V}-V$). This is commonly measured by the *condition* $\kappa[f_x(A)]$ of the computed object $X=f_x(A)$:

$$\kappa[f_x(A)] \triangleq \lim_{\delta\to 0} \sup_{\substack{\bar{A} \\ d_2(A,\bar{A})=\delta}} \left[\frac{d_1(X,\bar{X})}{\delta}\right]; \quad \bar{X}=f_x(\bar{A}) \tag{9}$$

where $d_1(\cdot,\cdot)$ and $d_2(\cdot,\cdot)$ are distance functions in the appropriate spaces [10]. When $\kappa[f_x(A)]$ is infinite the problem of determining X from A is called *ill-posed* (versus *well-posed*). When $\kappa[f_x(A)]$ is finite and "relatively large" or "relatively small" the problem is said to be *badly conditioned* and *well conditioned*, respectively.

Notice that backward stability is a property of an algorithm while conditioning is associated with a problem and the specific data for that problem. The errors E_x in the result depend on both the stability of the algorithm used and the conditioning of the specific problem. A "good" algorithm therefore is supposed to be backward stable since the size of the errors E_x in the result is then mainly affected by the condition of the problem, not by the algorithm. An unstable algorithm, on the other hand, may yield a large error E_x even when the problem is well conditioned.

About the singular values of any matrix A it is known that [11]

$$\kappa[f_\Sigma(A)]=1. \tag{10}$$

Hence, from (7) the diagonal elements $\bar{\sigma}_i$ of $\bar{\Sigma}$ are ϵ_A-close to the exact σ_i. *Generically*, i.e., for random E_A, all the diagonal elements of Σ will be perturbed and the rank of $\bar{\Sigma}$ then equals $k \triangleq \min(m,n)$. Yet if in (6), $\rho<k$, then the $(k-\rho)$ last diagonal elements of $\bar{\Sigma}$ are ϵ_A-small.

The number of $\bar{\sigma}_i > \epsilon_A$ is called the *numerical rank* or ϵ_A-rank of A (more sophisticated definitions of ϵ-rank are given in [12], [13]). When putting the ϵ_A-small singular values equal to zero, one obtains thus an ϵ_A-close matrix $\hat{A}$ whose rank is the numerical rank of A.

While in general there may be little justification for putting the ϵ_A-small singular values equal to zero, it is indeed a sound choice in several applications. A typical example is the determination of $\operatorname{im} A$ and $\ker A$, which, e.g., plays a fundamental role in least-squares solutions of the equation $Ax=b$ [14]. Notice that orthonormal bases for $\operatorname{im} A$ and $\ker A$ are given by the first ρ columns of U, respectively, the last $(n-\rho)$ columns of V in (6). The condition of $\operatorname{im} A$ and $\ker A$ is thus connected to the sensitivity of the transformation matrices U and V of the SVD. Consider, for example, the computation of $\operatorname{im} A$ (where we assume $m>n$). As distance function between two spaces X and Y we use the *gap* $\gamma(\mathsf{X},\mathsf{Y}) \triangleq \|P_\mathsf{X}-P_\mathsf{Y}\|_2$ where $P_.$ is the orthogonal projector on the indexed space. If A has full rank n, then

$$\kappa[\operatorname{im} A]=\sigma_n^{-1} \tag{11}$$

as illustrated by the following example ($\sigma_1=1$, $\sigma_2=a<1$):

$$A \triangleq \begin{bmatrix} 1 & 0 \\ 0 & a \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad E_A \triangleq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \delta \\ 0 & 0 \end{bmatrix};$$

$$\bar{A} \triangleq A+E_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \bar{a} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

with $\bar{a} \triangleq \sqrt{a^2+\delta^2}$, $c \triangleq a/\bar{a}$, $s \triangleq \delta/\bar{a}$. The second basis vector of $\operatorname{im} A$ is rotated in $\operatorname{im} \bar{A}$ over an angle θ, where $\sin\theta=s$, and one may check that $\gamma(\operatorname{im}\bar{A},\operatorname{im} A)=s$ whence (9) indeed converges to

$$\kappa[\operatorname{im} A]=a^{-1}=\sigma_2^{-1}.$$

In other words, the smallest singular value σ_n of A says how sensitive $\operatorname{im} A$ is to perturbations in A. When σ_n tends to zero the condition of $\operatorname{im} A$ gets infinitely large. When $\operatorname{rank}(A)<n<m$, $\operatorname{im} A$ is therefore ill-posed. Arbitrarily small perturbations can even change the dimension of $\operatorname{im} A$ as is seen in the following example ($\sigma_1=1$, $\sigma_2=a$, $\sigma_3=0$):

$$A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A δ-perturbation of the (3, 3)-element in A causes $\operatorname{im}\bar{A}$ to have dimension three. Therefore, $\gamma(\operatorname{im}\bar{A},\operatorname{im} A)=1$ no matter how small δ is, and $\kappa[\operatorname{im} A]=\infty$. Yet, when restricting us to those E_A such that $\rho \triangleq \operatorname{rank}(A)=\operatorname{rank}(A+E_A)$, then $\kappa_\rho[\operatorname{im} A]$ becomes finite. Indeed, this "restricted" condition number equals [4]

$$\kappa_\rho[\operatorname{im} A] \triangleq \lim_{\delta\to 0} \sup_{\substack{\|E_A\|_2=\delta \\ \operatorname{rank}(A+E_A)=\rho}} \frac{\gamma(\operatorname{im}(A+E_A),\operatorname{im} A)}{\delta} = \sigma_\rho^{-1}. \tag{12}$$

This is, e.g., obtained by δ perturbing the (3, 2)-element in A. We are then indeed in the same situation as in the previous example. The smallest singular value of A which is not zero says thus how sensitive $\operatorname{im} A$ is to perturbations in A preserving its rank ρ; $\operatorname{im} A$ is thus well conditioned in the restricted sense if all its singular values are either ϵ-small or close to 1. Then the ϵ-rank ρ of A is well defined and $\kappa_\rho[\operatorname{im} A] \cong 1$ according to (12).

Numerically this reflects the fact that ϵ_A-small singular values should be put equal to zero in order to have a "robust" numerical definition of im A. Indeed, in practical computations one cannot distinguish between ϵ_A-small and zero singular values [see (6)–(8)] and the determination of im A might be ill-posed or will be at least very badly posed [see (11)] when such pathologically small singular values occur. Fixing the rank such that the smallest singular value σ_ρ is significantly larger than ϵ_A makes $\kappa_\rho[\text{im } A]$ well bounded. In case ρ would have been the exact rank of A, the computed im $\bar{A}$ would be "close" to the real im A; in case rank A was not equal to ρ it would be impossible to compute a reasonable approximation of im A with a computer using ϵ-precision.

The above analysis was given to justify in a certain sense the computation of an "nongeneric" result for the system theoretic problems considered hereafter. Since arbitrarily small perturbations change this result to a "generic" one, and therefore cause drastic changes in this result, the related problem is ill-posed (compare to the above example where A has defective rank). In such a case, computed results only make sense if they are "chosen" to be nongeneric. If the choice happens to be correct, the error of the computed result can eventually be bounded; otherwise this is impossible because of the finite precision of the computer. In some applications we will see that nongeneric results yield useful information even when the data were generic but close to nongeneric data. Similar observations have been made in the numerical literature for possibly ill-posed problems such as the computation of multiple roots, generalized inverses, Jordan form, Kronecker form, least-squares solution, etc. (see [3], [15], [6], [14]), and successful algorithms were derived using these principles (see [15], [6], [16], [3]).

Unitary transformations certainly deserve some preference in the development of numerically stable algorithms and are extensively used in this paper. Exceptions have to be made, however, for cases where a special structure is apparent in the data. When, e.g., the elements of a matrix vary significantly in size, a "preconditioning" [64] or "balancing" [8] might be recommended first, in order to neutralize a possible bad scaling of the variables involved (see also [46]). In the case of sparse matrices one might prefer to use stable elementary transformations since they are more likely to preserve sparsity and they also save computing time. In some special cases, such as, e.g., canonical forms, a completely different—and probably better—approach may be found because of the special structure of the data.

III. Controllability and Observability

In several applications a linear time-invariant system is described by a state-space realization

$$\{\lambda I_n - A_{nn}, B_{np}, C_{mn}, D_{mp}(\lambda)\} \tag{13}$$

where A, B, and C are constant and $D(\lambda)$ is polynomial (the indexes here denote the dimensions of each matrix).

A. Controllable and Unobservable Subspaces

The *controllable subspace* $\mathsf{C}(A, B)$ and the *unobservable subspace* $\bar{\mathsf{O}}(A, C)$ of the system (13) are subspaces of the state-space X defined, respectively, as [17], [18]

$$\mathsf{C}(A, B) \triangleq \inf\{\mathsf{S} | A\mathsf{S} \subset \mathsf{S}; \text{im } B \subset \mathsf{S}\}$$

$$\bar{\mathsf{O}}(A, C) \triangleq \sup\{\mathsf{S} | A\mathsf{S} \subset \mathsf{S}; \mathsf{S} \subset \ker C\}. \tag{14a}$$

The above infimum and supremum can be proved to exist by standard techniques and are also equal to [17]

$$\mathsf{C}(A, B) = \text{im}\left[\, B \mid AB \mid \cdots \mid A^{n-1}B \,\right];$$

$$\bar{\mathsf{O}}(A, C) = \ker \begin{bmatrix} C \\ \hline CA \\ \hline \vdots \\ \hline CA^{n-1} \end{bmatrix}. \tag{14b}$$

When $\mathsf{C}(A, B) = \mathsf{X}$ the (A, B)-pair is called *controllable*; when $\bar{\mathsf{O}}(A, C) = \{0\}$ the (A, C)-pair is called *observable*. When both controllability and observability are satisfied the system (13) is called *irreducible* or minimal. Note that generically (13) is indeed irreducible (see, e.g., [17]). If, on the other hand, $\dim \mathsf{C}(A, B) \triangleq c < n$ and/or $\dim \bar{\mathsf{O}}(A, C) \triangleq \bar{o} > 0$, then these two spaces play a fundamental role in the state-space structure of the system (13) (see, e.g., [17], [18]).

Let the last c columns of the unitary matrix T span $\mathsf{C}(A, B)$; then after the state-space transformation T, we have that $\mathsf{C}(T^{-1}AT, T^{-1}B)$ is displayed as

$$\text{im}\begin{bmatrix} 0 \\ I_c \end{bmatrix}$$

and (see [17])

$$T^{-1}\left[\, \lambda I_n - A \mid B \,\right]\left[\begin{array}{c|c} T & \\ \hline & I_p \end{array}\right] \triangleq \left[\begin{array}{c|c} \lambda I_{\bar{c}} - A_{\bar{c}} & 0 \\ \hline \times & \lambda I_c - A_c \end{array} \;\middle|\; \begin{array}{c} 0 \\ \hline B_c \end{array}\right] \tag{15}$$

where (A_c, B_c) is controllable. The eigenvalues of $A_{\bar{c}}$ are called uncontrollable modes or also *input decoupling zeros* [18], [19]. A dual result holds for $\bar{\mathsf{O}}(A, C)$. Let the first $\bar{o}$ columns of a unitary matrix T span $\bar{\mathsf{O}}(A, C)$. Then after the state-space transformation T we have that $\bar{\mathsf{O}}(T^{-1}AT, CT)$ is displayed as

$$\text{im}\begin{bmatrix} I_{\bar{o}} \\ 0 \end{bmatrix}$$

and (see [17])

$$\left[\begin{array}{c|c} T^{-1} & \\ \hline & I_m \end{array}\right]\left[\begin{array}{c} \lambda I_n - A \\ \hline -C \end{array}\right] T \triangleq \left[\begin{array}{c|c} \lambda I_{\bar{o}} - A_{\bar{o}} & \times \\ \hline 0 & \lambda I_o - A_o \\ \hline 0 & -C_o \end{array}\right] \tag{16}$$

where (A_o, C_o) is observable. The eigenvalues of $A_{\bar{o}}$ are called the unobservable modes or also *output decoupling zeros* of the (A, C)-pair [18], [19].

Because of the duality of both problems (the decompositions (15) and (16) are one another's conjugate transpose) we only focus on the computation of $\mathrm{C}(A, B)$. Let U_1 be a unitary transformation compressing the rows of B and let ρ_1 be the rank of B; then A_1, B_1, X_1, Y_1, and Z_1 are matrices of appropriate dimension defined by

$$U_1^* A U_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline X_1 & Y_1 \end{array}\right] \begin{matrix} \} \tau_1 \\ \} \rho_1 \end{matrix}; \quad U_1^* B = \left[\begin{array}{c} 0 \\ \hline Z_1 \end{array}\right] \begin{matrix} \} \tau_1 \\ \} \rho_1 \end{matrix} \tag{17}$$

where Z_1 has full row rank ρ_1. Notice that a state-space transformation of the type

$$\left[\begin{array}{c|c} U_2 & \\ \hline & I_{\rho_1} \end{array}\right]$$

applied to (17), will only effect A_1, B_1, and X_1. If B_1 has neither zero rank nor full row rank, then we can use U_2 to compress the rows of B_1 and repeat a partitioning of the type (17) on $U_2^* A_1 U_2$ and $U_2^* B_1$. Algorithm 1 continues this recursion until a matrix B_k is obtained with either zero rank ($\rho_k = 0$) or full row rank ($\tau_k = 0$). This algorithm can be viewed as a stabilized version of Rosenbrock's minimal realization procedure (see also the Conclusion).

Algorithm 1

comment initialization;

$$c := 0;\ T := I_n;\ A_0 := A;\ B_0 := B;\ j := 1;$$

step_j: *comment* construct a unitary transformation U_j to compress the rows of B_{j-1} as follows (ρ_j and τ_j are defined by the compression);

$$\begin{matrix} \tau_j \\ \rho_j \end{matrix} \left\{ \left[\begin{array}{c} 0 \\ \hline Z_j \end{array}\right] := U_j^* B_{j-1}; \right. \quad \begin{array}{l} \text{if } \rho_j = 0 \text{ then go to exit\_1;} \\ \text{if } \tau_j = 0 \text{ then go to exit\_2;} \end{array}$$

comment transform and partition analogously A_{j-1} and rename obtained blocks as follows;

$$\begin{matrix} \tau_j \\ \rho_j \end{matrix} \left\{ \left[\begin{array}{c|c} A_j & B_j \\ \hline X_j & Y_j \end{array}\right] := U_j^* A_{j-1} U_j; \right.$$

comment update;

$$T := T \left[\begin{array}{c|c} U_j & \\ \hline & I_c \end{array}\right]; \quad c := c + \rho_j;$$

$$j := j+1; \quad \text{go to step\_}j;$$

comment k is the number of full rank "stairs" on ending of the algorithm;

exit_1: $k := j-1$; $\bar{c} := \tau_k$; $c := n - \bar{c}$; *stop*;

exit_2: $k := j$; $Z_k := B_k$; $Y_k := A_k$; $\bar{c} := 0$; $c := n$; *stop*

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Note that we allow T and c to be replaced by their new value in the recursion. The state-space transformation T constructed by this algorithm reduces the pencil $[\lambda I - A \mid B]$ to the form (at exit_1 with $\rho_{k+1} = 0$)

$$\left[\begin{array}{c|c} T^{-1}(\lambda I_n - A)T & T^{-1}B \end{array}\right] \triangleq \left[\begin{array}{c:c|c} \lambda I_{\bar{c}} - A_{\bar{c}} & 0 & 0 \\ \hdashline \cancel{X} & \lambda I_c - A_c & B_c \end{array}\right]$$

$$\triangleq \left[\begin{array}{c:ccc|c} \lambda I_{\tau_k} - A_k & & 0 & & 0 \\ \hdashline -X_k & \lambda I_{\rho_k} - Y_k & -Z_k & & \\ & & \ddots & \ddots & 0 \\ \cancel{X} & \cancel{X} & & & -Z_2 \\ & & & & \lambda I_{\rho_1} - Y_1 \end{array}\right] \begin{matrix} 0 \\ \\ 0 \\ 0 \\ Z_1 \end{matrix}. \tag{18}$$

If $\tau_k = 0$ would have been the stopping rule (exit_2), then the obtained decomposition would be similar to (18) but without the uncontrollable part. The Z_i have full row rank by construction, whence

$$\left[\begin{array}{c|c} \lambda I_c - A_c & B_c \end{array}\right]$$

in (18) has full row rank for any value of λ. According to the Popov–Hautus test [17] the pair (A_c, B_c) is thus controllable and (18) is the required decomposition (15). Moreover, one can check that the rank property of the Z_i ensures that

$$\operatorname{rank}\left[\begin{array}{c|c|c|c} B_c & A_c B_c & \cdots & A_c^{j-1} B_c \end{array}\right] = \sum_{i=1}^{j} \rho_i$$

$$j = 1, \cdots, k-1 \tag{19}$$

whence the Kronecker indexes of the (A_c, B_c)-pair—and thus also of the (A, B)-pair—are determined by the index set $\{\rho_i\}$ (see [6], [17]).

Remarks: Several remarks ought to be made about the above algorithm (see also [1], [21], [46]).

- In its actual form the algorithm computes the state-space transformation T that reduces $\mathrm{C}(A, B)$ to $\mathrm{C}(T^{-1}AT, T^{-1}B) = \operatorname{im}\left[\begin{array}{c} 0 \\ \hline I_c \end{array}\right]$ and hence computes an orthonormal basis for $\mathrm{C}(A, B)$, namely, the last c columns of T. This is done implicitly by transforming $[\lambda I - A \mid B]$ to the "staircase form" (18), but its irrelevant elements (denoted by $\cancel{X}$) are not computed. A simple modification of the algorithm will also compute these elements if required, but this was left out in the interests of brevity.
- The numerical stability of the algorithm can be proved. The key idea in the proof is that each unitary transformation U_j can be implemented in a (backward) numerically stable way [2]. A sequence of such transformations is also backward stable because the norm of each transformation U_j equals 1 (the choice of *unitary* transformations is thus of basic importance). For more details we refer to [1]. We then have that the right-hand side of (18) is the exact decomposition of a slightly perturbed pencil

$$\bar{T}^{-1}\left[\lambda I_n - \bar{A}_n \mid \bar{B}\right]\left[\begin{array}{c|c}\bar{T} & \\ \hline & I_p\end{array}\right] = \left[\begin{array}{c:c|c}\lambda I_{\bar{c}} - A_{\bar{c}} & 0 & 0 \\ \hline \times & \lambda I_c - A_c & B_c\end{array}\right] \tag{20}$$

with

$$\|\bar{X} - X\| \leqslant \epsilon_x \triangleq \Pi_x \cdot \epsilon \cdot \|X\| \qquad \text{for } X = A, B.$$

Moreover, $\bar{T}$ is nearly unitary (although not close to T), and its last c columns form a nearly orthonormal basis for $\mathrm{C}(\bar{A}, \bar{B})$ [1], [3].

- Numerically, the (A, B)-pair is only known up to an uncertainty region depending on the machine accuracy ϵ. If in that region there is an $(\bar{A}, \bar{B})$-pair with uncontrollable modes (thus, $\bar{c} \neq 0$) it makes much sense to accept $(\bar{A}, \bar{B})$ to be the "true" value of (A, B) since any of the ϵ-neighbors of $(\bar{A}, \bar{B})$ would at least have $\bar{c}$ modes that are "hardly" controllable. Any attempt, e.g., to move these "ϵ-controllable" modes by feedback would indeed require a feedback matrix F of approximate norm $1/\epsilon$! Moreover, when accepting this nongeneric result, i.e., when neglecting ϵ-small singular values in the compressions of Algorithm 1, the resulting decomposition (18) can be well defined. This is illustrated by the simple example

$$\left[\begin{array}{cc|c}\lambda - 1 & -1 & 1 \\ 0 & \lambda - 2 & 0\end{array}\right] \tag{21}$$

where $\mathrm{C}(A, B) = \operatorname{im}\begin{bmatrix}1\\0\end{bmatrix}$. For any ϵ-perturbation $(\bar{A}, \bar{B})$ preserving the dimension of the controllable subspace we have that $\gamma(\mathrm{C}(A, B), \mathrm{C}(\bar{A}, \bar{B})) \approx \epsilon$. Indeed, since $\dim \mathrm{C}(A, B) = \dim \mathrm{C}(\bar{A}, \bar{B}) = 1$ we have $\mathrm{C}(A, B) = \operatorname{im} B$ and $\mathrm{C}(\bar{A}, \bar{B}) = \operatorname{im} \bar{B}$. Moreover, $\operatorname{im} B$ is well conditioned according to (11) since its unique singular value is 1. The transformation T reducing (21) to its staircase form is also well conditioned since its two columns are spanned by $\operatorname{im} B^{\perp}$ and $\operatorname{im} B$, respectively. In general, $\mathrm{C}(A, B)$ will be well conditioned in this *restricted* sense (of preserving the dimension of the controllable subspace) if in Algorithm 1 the $\operatorname{im} B_j$ are well conditioned in that sense, i.e., if the "nonzero" singular values of the B_j are close to 1. Indeed, $\operatorname{im} B_{j-1}$ determines the span of the last ρ_j columns of U_j for $j = 1, \cdots, k$ and together these determine the span of the last c columns of T, hence also $\mathrm{C}(A, B)$.

- We finally stress the importance of working directly on the (A, B)-pair instead of on the controllability matrix

$$C_n \triangleq \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

or even the "infinite controllability matrix" when A has eigenvalues bounded by 1 (see, e.g., [20])

$$C_\infty \triangleq \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix}.$$

For the following example,

$$\left[\begin{array}{cc|c}\lambda + \frac{1}{2} & \epsilon^{1/2} & 0 \\ 0 & \lambda + \frac{1}{2} & \epsilon^{1/2}\end{array}\right], \tag{22}$$

both C_2 and C_∞ have one singular value of the order of $\epsilon^{1/2}$ and one of the order of ϵ, which suggests that some ϵ-perturbation of (22) might yield $\dim \mathrm{C}(\bar{A}, \bar{B}) = 1$. Yet, one can prove that this requires perturbations in A and B of the order of $\epsilon^{1/2}$! This shows that arbitrary ϵ-perturbations of C_n and C_∞ (e.g., those making them singular) cannot be induced by appropriate ϵ-perturbations of A and B because of the specific structure of C_n and C_∞. Therefore, it is important that an algorithm evaluating $\mathrm{C}(A, B)$ respects this structure which is, e.g., done by directly working on the (A, B)-pair.

B. Irreducible Realization and Kalman Decomposition

Suppose the system (13) is not irreducible and one wants an irreducible state-space model having the same transfer function as (13). This can be done in two steps using Algorithm 1 [21]. Let the state-space transformation T_c satisfy (15). Then applying it to the system matrix of (13) we obtain

$$\left[\begin{array}{c|c}T_c^{-1} & \\ \hline & I_m\end{array}\right]\left[\begin{array}{c|c}\lambda I - A & B \\ \hline -C & D(\lambda)\end{array}\right]\left[\begin{array}{c|c}T_c & \\ \hline & I_p\end{array}\right] \triangleq \left[\begin{array}{c:c|c}\lambda I_{\bar{c}} - A_{\bar{c}} & 0 & 0 \\ \hline \times & \lambda I_c - A_c & B_c \\ \hline \times & -C_c & D(\lambda)\end{array}\right]. \tag{23}$$

The system $\{\lambda I_c - A_c, B_c, C_c, D(\lambda)\}$ is now controllable and has the same transfer function as (13). Let T_o now be a state-space transformation extracting the unobservable part of the (A_c, C_c)-pair as in (16); then with

$$T \triangleq T_c\left[\begin{array}{c|c}I_{\bar{c}} & \\ \hline & T_o\end{array}\right]$$

we have

$$\left[\begin{array}{c|c}T^{-1} & \\ \hline & I_m\end{array}\right]\left[\begin{array}{c|c}\lambda I_n - A & B \\ \hline -C & D(\lambda)\end{array}\right]\left[\begin{array}{c|c}T & \\ \hline & I_p\end{array}\right] = \left[\begin{array}{c:c:c|c}\lambda I_{\bar{c}} - A_{\bar{c}} & 0 & & 0 \\ \hline \times & \lambda I_{c\bar{o}} - A_{c\bar{o}} & \times & \times \\ \hline \times & 0 & \lambda I_{co} - A_{co} & B_{co} \\ \hline \times & 0 & -C_{co} & D(\lambda)\end{array}\right]. \tag{24}$$

The system $\{\lambda I_{co} - A_{co}, B_{co}, C_{co}, D(\lambda)\}$ is now controllable and observable, hence irreducible, and has the same transfer function as (13) (see, e.g., [19]). Notice that the decomposition (24) yielding this irreducible system can be obtained by unitary state-space transformations on the

original system. Therefore, one can again ensure that it can be obtained in a backward stable way. This means that the computed decomposition (24) holds for the slightly perturbed matrices $\bar{A}$, $\bar{B}$, $\bar{C}$ and a nearly unitary matrix $\bar{T}$ ($D(\lambda)$ is unaffected) where

$$\|\bar{X}-X\| \leq \epsilon_x \triangleq \Pi_x \cdot \epsilon \cdot \|X\| \qquad \text{for } X=A, B, C. \quad (25)$$

The Kalman decomposition [17], [18] displays the four fundamental parts of an arbitrary state-space system (13). Let the state-space X be decomposed into the linearly independent subspaces $\mathsf{X}_{c\bar{o}}$, $\mathsf{X}_{\bar{c}\bar{o}}$, X_{co}, and $\mathsf{X}_{\bar{c}o}$ defined by ($\cap$ denoting the intersection)

i) $\mathsf{X}_{c\bar{o}} \triangleq \mathsf{C}(A, B) \cap \overline{\mathsf{O}}(A, C)$

ii) $\mathsf{X}_{\bar{c}\bar{o}} \oplus \mathsf{X}_{c\bar{o}} \triangleq \overline{\mathsf{O}}(A, C)$

iii) $\mathsf{X}_{co} \oplus \mathsf{X}_{c\bar{o}} \triangleq \mathsf{C}(A, B)$

iv) $\mathsf{X}_{\bar{c}o} \oplus (\mathsf{X}_{c\bar{o}} \oplus \mathsf{X}_{\bar{c}\bar{o}} \oplus \mathsf{X}_{co}) \triangleq \mathsf{X}$ (26)

and let $c\bar{o}$, $\bar{c}\bar{o}$, co, and $\bar{c}o$ denote their respective dimensions. When choosing a coordinate system in which $\mathsf{X}_{c\bar{o}}$ is spanned by the first $c\bar{o}$ unit vectors, $\mathsf{X}_{\bar{c}\bar{o}}$ by the next $\bar{c}\bar{o}$ ones etc., then the corresponding state-space realization $\{\lambda I_n - A_t, B_t, C_t, D(\lambda)\}$ has the following form (see, e.g., [18]):

$$\left[\begin{array}{c|c} \lambda I_n - A_t & B_t \\ \hline C_t & D(\lambda) \end{array}\right] \triangleq \left[\begin{array}{cccc|c} \lambda I_{c\bar{o}} - A_{c\bar{o}} & \times & \times & \times & B_{c\bar{o}} \\ 0 & \lambda I_{\bar{c}\bar{o}} - A_{\bar{c}\bar{o}} & 0 & \times & 0 \\ 0 & 0 & \lambda I_{co} - A_{co} & \times & B_{co} \\ 0 & 0 & 0 & \lambda I_{\bar{c}o} - A_{\bar{c}o} & 0 \\ \hline 0 & 0 & C_{co} & C_{\bar{c}o} & D(\lambda) \end{array}\right]. \quad (27)$$

It can be proved that, in general, the state-space transformation T transforming $\{\lambda I_n - A, B, C, D(\lambda)\}$ to a decomposition of the type (27), cannot be chosen unitary (see [22] for an "optimal" T), while the decomposition (24) can be obtained by a unitary state-space transformation. If one is only interested in computing an irreducible transformation, the earlier decomposition (24) is therefore preferable (both for the numerical stability as for the number of computations).

IV. System Matrix of a Proper Transfer Function

Let $\{\lambda I_n - A_{nn}, B_{np}, C_{mn}, D_{mp}\}$ be a system with "proper" transfer function $R(\lambda)$ (i.e., $R(\infty)=D$ is bounded [42]). The eigenstructure of its system matrix

$$P(\lambda) \triangleq \left[\begin{array}{c|c} \lambda I_n - A & B \\ \hline -C & D \end{array}\right] \quad (28)$$

yields information such as the supremal (A, B)-invariant subspace $\hat{\mathsf{V}}$ in $\ker C$ and the supremal (A, B)-controllability subspace $\hat{\mathsf{R}}$ in $\ker C$ when $D=0$ [17], [23], the invariant zeros of $P(\lambda)$ [19], [24], the factorizability of $R(\lambda)$ when $P(\lambda)$ is regular and irreducible [26], the pole/zero structure, and the left and right null space structure of $R(\lambda)$ when $P(\lambda)$ is irreducible [23], [25] (a discussion of this last property is postponed to Section V).

A. Supremal (A, B)-Invariant Subspace in $\ker C$

Let us first consider $D=0$ (i.e., $R(\lambda)$ strictly proper). The subspace V_F is then called an *(A, B)-invariant subspace in $\ker C$* if there exists a feedback matrix F such that $\mathsf{V}_F = \overline{\mathsf{O}}(A+BF, C)$. It is known that there exists a unique (A, B)-invariant subspace $\hat{\mathsf{V}}$ in $\ker C$ including all others. It is therefore called the *supremal (A, B)-invariant subspace in $\ker C$* (see [17], [23]). Its computation can be obtained as follows. We show constructively that there exists a unitary state-space transformation T that transforms the system

$$\{\lambda I_n - A, B, C, 0\} \quad (29)$$

to the form

$$\left[\begin{array}{c|c} \lambda I_n - T^{-1}AT & T^{-1}B \\ \hline -CT & 0 \end{array}\right] \triangleq \left[\begin{array}{c|c|c} \lambda I_{wo} - A_{wo} & \times & B_{wo} \\ \hline -C_{wo} & \lambda I_{so} - A_{so} & B_{so} \\ \hline 0 & -C_{so} & 0 \end{array}\right]$$

$$\triangleq \left[\begin{array}{c|c|c} \lambda I_{\nu_k} - A_k & \times & B_k \\ \hline -C_k & \lambda I_{\mu_k} - \times & D_k \\ 0 & -W_k & \lambda I_{\delta_k} - X_k \\ \hline 0 & -Y_k & -Z_k \end{array}\right] \quad (30a)$$

where $\operatorname{im} C_k \subset \operatorname{im} D_k$ and the matrix

$$\left[\begin{array}{cc} -W_k & \lambda I_{\delta_k} - X_k \\ -Y_k & -Z_k \end{array}\right] \quad (30b)$$

has full column rank for all λ. Since $\operatorname{im} C_k \subset \operatorname{im} D_k$, the equation $D_k \hat{F}_1 = -C_k$ has the solution $\hat{F}_1 \triangleq -D_k^+ C_k$ where D_k^+ is the Moore–Penrose inverse of D_k [14]. Applying the feedback $\hat{F}_t \triangleq [\hat{F}_1 \;\; 0]$ in the coordinate system of (30a) we obtain

$$\left[\begin{array}{c|c|c} \lambda I_{wo} - (A_{wo} + B_{wo}\hat{F}_1) & \times & B_{wo} \\ \hline 0 & \lambda I_{so} - A_{so} & B_{so} \\ \hline 0 & -C_{so} & 0 \end{array}\right]. \quad (31)$$

Because of the full column rank of (30b), the (A_{so}, C_{so})-pair is observable (Popov–Hautus test [17]). Hence, $V_{\hat{F}_t} = \operatorname{im}\begin{bmatrix} I_{wo} \\ 0 \end{bmatrix}$ is the unobservable space of (31). Moreover, no feedback can affect (30b) in (30a) and thus neither the observability of the (A_{so}, C_{so})-pair, whence $V_{\hat{F}_t} = \hat{V}_t$. In the original coordinate system $\hat{V}$ is spanned by the *wo* first columns of T since $\hat{V} = T \cdot \operatorname{im}\begin{bmatrix} I_{wo} \\ 0 \end{bmatrix}$. The subsystems $\{\lambda I_{so} - A_{so}, B_{so}, C_{so}, 0\}$ and $\{\lambda I_{wo} - A_{wo}, B_{wo}, C_{wo}, 0\}$ are sometimes called the *strongly observable* and *weakly observable* part of the system, respectively (see [27], [28] for an extensive discussion). Algorithm 2 will be shown to construct the state-space transformation T in (30a) recursively by implicitly reducing (29) to (30a). This algorithm can also be viewed as a stabilized version of several algorithms available in the system theoretic literature (see the Conclusion).

Algorithm 2

comment initialization: step $j = 1$;
begin comment compress the columns of C with V_1 to S_1 with full column rank ρ_1 and apply the state-space transformation V_1;

$$\begin{matrix} \nu_1 \\ \rho_1 \end{matrix}\left\{\left[\begin{array}{cc|c} A_1 & \not{\times} & B_1 \\ C_1 & \not{\times} & D_1 \\ \hline \underbrace{0}_{\nu_1} & \underbrace{S_1}_{\rho_1} & 0 \end{array}\right] := \left[\begin{array}{c|c} V_1^* & \\ \hline & I_m \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]\left[\begin{array}{c|c} V_1 & \\ \hline & I_p \end{array}\right];$$

$$T := V_1;\ \delta_1 := 0;\ \mu_1 := \rho_1;\ j := 2$$

end;

step_ j: *comment* check if $\operatorname{im} C_{j-1} \subset \operatorname{im} D_{j-1}$ and stop if so;
comment compress the rows of D_{j-1} with U_j^* and transform simultaneously the rows of C_{j-1};

$$\begin{matrix} \sigma_j \\ \tau_j \end{matrix}\left\{\left[\begin{array}{c|c} \bar{C}_{j-1} & \bar{D}_{j-1} \\ \hline \tilde{C}_{j-1} & 0 \end{array}\right] := U_j^*\left[\begin{array}{c|c} C_{j-1} & D_{j-1} \end{array}\right];\right.$$

if $\tau_j = 0$ *then go to* exit_ 1;
comment compress the columns of $\tilde{C}_{j-1}$ with V_j (if $\sigma_j = 0$ then of course $\tilde{C}_{j-1} = C_{j-1}$);

$$\left[\underbrace{0}_{\nu_j} \quad \underbrace{S_j}_{\rho_j}\right] := \tilde{C}_{j-1} V_j;\ \textit{if } \rho_j = 0 \textit{ then go to } \text{exit\_}1;$$

comment update T and stop if $\tilde{C}_{j-1}$ has full column rank;

$$T := T \cdot \operatorname{diag}\{V_j, U_j, I_{\delta_{j-1}}\};$$

$$\mu_j := \rho_j + \sigma_j;\quad \delta_j := \delta_{j-1} + \tau_j;$$

if $\nu_j = 0$ *then go to* exit_ 2;
comment update;

$$\begin{matrix} \nu_j \\ \mu_j \end{matrix}\left\{\left[\begin{array}{ccc|c} A_j & \not{\times} & & B_j \\ C_j & \not{\times} & & D_j \\ \hline \underbrace{\quad}_{\nu_j} & \underbrace{\quad}_{\rho_j} & & \underbrace{\quad}_{p} \end{array}\right] := \left[\begin{array}{c|c} V_j^* & \\ \hline & I_{\sigma_j} \end{array}\right]\cdot\left[\begin{array}{c|c} A_{j-1} & B_{j-1} \\ \hline \bar{C}_{j-1} & \bar{D}_{j-1} \end{array}\right]\left[\begin{array}{c|c} V_j & \\ \hline & I_p \end{array}\right];\right.$$

$j := j + 1$; *go to* step_ j;
comment k is the number of steps completed on ending of the algorithm;
exit_ 1: $k := j - 1$; $wo := \nu_k$; $so := n - wo$; *stop*;
exit_ 2: $k := j$; $wo := 0$; $so := n$; *stop* ∇∇∇

After the initializing step 1 we obtain the form

$$\left[\begin{array}{c:c|c} \lambda I_{\nu_1} - A_1 & \not{\times} & B_1 \\ \hdashline -C_1 & \lambda I_{\mu_1} - \not{\times} & D_1 \\ \hline 0 & -S_1 & 0 \end{array}\right].$$

This satisfies (30a), (30b) with $\delta_1 = 0$, $Y_1 = S_1$. If at this stage $\operatorname{im} C_1 \subset \operatorname{im} D_1$ the algorithm stops. Otherwise, an additional state-space tansformation is performed. We now prove inductively that the from (30a), (30b) is maintained in each step of Algorithm 2. Let (30a), (30b) hold for $k-1$ but $\operatorname{im} C_{k-1} \not\subset \operatorname{im} D_{k-1}$. Then a state-space transformation $T_k \triangleq \operatorname{diag}\{V_k, U_k, I_{\delta_{k-1}}\}$ is performed:

$$\left[\begin{array}{c:c:c|c} V_k^* & & & \\ \hdashline & U_k^* & & \\ \hdashline & & I_{\delta_{k-1}} & \\ \hline & & & I_m \end{array}\right]\left[\begin{array}{c:c:c|c} \lambda I_{\nu_{k-1}} - A_{k-1} & \not{\times} & \not{\times} & B_{k-1} \\ \hdashline -C_{k-1} & \lambda I_{\mu_{k-1}} - \not{\times} & \not{\times} & D_{k-1} \\ \hdashline 0 & -W_{k-1} & \lambda I_{\delta_{k-1}} - X_{k-1} & 0 \\ \hline 0 & -Y_{k-1} & -Z_{k-1} & 0 \end{array}\right]\left[\begin{array}{c:c:c|c} V_k & & & \\ \hdashline & U_k & & \\ \hdashline & & I_{\delta_{k-1}} & \\ \hline & & & I_p \end{array}\right]$$

$$\triangleq \left[\begin{array}{ccc:c|c} \lambda I_{\nu_k} - A_k & \not{\times} & \not{\times} & \not{\times} & \not{\times} & B_k \\ -C_k^+ & \lambda I_{\rho_k} - \not{\times} & \not{\times} & \not{\times} & & D_k^+ \\ \hdashline -C_k^- & \not{\times} & \lambda I_{\sigma_k} - \not{\times} & \not{\times} & \not{\times} & D_k^- \\ 0 & -S_k & \not{\times} & \lambda I_{\tau_k} - \not{\times} & & 0 \\ \hdashline & 0 & & -W_{k-1}U_k & \lambda I_{\delta_{k-1}} - X_{k-1} & 0 \\ \hline & 0 & & -Y_{k-1}U_k & -Z_{k-1} & 0 \end{array}\right] \begin{matrix} \\ \mu_k \\ \\ \delta_k \\ \\ \end{matrix}$$

$$\triangleq \left[\begin{array}{c|c|c|c} \lambda I_{\nu_k} - A_k & \times & \times & B_k \\ \hline -C_k & \lambda I_{\mu_k} - \times & \times & D_k \\ 0 & -W_k & \lambda I_{\delta_k} - X_k & 0 \\ \hline 0 & -Y_k & -Z_k & 0 \end{array}\right].$$

This is again of the form (30a) and rank condition (30b) is satisfied because S_k has full column rank and because (30b) holds for $k-1$. This recursion is continued until

- (exit 1) im $C_k \subset$ im D_k and $\hat{\mathcal{V}}$ is then spanned by the ν_k first columns of T; or
- (exit 2) $\nu_k = 0$ and $\hat{\mathcal{V}} = \{0\}$.

One of these two situations must occur after a finite number of steps since ν_k decreases each time that im $C_k \not\subset$ im D_k.

Remarks:

- $\hat{\mathcal{V}}$ is computed by constructing a unitary state-space transformation T that reduces the system (29) to the special form (30a), (30b), but irrelevant elements in this form are not computed. Indeed, only A_j, B_j, C_j, and D_j are carried along in the recursion but the algorithm can easily be modified in order to compute completely the final form (30a) if requested. The feedback matrix

$$\hat{F}_t \triangleq \left[\begin{array}{c|c} -D_k^+ C_k & 0 \end{array}\right] \tag{32}$$

is easily seen to be a minimum norm feedback F_t (both in Frobenius and spectral norm) satisfying $\hat{\mathcal{V}}_t = \overline{\mathcal{O}}(T^{-1}AT + T^{-1}BF_t)$. Since T is unitary, the feedback $\hat{F} \triangleq \hat{F}_t \cdot T^{-1}$ in the original coordinate system has also minimum norm for $\hat{\mathcal{V}} = \overline{\mathcal{O}}(A+BF)$. Notice that in the construction of a basis for $\hat{\mathcal{V}}$ we did *not* use any feedback operation since their norm can be very large (if D_k^+ is large in norm).

- Numerical stability of Algorithm 2 can again be proved because of the choice of unitary state-space transformations (see [1]). We thus have that (30a) holds exactly but for slightly perturbed matrices $\bar{A}$, $\bar{B}$, and $\bar{C}$ satisfying

$$\|\bar{X} - X\| \leq \Pi_x \cdot \epsilon \cdot \|X\|; \qquad \text{for } X = A, B, C. \tag{33}$$

Moreover, the computed matrix $\bar{T}$ is nearly unitary and its first wo columns form a nearly orthonormal basis for $\hat{\mathcal{V}}(\bar{A}, \bar{B}, \bar{C})$. Again it is known that generically $\hat{\mathcal{V}}$ is $\{0\}$ when $m > p$ and ker C when $m \leq p$, whence any different result for $\hat{\mathcal{V}}$ is ill-posed [17]. Yet, in applications such as, e.g., disturbance decoupling [17], nongeneric results do make sense numerically. Indeed, let $Dv(t)$ be an input disturbance to the system (29); then it can be decoupled if im $D \subset \hat{\mathcal{V}}$ [17]. In the case $m > p$ one expects generically $\hat{\mathcal{V}}$ to be $\{0\}$ whence nothing can be decoupled. Yet if the computed $\hat{\mathcal{V}}(\bar{A}, \bar{B}, \bar{C})$ is nonempty and contains im D the input noise $Dv(t)$ can be decoupled for the slightly perturbed system $\{\lambda I_n - \bar{A}, \bar{B}, \bar{C}, 0\}$ and the feedback $\hat{F}$ decoupling $Dv(t)$ In this perturbed system will also ϵ-decouple, i.e., severely lower in norm the influence of, the disturbance $Dv(t)$ in the true system (29). In the case $m \leq p$ generically $\hat{\mathcal{V}} =$ ker C and hence a disturbance with im $D =$ ker C could be decoupled. But a computed $\hat{\mathcal{V}}(\bar{A}, \bar{B}, \bar{C})$ smaller than ker C indicates that this would require a feedback $\hat{F}$ of approximate norm $1/\epsilon$ for the true system (29).

Nongeneric results are thus meaningful since they yield relevant information about the system under some uncertainty of the data (which is unavoidable in the computer). Moreover, as discussed in Section II, nongeneric results can be well conditioned in a restricted sense, namely, if all the compressions in Algorithm 2 are well conditioned in this sense.

- Define $\tilde{A}_{wo} \triangleq A_{wo} + B_{wo}\hat{F}_1$ in (31) and let V be a unitary transformation compressing the columns of B_{so} to $[\bar{B}_{so}\ \ 0]$. Then the input-space transformation V applied to (31) gives

$$\left[\begin{array}{c|c|cc} \lambda I_{wo} - \tilde{A}_{wo} & \times & \bar{B}_{wo} & \tilde{B}_{wo} \\ \hline 0 & \lambda I_{so} - A_{so} & \bar{B}_{so} & 0 \\ \hline 0 & -C_{so} & 0 & 0 \end{array}\right]. \tag{34}$$

It is clear that $\tilde{B}_{wo}$ can now be used for an additional feedback operation in order to freely assign part of the spectrum of the weakly observable part $\tilde{A}_{wo}$ and yet keep it decoupled from the output. In how far this is possible is determined by the controllable subspace $\mathcal{C}(\tilde{A}_{wo}, \tilde{B}_{wo})$. The corresponding subspace

$$\hat{\mathcal{R}} \triangleq \left[\begin{array}{c} \mathcal{C}(\tilde{A}_{wo}, \tilde{B}_{wo}) \\ \hline 0 \end{array}\right]$$

of the state-space $\mathcal{X}$ of (34) is called the *supremal (A, B)-contollability subspace in ker C*. $\hat{\mathcal{R}}$ is thus the largest (A, B)-invariant subspace that can be decoupled from the output and in which, at the same time, modes can be freely assigned [17]. The computation of $\mathcal{C}(\tilde{A}_{wo}, \tilde{B}_{wo})$ using Algorithm 1 yields thus a basis for it but the construction of $\tilde{A}_{wo}$ requires a feedback operation which may be numerically unstable. In Section IV-C we show how to circumvent this difficulty.

B. Zeros of a State-Space System

Suppose $R(\lambda)$ is proper and let $P(\lambda)$, given in (28), be its system matrix. The Smith zeros of this polynomial matrix are called the *invariant zeros* of the system $\{\lambda I_n - A, B, C, D\}$. We show in the sequel how to compute these in a numerically stable way. If the system is not irreducible then its input and output decoupling zeros can be computed using Algorithm 1 and an irreducible system $\{\lambda I_{co} - A_{co}, B_{co}, C_{co}, D\}$ can be obtained at the same time. The invariant zeros of the latter system are the finite McMillan zeros of $R(\lambda)$ [19] or, also, the transmission zeros of (28) [24]. They can thus be computed by combining Algorithm 1 and a method for computing invariant zeros. This method is conceptually close to Silverman's structure algorithm [49] as modified by Moylan [50], although these authors have a different application in mind.

Algorithm 3

comment initialization;

$$A_0 := A;\ B_0 := B;\ C_0 := C;\ D_0 := D;$$
$$\nu_0 := n;\ \delta_0 := 0;\ j := 1;$$

step__j: *comment* compress the rows of D_{j-1} with U_j^* and transform simultaneously the rows of C_{j-1};

$$\left.\begin{matrix}\sigma_j\\ \tau_j\end{matrix}\right\{ \left[\begin{array}{c|c} \bar{C}_{j-1} & \bar{D}_{j-1} \\ \hline \tilde{C}_{j-1} & 0 \end{array}\right] := U_j^* \left[\begin{array}{c|c} C_{j-1} & D_{j-1} \end{array}\right];$$

if $\tau_j = 0$ *then go to* exit__1;

comment compress the columns of $\tilde{C}_{j-1}$ with V_j ($\tilde{C}_{j-1} = C_{j-1}$ if $\sigma_j = 0$);

$$\underbrace{\begin{bmatrix} 0 & S_j \end{bmatrix}}_{\nu_j \quad \rho_j} := \tilde{C}_{j-1} V_j; \quad \text{if } \rho_j = 0 \text{ then go to exit\_\_1;}$$
$$\text{if } \nu_j = 0 \text{ then go to exit\_\_2;}$$

comment update;

$$\mu_j := \rho_j + \sigma_j; \quad \delta_j := \delta_{j-1} + \rho_j;$$

$$\left.\begin{matrix}\nu_j\\ \mu_j\end{matrix}\right\{ \left[\begin{array}{ccc|c} A_j & \times & & B_j \\ C_j & \times & & D_j \\ \hline & & I_{\sigma_j} & \end{array}\right] := \left[\begin{array}{c|c} V_j^* & \\ \hline & I_{\sigma_j} \end{array}\right]$$
$$\underbrace{}_{\nu_j}\ \underbrace{}_{\rho_j}\ \underbrace{}_{p}$$

$$\cdot \left[\begin{array}{c|c} A_{j-1} & B_{j-1} \\ \hline \bar{C}_{j-1} & \bar{D}_{j-1} \end{array}\right] \left[\begin{array}{c|c} V_j & \\ \hline & I_p \end{array}\right];$$

$$j := j+1;\ \textit{go to}\ \text{step}\_\_j;$$

exit__1: *comment* $\{\lambda I_n - A, B, C, D\}$ and $\{\lambda I_{n_r} - A_r, B_r, C_r, D_r\}$ have the same invariant zeros;

$$k := j-1;\ A_r := A_k;\ B_r := B_k;\ C_r := \bar{C}_k;$$
$$D_r := \bar{D}_k;\ n_r := \nu_k;\ m_r := \sigma_j;\ p_r := p;$$

stop;

exit__2: *comment* $\{\lambda I_n - A, B, C, D\}$ has no invariant zeros;
$k := j;\ n_r := 0;$ *stop*

$\nabla\nabla\nabla$

Step j of the above algorithm reduces the system matrix of $\{\lambda I_{\nu_{j-1}} - A_{j-1}, B_{j-1}, C_{j-1}, D_{j-1}\}$ to the form

$$\left[\begin{array}{c|c} V_j^* & \\ \hline & U_j^* \end{array}\right] \left[\begin{array}{c|c} \lambda I_{\nu_{j-1}} - A_{j-1} & B_{j-1} \\ \hline -C_{j-1} & D_{j-1} \end{array}\right] \left[\begin{array}{c|c} V_j & \\ \hline & I_p \end{array}\right]$$
$$= \left[\begin{array}{ccc|c} \lambda I_{\nu_j} - A_j & \vdots & \times & B_j \\ \hline -C_j & \vdots & \times & D_j \\ \hline 0 & \vdots & -S_j & 0 \end{array}\right] \quad (35)$$

where S_j has full column rank ρ_j. Using S_j as pivot, (35) can be transformed by unimodular row and column transformations (denoted by $\sim$) to

$$\left[\begin{array}{c|c} \lambda I_{\nu_{j-1}} - A_{j-1} & B_{j-1} \\ \hline -C_{j-1} & D_{j-1} \end{array}\right] \sim \left[\begin{array}{c|c:c} \lambda I_{\nu_j} - A_j & B_j & \\ \hline -C_j & D_j & \\ \hdashline & & I_{\rho_j} \\ & & 0 \end{array}\right] \quad (36)$$

and the systems $\{\lambda I - A_k, B_k, C_k, D_k\}$ for $k = j-1, j$ have the same Smith zeros, hence the same invariant zeros. Using (35) and (36) recursively we deduce that Algorithm 3 constructs unitary transformations Q_l and Q_r such that

$$Q_l^* \left[\begin{array}{c|c} \lambda I_n - A & B \\ \hline -C & D \end{array}\right] \left[\begin{array}{c|c} Q_r & \\ \hline & I_p \end{array}\right]$$
$$= \left[\begin{array}{c|ccc|c} \lambda I_{n_r} - A_r & \times & \cdots & \times & B_r \\ \hline -C_r & \times & \cdots & \times & D_r \\ & -S_k & \cdots & \times & \\ 0 & & \ddots & \vdots & 0 \\ & 0 & & -S_1 & \end{array}\right] \quad (37)$$

where the S_j have full column rank. Hence, we have

$$\left[\begin{array}{c|c} \lambda I_n - A & B \\ \hline -C & D \end{array}\right] \sim \left[\begin{array}{cc:c} \lambda I_{n_r} - A_r & B_r & \\ -C_r & D_r & \\ \hdashline & & I_{\delta_k} \\ & & 0 \end{array}\right]. \quad (38)$$

Note that since D_r has full row rank m_r, the system matrix

$$P_r(\lambda) \triangleq \left[\begin{array}{c|c} \lambda I_{n_r} - A_r & B_r \\ \hline -C_r & D_r \end{array}\right] \quad (39)$$

has no left null space. A "dual" Algorithm 3* can now be defined by interchanging the role of rows and columns in Algorithm 3 and would yield a system matrix

$$P_c(\lambda) \triangleq \left[\begin{array}{c|c} \lambda I_{n_c} - A_c & B_c \\ \hline -C_c & D_c \end{array}\right] \quad (40)$$

with the same transmission zeros as $P(\lambda)$ but now with D_c of full column rank. This dual algorithm consists basically of Algorithm 3 running on $P^*(\lambda)$ yielding $P_c^*(\lambda)$. When running this Algorithm 3* on $P_r(\lambda)$ we obtain a system matrix

$$P_{rc}(\lambda) \triangleq \left[\begin{array}{c|c} \lambda I_{n_{rc}} - A_{rc} & B_{rc} \\ \hline -C_{rc} & D_{rc} \end{array}\right] \quad (41)$$

with D_{rc} square invertible (D_r has full row rank to start with, which is not affected by Algorithm 3*). Hence, $P_{rc}(\lambda)$ is invertible and has the same Smith zeros as $P(\lambda)$. These are known to be the eigenvalues of $\hat{A} \triangleq A_{rc} -$

$B_{rc}D_{rc}^{-1}C_{rc}$ as is easily seen from

$$\left[\begin{array}{c|c} I & -B_{rc}D_{rc}^{-1} \\ \hline & I \end{array}\right]\cdot P_{rc}(\lambda)=\left[\begin{array}{c|c} \lambda I-\hat{A} & 0 \\ \hline -C_{rc} & D_{rc} \end{array}\right]. \quad (42)$$

Unfortunately, this may be an unstable transformation. It is preferable to use a unitary transformation Z compressing the rows of $\begin{bmatrix} B_{rc} \\ D_{rc} \end{bmatrix}$ to $\tilde{D}$ (which, of course, will be invertible):

$$Z\cdot P_{rc}(\lambda)\triangleq\left[\begin{array}{c|c} \lambda\tilde{B}-\tilde{A} & 0 \\ \hline \times & \tilde{D} \end{array}\right]. \quad (43)$$

It is easy to derive from (42) and (43) that $\tilde{B}$ is invertible and that

$$(\lambda I-\hat{A})=\tilde{B}^{-1}(\lambda\tilde{B}-\tilde{A}). \quad (44)$$

Hence, $\lambda\tilde{B}-\tilde{A}$ carries the same information as $\hat{A}$ but can be obtained in a numerically stable way. The generalized eigenvalues of $\lambda\tilde{B}-\tilde{A}$ are all finite since $\tilde{B}$ is invertible and can be computed using the backward stable QZ algorithm. Embedding (4) in (43) gives ($k\triangleq n_{rc}$)

$$Z_lP_{rc}(\lambda)Z_r\triangleq\lambda B_z-A_z$$

$$\triangleq\lambda\left[\begin{array}{ccc|c} b_{11} & & 0 & \\ \times & \ddots & & 0 \\ & & b_{kk} & \\ \hline & \times & & 0 \end{array}\right]-\left[\begin{array}{ccc|c} a_{11} & & 0 & \\ \times & \ddots & & 0 \\ & & a_{kk} & \\ \hline & \times & & -\tilde{D} \end{array}\right] \quad (45)$$

where Z_l and Z_r are unitary and $\lambda_i=A_{ii}/b_{ii}$ are the zeros of $P_{rc}(\lambda)$.

Remarks:

- Numerical stability of Algorithms 3 and 3* can be proved again because of the use of unitary transformations. Although the transformations do *not* preserve the coefficient of λ [see (37)], only A, B, C, and D in $P(\lambda)$ are used in the algorithms. Hence, the transformations of the coefficient of λ can be assumed to be exactly computed in the decomposition (37) since they are never actually performed. Therefore, one can prove that (37) holds *exactly* for slightly perturbed matrices $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ satisfying

$$\|\bar{X}-X\|\leq\Pi_x\cdot\epsilon\cdot\|X\| \qquad \text{for } X=\begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (46)$$

A similar result holds for the dual decomposition obtained by Algorithm 3*. Note that A, B, C, and D are transformed as a compound matrix whence (46) does not hold for $X=A$, B, C, D separately. In the decomposition (45) the coefficient of λ is effectively transformed but because its rank is thereby preserved, one can still prove that (45) holds exactly for the perturbed matrices $\bar{A}_{rc}$, $\bar{B}_{rc}$, $\bar{C}_{rc}$, and $\bar{D}_{rc}$ satisfying

$$\|\bar{X}-X\|\leq\Pi_x\cdot\epsilon\cdot\|X\| \qquad \text{for } X=\begin{bmatrix} A_{rc} & B_{rc} \\ C_{rc} & D_{rc} \end{bmatrix}. \quad (47)$$

Together, (46) and (47) say that the computed zeros $\lambda_i=a_{ii}/b_{ii}$ are the exact zeros of the perturbed system $\{\lambda I_n-\bar{A},\bar{B},\bar{C},\bar{D}\}$ with

$$\left\|\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}-\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right\|\leq\Pi\cdot\epsilon\cdot\left\|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right\|. \quad (48)$$

- Remarks about restricted conditioning for zeros of a system can only be made with a specific application in mind. We will see how they are connected to the supremal (A, B)-controllability subspace in $\ker C$. See also Section VI for an illustrative example.

C. Supremal (A, B)-Controllability Subspace in ker C

Algorithm 3 is clearly similar to Algorithm 2 and could be used for the computation of $\hat{V}$ when $D=0$. Similarly, we now show that Algorithm 3* running on the form (30a), obtained by Algorithm 2, yields a basis for $\hat{R}$ (see [17] or Section IV-A for a definition). Without loss of generality we can assume that (30a) is in the form

$$\left[\begin{array}{ccc|cc} \lambda I_{wo}-A_{wo} & \times & \times & \bar{B}_{wo} & \tilde{B}_{wo} \\ \hline -C_{wo} & \lambda I-\times & \times & \bar{D}_{wo} & 0 \\ 0 & \times & \lambda I-\times & 0 & 0 \\ \hline 0 & \times & \times & \underbrace{0}_{\bar{p}} & \underbrace{0}_{\tilde{p}} \end{array}\right] \quad (49)$$

where $\bar{D}_{wo}$ is square invertible (this situation can always be obtained by updating (30a) with unitary state-space and input-space transformations). A straightforward method would be now to perform the feedback $F=-[\bar{D}_{wo}^{-1}\ \ 0]C_{wo}$ on the system (49) and to compute the subspace $\mathcal{C}(A_{wo}+\bar{B}_{wo}\bar{D}_{wo}^{-1}C_{wo},\ \tilde{B}_{wo})$ (see IV-A). Depending on the conditioning of the inversion of $\bar{D}_{wo}$, this procedure may be unstable. Below, we show how the feedback operation can be avoided in order to yield a stable method for computing $\hat{R}$.

Algorithm 3* applied to the reduced system $\{\lambda I_{wo}-A_{wo}, [\bar{B}_{wo}\mid\tilde{B}_{wo}], C_{wo}, [\bar{D}_{wo}\mid 0]\}$ constructs unitary transformations P and Q giving [dual form to (37)]

$$\left[\begin{array}{c|c} P^* & \\ \hline & I_{\tilde{p}} \end{array}\right]\left[\begin{array}{c|cc} \lambda I_{wo}-A_{wo} & \bar{B}_{wo} & \tilde{B}_{wo} \\ \hline -C_{wo} & \bar{D}_{wo} & 0 \end{array}\right]\left[\begin{array}{c|c} Q & \\ \hline & I_{\tilde{p}} \end{array}\right]$$

$$\triangleq\left[\begin{array}{c:c:c:c} \lambda I_{\bar{c}}-A_{\bar{c}} & B_{\bar{c}} & 0 & 0 \\ \hdashline -R & \lambda S-T & \lambda X-Y & \tilde{B}_c \\ \hline \underbrace{-C_{\bar{c}}}_{\bar{c}} & \underbrace{D_{\bar{c}}}_{\bar{p}} & \underbrace{0}_{c} & \underbrace{0}_{\tilde{p}} \end{array}\right]\begin{matrix} \} \bar{c} \\ \} c \end{matrix} \quad (50)$$

where $[\lambda X-Y\mid\tilde{B}_c]$ has full row rank for all λ.

Let P, c, $\bar{c}$ be defined as above; then we transform and partition $\{\lambda I_{wo}-A_{wo}, [\bar{B}_{wo}\mid\tilde{B}_{wo}], C_{wo}, [\bar{D}_{wo}\mid 0]\}$ as follows:

$$\left[\begin{array}{c|c} P^* & \\ \hline & I_{\tilde{p}} \end{array}\right]\left[\begin{array}{c|cc} \lambda I_{wo}-A_{wo} & \bar{B}_{wo} & \tilde{B}_{wo} \\ \hline -C_{wo} & \bar{D}_{wo} & 0 \end{array}\right]\left[\begin{array}{c|cc} P & & \\ \hline & I_{\tilde{p}} & \\ & & I_{\tilde{p}} \end{array}\right]$$

$$\triangleq \left[\begin{array}{cc|cc} \lambda I_{\tilde{c}}-A_{11} & -A_{12} & \bar{B}_1 & 0 \\ -A_{21} & \lambda I_c-A_{22} & \bar{B}_2 & \tilde{B}_2 \\ \hline -C_1 & -C_2 & \bar{D}_{wo} & 0 \end{array}\right] \tag{51}$$

where $\tilde{B}_1=0$ because of (50). We now prove that there exists a feedback F such that

$$\left[\begin{array}{cc|cc} \lambda I_{\tilde{c}}-A_{11} & -A_{12} & \bar{B}_1 & 0 \\ -A_{21} & \lambda I_c-A_{22} & \bar{B}_2 & \tilde{B}_2 \\ \hline -C_1 & -C_2 & \bar{D}_{wo} & 0 \end{array}\right]\left[\begin{array}{ccc|} I_{\tilde{c}} & & \\ & I_c & \\ \hline 0 & F & I_{\tilde{p}} \\ 0 & 0 & \end{array}\right.\left.\begin{array}{c} \\ \\ \\ I_{\tilde{p}} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} \lambda I_{\tilde{c}}-A_{11} & 0 & \bar{B}_1 & 0 \\ -A_{21} & \lambda I_c-\tilde{A}_{22} & \bar{B}_2 & \tilde{B}_2 \\ \hline -C_1 & 0 & \bar{D}_{wo} & 0 \end{array}\right] \tag{52}$$

where $(\tilde{A}_{22}, \tilde{B}_2)$ is controllable. According to the discussion in Section IV-A we would then have constructed a coordinate system in which $C(\tilde{A}_{wo}, \tilde{B}_{wo})$ is displayed as $\operatorname{im}\begin{bmatrix} 0 \\ I_c \end{bmatrix}$. In order to prove the existence of F in (52) we define a unitary matrix E:

$$E \triangleq \left[\begin{array}{c|c} P^* & \\ \hline & I_{\tilde{p}} \end{array}\right]\cdot Q \cdot \begin{bmatrix} I_{\tilde{c}} & 0 & 0 \\ 0 & 0 & I_{\tilde{p}} \\ 0 & I_c & 0 \end{bmatrix}$$

and use (51) to rewrite (50) as

$$\left[\begin{array}{cc|cc} \lambda I_{\tilde{c}}-A_{11} & -A_{12} & \bar{B}_1 & 0 \\ -A_{21} & \lambda I_c-A_{22} & \bar{B}_2 & \tilde{B}_2 \\ \hline -C_1 & -C_2 & \bar{D}_{wo} & 0 \end{array}\right]\left[\begin{array}{c|c} E & \\ \hline & I_{\tilde{p}} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} \lambda I_{\tilde{c}}-A_{\tilde{c}} & 0 & B_{\tilde{c}} & 0 \\ -R & \lambda X-Y & \lambda S-T & \tilde{B}_c \\ \hline -C_{\tilde{c}} & 0 & D_{\tilde{c}} & 0 \end{array}\right]. \tag{53}$$

Equating the coefficient of λ in (53) we find that E can be written as

$$E \triangleq \begin{bmatrix} I_{\tilde{c}} & 0 & 0 \\ 0 & X & S \\ 0 & \underbrace{W}_{c} & \underbrace{Z}_{\tilde{p}} \end{bmatrix}\begin{matrix} \} c \\ \} \tilde{p} \end{matrix}$$

for some matrices W and Z. From the invertibility of $\bar{D}_{wo}$ and E and from

$$\begin{bmatrix} -C_2 & \bar{D}_{wo} \end{bmatrix}\begin{bmatrix} X \\ W \end{bmatrix}=0$$

it follows that X is also invertible. Indeed, $Xy=0$ for some $y\neq 0$ would imply that $\bar{D}_{wo}Wy=0$ and hence $Wy=0$ which finally would contradict the invertibility of E. Since X and E are invertible we can write, with $F=WX^{-1}$ and the invertible matrix $\bar{Z} \triangleq Z-WX^{-1}S$:

$$E \triangleq \left[\begin{array}{c|cc} I_{\tilde{c}} & & \\ \hline & I_c & 0 \\ & F & I_p \end{array}\right]\cdot\left[\begin{array}{c|cc} I_{\tilde{c}} & & \\ \hline & X & S \\ & 0 & \bar{Z} \end{array}\right].$$

From (53) we then define

$$\left[\begin{array}{cc|cc} \lambda I_{\tilde{c}}-A_{11} & 0 & \bar{B}_1 & 0 \\ -A_{21} & \lambda I_c-\tilde{A}_{22} & \bar{B}_2 & \tilde{B}_2 \\ \hline -C_1 & 0 & \bar{D}_{wo} & 0 \end{array}\right]$$

$$\triangleq \left[\begin{array}{cc|cc} \lambda I_{\tilde{c}}-A_{\tilde{c}} & 0 & B_{\tilde{c}} & 0 \\ -R & \lambda X-Y & \lambda S-T & \tilde{B}_c \\ \hline -C_{\tilde{c}} & 0 & D_{\tilde{c}} & 0 \end{array}\right]$$

$$\cdot\left[\begin{array}{ccc} I_{\tilde{c}} & & \\ & \begin{bmatrix} X & S \\ 0 & \bar{Z} \end{bmatrix}^{-1} & \\ & & I_{\tilde{p}} \end{array}\right].$$

This satisfies (52) with F defined as above and $(\tilde{A}_{22}, \tilde{B}_2)$ being controllable since

$$\begin{bmatrix} \lambda I_c-\tilde{A}_{22} & \tilde{B}_2 \end{bmatrix}=\begin{bmatrix} \lambda X-Y & \tilde{B}_c \end{bmatrix}\begin{bmatrix} X^{-1} & \\ & I_{\tilde{p}} \end{bmatrix}$$

has full row rank for all λ.

From the above it follows that the unitary state-space transformation $T=\operatorname{diag}\{P, I_{so}\}$ reduces (49) to a coordinate system in which

$$\hat{\mathcal{R}}=\operatorname{im}\begin{bmatrix} 0 \\ I_c \\ 0 \end{bmatrix}\begin{matrix} \} \bar{c} \\ \\ \} so. \end{matrix}$$

The last c columns of P appropriately completed with zeros thus constitute an orthonormal basis of $\hat{\mathcal{R}}$ in the coordinate system of (49). The feedback F actually decoupling this subspace can be computed directly from (52) as $F=-\bar{D}_{wo}^{-1}C_2$ and can again be checked to be minimum norm. Remarks similar to those made for $\hat{\mathcal{V}}$ yield that the computed basis is a nearly orthonormal basis of $\hat{\mathcal{R}}(\bar{A}, \bar{B}, \bar{C})$, where the slightly perturbed system $\{\lambda I_n-\bar{A}, \bar{B}, \bar{C}, 0\}$ satisfies (48).

It can be checked (see, e.g. [27]) that the invariant zeros of the subsystem

$$\left[\begin{array}{c|c} \lambda I_{\bar{c}} - A_{11} & \bar{B}_1 \\ \hline -C_1 & \bar{D}_{wo} \end{array}\right] \tag{54}$$

in (52) are the transmission zeros of the system $\{\lambda I_n - A, B, C, 0\}$ [and are the uncontrollable modes of the $(\tilde{A}_{wo}, \tilde{B}_{wo})$-pair in (34)]. The form is indeed close to the "zero pencil" $P_{rc}(\lambda)$ in (41) as one would expect from the connections of $\hat{\mathbf{V}}$ and $\hat{\mathbf{R}}$ with Algorithms 3 and 3*. Comments on "restricted conditioning" for $\hat{\mathbf{R}}$ and the occurrence of zeros in the different cases $m \lessgtr p$ can again be given in the same spirit as those for $\hat{\mathbf{V}}$ in Section IV-A.

D. Invariant Pole/Zero Directions

Let $\{\lambda I_n - A, B, C, D\}$ be an irreducible state-space system of a regular transfer function $R(\lambda)$ and assume for the moment that $D = R(\infty)$ is regular. The system matrix $P(\lambda)$ of such a system is regular and identical to its "zero pencil" $P_{rc}(\lambda)$ deduced in (41).

The poles (respectively, zeros) of $R(\lambda)$ are then finite and n in number, multiplicity counted, and are the eigenvalues of A (respectively, $\hat{A} \triangleq A - BD^{-1}C$); n is then also the McMillan degree of $R(\lambda)$. One can associate to A (respectively, $\hat{A}$) so-called invariant pole (respectively, zero) directions which are the eigenvectors, or in the defective case the principal vectors, of A (respectively, $\hat{A}$) (see [24]). These vectors, or more generally, the invariant subspaces of A and $\hat{A}$ (which are spanned by them) play a fundamental role in the problem of minimal cascade factorization of $R(\lambda)$:

$$R(\lambda) = R_1(\lambda) \cdot R_2(\lambda)$$

where the degrees of $R_1(\lambda)$ and $R_2(\lambda)$ add up to the degree of $R(\lambda)$. Indeed, to each pair of subspaces $\mathbf{X}$, $\mathbf{Y}$ satisfying

$$A\mathbf{X} \subset \mathbf{X}; \quad \hat{A}\mathbf{Y} \subset \mathbf{Y}; \quad \mathbf{X} \oplus \mathbf{Y} = \mathbb{C}^n \tag{55}$$

there corresponds such a factorization and conversely (see [26]). Stable methods for the computation of orthonormal bases for invariant subspaces are available in the numerical literature [29] but the construction of $\hat{A}$ requires a transformation that can be unstable as shown in (42). On the other hand, the regular pencil defined in (43),

$$\lambda \tilde{B} - \tilde{A} = \tilde{B}(\lambda I - \hat{A}), \tag{56}$$

can be obtained in a stable way as shown there. One easily sees that because of (56) a subspace $\mathbf{Y}$ is invariant under $\hat{A}$ iff

$$\dim(\tilde{B}\mathbf{Y} + \tilde{A}\mathbf{Y}) = \dim \mathbf{Y}$$

whence it is also a deflating subspace of $\lambda \tilde{B} - \tilde{A}$. In order to compute all possible deflating subspaces of $\lambda \tilde{B} - \tilde{A}$ one can make use of the QZ algorithm and of an appropriate updating of the decomposition (4) obtained by this algorithm [30].

When D is singular but $P(\lambda)$ is still irreducible and invertible then $R(\lambda)$ has some infinite zeros. The pencil $\lambda \tilde{B} - \tilde{A}$ obtained in (43) is then still regular but $\tilde{B}$ is not, whence it has infinite generalized eigenvalues which are the infinite zeros of $R(\lambda)$ (see Section V-B for more information). The generalized eigenvectors corresponding to these infinite eigenvalues of $\lambda \tilde{B} - \tilde{A}$ could be defined as "infinite zero directions" (compare [24]). The condition (55) for factorizability can also be adapted by requiring $\mathbf{Y}$ to be a deflating subspace of $\lambda \tilde{B} - \tilde{A}$ instead of an invariant subspace of $\hat{A}$.

V. Generalized State-Space Systems

State-space models can be generalized to models of the type [31]–[33]

$$\{\lambda E_{nn} - A_{nn}, B_{np}, C_{mn}, D_{mp}\} \tag{57}$$

with A, B, C, D, and E constant and $\lambda E - A$ regular but no such assumption on E. Any rational transfer function can be represented by such a generalized state-space model (GSSM), whereas state-space models in general require $D(\lambda)$ to be polynomial [31]. Moreover, a GSSM is easy to derive when, e.g., the system is given by a polynomial system $\{T_{nn}(\lambda), U_{np}(\lambda), V_{mn}(\lambda), W_{mp}(\lambda)\}$. Let indeed d be the highest power of λ occurring in these polynomial matrices and let T_i, U_i, V_i, W_i be their coefficients of λ^i. Let us define

$$A_0 \triangleq \begin{bmatrix} T_0 & 0_{nm} \\ -V_0 & -I_m \end{bmatrix} \quad A_i \triangleq \begin{bmatrix} T_i & 0_{nm} \\ -V_i & 0_m \end{bmatrix} \quad i = 1, \cdots, d$$

$$C_0 \triangleq [0_{mn} \quad I_m] \quad B_i \triangleq \begin{bmatrix} U_i \\ W_i \end{bmatrix} \quad i = 0, \cdots, d. \tag{58a}$$

Then a GSSM for the transfer function of $\{T(\lambda), U(\lambda), V(\lambda), W(\lambda)\}$ is given by the system matrix

$$\left[\begin{array}{cccc|c} -I & & & A_d & B_d \\ \lambda I & \ddots & & \vdots & \vdots \\ & \ddots & -I & A_1 & B_1 \\ & & \lambda I & A_0 & B_0 \\ \hline 0 & \cdots & 0 & C_0 & 0 \end{array}\right]. \tag{58b}$$

This can be derived using system equivalence techniques explained in [19] and is omitted here in the interests of brevity. Notice that the role of state-space transformations is here replaced by invertible transformations P and Q giving a GSSM $\{P(\lambda E - A)Q, PB, CQ, D\}$ with the same transfer function [31].

A. Controllable and Unobservable Subspace

Definitions of controllability and observability of a GSSM may differ in the literature (see, e.g., [31]–[34], [57]) because of the different contexts in which these concepts can be defined (discrete time versus continuous time, allowing impulsive solutions, etc.). Below we give

definitions which follow more the work of Verghese [33], [34], because of its elegant implications described in Section V-B. Nevertheless, the procedure described below could very well be modified to meet the definitions given by other authors (e.g., [31], [57]), using the decomposition (3).

The controllable and unobservable subspace of the state-space X of a GSSM $\{\lambda E_{nn}-A_{nn}, B_{np}, C_{mn}, D_{mp}\}$ can be defined, respectively, as

$$\begin{aligned}\mathsf{C}(E,A,B)&=\inf\{\mathsf{S}|\dim(E\mathsf{S}+A\mathsf{S})=\dim\mathsf{S};\\ &\qquad \operatorname{im}B\subset E\mathsf{S}+A\mathsf{S}\}\\ \overline{\mathsf{O}}(E,A,C)&=\sup\{\mathsf{S}|\dim(E\mathsf{S}+A\mathsf{S})=\dim\mathsf{S};\\ &\qquad \mathsf{S}\subset\ker C\}.\end{aligned}\tag{59}$$

That the above infimum and supremum indeed exist can be proved with standard techniques. When $E=I$ these definitions coincide with the controllable subspace $\mathsf{C}(A,B)$ and unobservable subspace $\overline{\mathsf{O}}(A,C)$ of the state-space system $\{\lambda I_n-A, B, C, D\}$, since the deflating subspaces of $\lambda I-A$ are nothing but the invariant subspaces of A.

Let c be the dimension of the controllable subspace C and let V and U be unitary transformations whose last c columns span C and $E\mathsf{C}+A\mathsf{C}$, respectively. Then we can transform the $(\lambda E-A, B)$-pair as

$$\begin{aligned}U^*[\lambda E-A \mid B]\left[\begin{array}{c|c}V&\\\hline &I_p\end{array}\right]&\triangleq[\lambda E_t-A_t \mid B_t]\\ &\triangleq\left[\begin{array}{c:c|c}\lambda E_{\bar c}-A_{\bar c}&0&0\\\hdashline \underbrace{\ast}_{\bar c}&\underbrace{\lambda E_c-A_c}_{c}&B_c\end{array}\right]\begin{array}{l}\}\ \bar c\\ \}\ c\end{array}\end{aligned}\tag{60}$$

since the transformed $(\lambda E_t-A_t, B_t)$-pair has the properties

$$\operatorname{im}B_t\subset E_t\mathsf{C}+A_t\mathsf{C}=\mathsf{C}=\operatorname{im}\left[\begin{array}{c}0\\I_c\end{array}\right].$$

Note that both $\lambda E_{\bar c}-A_{\bar c}$ and λE_c-A_c are regular since $\lambda E-A$ is regular. The $(\lambda E_c-A_c, B_c)$-pair is said to be *controllable* since its controllable subspace has full dimension c.

Analogously, let $\bar o$ be the dimension of the unobservable subspace $\overline{\mathsf{O}}$ and let V and U be unitary transformations whose first $\bar o$ columns span $\overline{\mathsf{O}}$ and $E\overline{\mathsf{O}}+A\overline{\mathsf{O}}$, respectively. Then the $(\lambda E-A, C)$-pair can be transformed to

$$\begin{aligned}\left[\begin{array}{c|c}U^*&\\\hline &I_m\end{array}\right]\left[\begin{array}{c}\lambda E-A\\\hline -C\end{array}\right]\cdot V&\triangleq\left[\begin{array}{c}\lambda E_t-A_t\\\hline -C_t\end{array}\right]\\ &\triangleq\left[\begin{array}{c:c}\lambda E_{\bar o}-A_{\bar o}&\ast\\\hdashline 0&\lambda E_o-A_o\\\hline 0&-C_o\end{array}\right]\begin{array}{l}\}\ \bar o\\ \}\ o\end{array}\\ &\qquad\underbrace{}_{\bar o}\quad\underbrace{}_{o}\end{aligned}\tag{61}$$

because for this transformed $(\lambda E_t-A_t, C_t)$-pair we have

$$\operatorname{im}\left[\begin{array}{c}I_{\bar o}\\0\end{array}\right]=E_t\overline{\mathsf{O}}+A_t\overline{\mathsf{O}}=\overline{\mathsf{O}}\subset\ker C_t.$$

Again $\lambda E_{\bar o}-A_{\bar o}$ and λE_o-A_o are regular and the $(\lambda E_o-A_o, C_o)$-pair is called *observable* since its unobservable subspace has zero dimension.

The decompositions (60) and (61) can be obtained by the following procedure. Since both forms are dual we only treat the first one.

Let us compress the rows of B in the $(\lambda E-A, B)$-pair to B_1 of full row rank s:

$$U_1^*[\lambda E-A \mid B]\triangleq\left[\begin{array}{c}\lambda E_2-A_2\\\hline \underbrace{\lambda E_1-A_1}_{n}\end{array}\ \middle|\ \begin{array}{c}0\\\hline B_1\end{array}\right]\begin{array}{l}\}\ r\\ \}\ s.\end{array}\tag{62}$$

Since $\lambda E-A$ has no left null space, neither does λE_2-A_2 and according to (3) the latter then has a decomposition (obtained by the pencil algorithm [6]):

$$U_2^*[\lambda E_2-A_2]V_2\triangleq\left[\begin{array}{c:c}\lambda E_{\bar c}-A_{\bar c}&0\\\hdashline \underbrace{\ast}_{\bar c}&\underbrace{\lambda E_r-A_r}_{n-\bar c}\end{array}\right]\begin{array}{l}\}\ \bar c\\ \}\ r-\bar c\end{array}\tag{63}$$

where $\lambda E_{\bar c}-A_{\bar c}$ is regular (contains the finite and infinite elementary divisors of λE_2-A_2) and λE_r-A_r has only right Kronecker indexes. Embedding (63) in (62), we obtain, with $U\triangleq U_1\operatorname{diag}\{U_2, I_s\}$ and $V\triangleq V_2$

$$\begin{aligned}&U^*[\lambda E-A \mid B]\left[\begin{array}{c|c}V&\\\hline &I_p\end{array}\right]\\ &\quad=\left[\begin{array}{c:c|c}\lambda E_{\bar c}-A_{\bar c}&0&0\\\hdashline &\lambda E_r-A_r&0\\ \underbrace{\ast}_{\bar c}&\underbrace{\ast}_{c}&B_1\end{array}\right]\begin{array}{l}\}\ \bar c\\ \}\ c.\end{array}\end{aligned}\tag{64}$$

This is the required decomposition (60) if we can show that the subsystem

$$[\lambda E_c-A_c \mid B_c]\triangleq\left[\begin{array}{c:c|c}\lambda E_r-A_r&0\\\hdashline \underbrace{\ast}_{c}&B_1\end{array}\right]\begin{array}{l}\}\ c-s\\ \}\ s\end{array}\tag{65}$$

is *controllable*, which we now prove by contradiction.

If the $(\lambda E_c-A_c, B_c)$-pair defined in (65) is not controllable, then an additional reduction of the type (60) can be performed on it,

$$\begin{aligned}\tilde U^*[\lambda E_c-A_c \mid B_c]\left[\begin{array}{c|c}\tilde V&\\\hline &I_p\end{array}\right]&\\ \triangleq\left[\begin{array}{c:c|c}\lambda\tilde E_{\bar c}-\tilde A_{\bar c}&0&0\\\hdashline \ast&\lambda\tilde E_c-\tilde A_c&\tilde B_c\end{array}\right]&\end{aligned}\tag{66}$$

where $\lambda\tilde{E}_{\bar{c}}-\tilde{A}_{\bar{c}}$ is regular. Let us assume that $\tilde{U}$ is chosen to compress at the same time $\tilde{B}_c$ to the form $\begin{bmatrix}0\\B_1\end{bmatrix}$ with B_1 as given in (65) [because of (65) this can always be satisfied by updating $\tilde{U}$ in (66)]. A comparison of (65) and (66) then yields that $\tilde{U}$ is of the form $\mathrm{diag}\{\hat{U}, I_s\}$, whence

$$\left[\begin{array}{c|c}\hat{U}^* & \\ \hline & I_s\end{array}\right]\left[\begin{array}{c|c|c}\lambda E_r-A_r & 0 & \\ \hline \times & B_1 & \\ \hline & & I_p\end{array}\right]\left[\begin{array}{c|c}\tilde{V} & \\ \hline & I_p\end{array}\right]$$

$$=\left[\begin{array}{cc|c}\lambda\tilde{E}_{\bar{c}}-\tilde{A}_{\bar{c}} & 0 & 0\\ \times & \times & 0\\ \hline \times & \times & B_1\end{array}\right].$$

This also says that $\hat{U}^*(\lambda E_r-A_r)\tilde{V}$ has a regular part $\lambda\tilde{E}_{\bar{c}}-\tilde{A}_{\bar{c}}$ and thus contradicts the construction of λE_r-A_r, which according to (3) has no elementary divisors or regular part.

Since (64) is the required decomposition (60) we thus have that the last c columns of V constructed by this procedure, span the controllable subspace $\mathsf{C}(E, A, B)$. Dual results hold for the space $\overline{\mathsf{O}}(E, A, C)$ and the decomposition (61). In analogy to Section III-B one can use the previous results to construct an *irreducible* GSSM $\{\lambda E_{co}-A_{co}, B_{co}, C_{co}, D\}$ with the same transfer function as the given GSSM $\{\lambda E-A, B, C, D\}$, by using only unitary transformations

$$\left[\begin{array}{c|c}U^* & \\ \hline & I_m\end{array}\right]\left[\begin{array}{c|c}\lambda E-A & B\\ \hline -C & D\end{array}\right]\left[\begin{array}{c|c}V & \\ \hline & I_p\end{array}\right]$$

$$\triangleq\left[\begin{array}{c|c|c}\lambda E_{\bar{c}}-A_{\bar{c}} & 0 & 0\\ \hline \times & \lambda E_{c\bar{o}}-A_{c\bar{o}} & \times\\ & 0 & \lambda E_{co}-A_{co} \quad B_{co}\\ \hline \times & 0 & -C_{co} \quad D\end{array}\right]. \tag{67}$$

That the GSSM's $\{\lambda E_{co}-A_{co}, B_{co}, C_{co}, D\}$ and $\{\lambda E-A, B, C, D\}$ have the same transfer function can be proved with standard techniques [19].

Due to the use of unitary transformations it is possible to show that this decomposition holds exactly for slightly perturbed matrices $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{E}$ (and nearly unitary $\bar{U}$ and $\bar{V}$) satisfying

$$\|\bar{X}-X\|\leqslant\Pi_x\cdot\epsilon\cdot\|X\| \qquad \text{for } X=A, B, C, E. \tag{68}$$

Similar results hold for the decompositions (60) and (61) and for the computed bases for $\mathsf{C}(\bar{E}, \bar{A}, \bar{B})$ and $\overline{\mathsf{O}}(\bar{E}, \bar{A}, \bar{C})$. The decomposition (67) can also be elaborated more in a Kalman type decomposition displaying the four fundamental parts of the system (see [34]) but the transformations required for that will in general not be unitary as discussed in Section III-B.

Notice that the decomposition (67) can also be used to construct an irreducible GSSM for the inverse transfer function $R^{-1}(\lambda)$ when $R(\lambda)$ is regular. Indeed, let $\{\lambda E-A, B, C, D\}$ be an arbitrary GSSM with transfer function $R(\lambda)$; then a GSSM with $R^{-1}(\lambda)$ as transfer function is given by the system matrix [19]

$$P_{\mathrm{inv}}(\lambda)\triangleq\left[\begin{array}{cc|c}\lambda E-A & B & 0\\ -C & D & I_m\\ \hline 0 & -I_m & 0\end{array}\right].$$

Extracting an irreducible GSSM of this can be done with the above method. This illustrates the elegance of the GSSM approach: the numerical tools needed for handling GSSM's are much the same as those used for state-space models but the above problem of inversion is, e.g., much more delicate when restricting oneself to the use of state-space models (see [49]).

B. Pole/Zero Structure and Null Space Structure

Definitions of pole/zero and null space structure of an arbitrary rational matrix are based on the Smith–McMillan canonical form [35] and on the concept of minimal polynomial bases [36], [37].

Following McMillan [35], every $m\times n$ rational matrix $R(\lambda)$ of normal rank r can be decomposed at any point λ_0 of the complex plane as

$$M_{mm}(\lambda)R_{mn}(\lambda)N_{nn}(\lambda)$$

$$\triangleq\left[\begin{array}{cc|c}(\lambda-\lambda_0)^{s_1} & & \\ & \ddots & 0_{r,n-r}\\ & (\lambda-\lambda_0)^{s_r} & \\ \hline 0_{m-r,r} & & 0_{m-r,n-r}\end{array}\right] \tag{69}$$

where $M(\lambda)$ and $N(\lambda)$ are rational matrices which are *regular at* λ_0 (i.e., $M(\lambda_0)$ and $N(\lambda_0)$ are regular) and the s_i are nondecreasing. The index set $\{s_1,\cdots,s_r\}$ is uniquely defined by this decomposition and describes the pole/zero structure of $R(\lambda)$ at λ_0: the positive indexes reflect the zero structure and the negative indexes the polar structure. The pole/zero structure at $\lambda=\infty$ is defined as above by replacing $\lambda-\lambda_0$ by $\mu\triangleq 1/\lambda$ in (69). Many authors seem to be unaware of this and give more involved but equivalent definitions of the structure at infinity [38].

The left and right null spaces of $R(\lambda)$ are vectorspaces over the field of rational functions $\mathbb{C}(\lambda)$ with coefficients in $\mathbb{C}$, defined, respectively, as

$$\mathsf{N}_l(R)\triangleq\{u(\lambda)|u^T(\lambda)R(\lambda)=0\}$$

$$\mathsf{N}_r(R)\triangleq\{v(\lambda)|R(\lambda)v(\lambda)=0\}$$

and have dimensions $m-r$ and $n-r$, respectively, where r is the normal rank of $R(\lambda)$. It is always possible to choose a polynomial basis

$$\{p_1(\lambda),\cdots,p_k(\lambda)\} \tag{70}$$

for any vectorspace S over $\mathbb{C}(\lambda)$ [36]. Let us define the *index* d_i of a polynomial vector $p_i(\lambda)$ as the maximum

polynomial degree in its components; then (70) is called a *minimal polynomial basis* of S if the sum of the indexes d_i is minimal over all polynomial bases for S. These indexes are invariant for a given space S, except for their ordering. When corresponding to the spaces $\mathsf{N}_l(R)$ and $\mathsf{N}_r(R)$ they are called, respectively, the *left* and *right minimal indexes* or $R(\lambda)$ (see [37] for an extensive discussion).

Applying the above definitions to a pencil $\lambda B-A$ with Kronecker canonical form (2) it is easy to prove the following (see, e.g., [1]).

i) The sizes s_i of the Jordan blocks at $\lambda=\lambda_0$ in the Jordan matrix J are the structural indexes at the zero $\lambda=\lambda_0$ of $\lambda B-A$.

ii) The sizes s_i, reduced by one, of the Jordan blocks at $\mu=0$ in the Jordan matrix N are the nonnegative structural indexes at the zero $\lambda=\infty$ of $\lambda B-A$.

iii) The negative structural indexes at the pole $\lambda=\infty$ of $\lambda B-A$ are all equal to -1 and are $k \triangleq$ rank B in number (notice that this is the only pole).

iv) The left and right Kronecker indexes are the left and right minimal indexes, respectively, of $\lambda B-A$.

Using the decompositions (3) and (4) we can thus compute the pole/zero structure and the null space structure of an arbitrary pencil in a stable way (this, of course, does not mean that the problem is well-posed [1]). The following discussion now shows how important the construction of an irreducible GSSM is in the computation of the structural elements of its transfer function. In [1] we show that the controllability of the $(\lambda E_{co}-A_{co}, B_{co})$-pair and the observability of the $(\lambda E_{co}-A_{co}, C_{co})$-pair imply the absence of finite and infinite zeros in the pencils

$$\left[\lambda E_{co}-A_{co} \mid B_{co}\right] \quad \text{and} \quad \left[\begin{array}{c} \lambda E_{co}-A_{co} \\ \hline -C_{co} \end{array}\right]. \tag{71}$$

In [25], [33] a GSSM which satisfies conditions (71) was termed *strongly irreducible* which is thus implied by the previous definition of *irreducibility* (the contrary is not true). For a strongly irreducible GSSM the following important connections are proved in [25] between the structural elements of

$$\left[\lambda E_{co}-A_{co}\right] \quad \text{and} \quad P_{co}(\lambda) \triangleq \left[\begin{array}{c|c} \lambda E_{co}-A_{co} & B_{co} \\ \hline -C_{co} & D \end{array}\right] \tag{72}$$

on one hand and the transfer function $R(\lambda)$ of the GSSM $\{\lambda E_{co}-A_{co}, B_{co}, C_{co}, D\}$ on the other hand.

i) The polar structure of $R(\lambda)$ is isomorphic to the zero structure of $\lambda E_{co}-A_{co}$ (at all points, infinity included).

ii) The zero structure of $R(\lambda)$ is isomorphic to the zero structure of $P_{co}(\lambda)$ (at all points, infinity included).

iii) The null spaces of $P_{co}(\lambda)$ and $R(\lambda)$ are isomorphic and their minimal indexes are equal.

Starting from an arbitrary GSSM it is thus possible to construct a strongly irreducible GSSM (72) and then compute the structural elements of its transfer function by using the decompositions (3) and (4) on the pencils (72). Note that this is also possible for irreducible state-space models $\{\lambda I-A, B, C, D\}$ when D is constant, but one is then restricted to proper transfer functions. The structural elements of irreducible state-space models are also closely connected to the objects discussed in Section IV [1].

VI. Polynomial and Rational Matrices

The structural elements of polynomial and rational matrices can be computed through the Smith canonical form and Smith–McMillan canonical form, respectively. Every rational matrix $R(\lambda)$ of normal rank r can be decomposed as [35]

$$M_{mm}(\lambda)R_{mn}(\lambda)N_{nn}(\lambda) \triangleq \left[\begin{array}{ccc} e_1(\lambda)/f_1(\lambda) & & 0_{r,n-r} \\ & \ddots & \\ & & e_r(\lambda)/f_r(\lambda) \\ \hline 0_{m-r,r} & & 0_{m-r,n-r} \end{array}\right] \tag{73}$$

where $M(\lambda)$ and $N(\lambda)$ are unimodular, the $e_i(\lambda)$ and $f_i(\lambda)$ are relatively prime monic polynomials, $e_i(\lambda)$ divides $e_{i+1}(\lambda)$, and $f_{j+1}(\lambda)$ divides $f_j(\lambda)$. Every polynomial matrix $P(\lambda)$ can be decomposed similarly but with the $f_j(\lambda)$ all being equal to 1. The finite pole/zero structure of $R(\lambda)$ follows immediately from (73) by computing the elementary factors $(\lambda-\lambda_0)^{s_i}$ from each $e_i(\lambda)/f_i(\lambda)$ [35] [19]. For the pole/zero structure at $\lambda=\infty$ one may transform λ to $1/\mu$ and apply (73) to $R(1/\mu)$ (see [35], [19] for more details). Similar results hold for a polynomial matrix $P(\lambda)$.

Polynomial bases for $\mathsf{N}_l(R)$ and $\mathsf{N}_r(R)$ are obviously given by the last $m-r$ rows of $M(\lambda)$ and the last $n-r$ columns of $N(\lambda)$, respectively. Reducing these bases to a row proper and column proper form then yield the required minimal polynomial bases for $\mathsf{N}_l(R)$ and $\mathsf{N}_r(R)$, respectively (see [37], [42]). The same method applies to a polynomial matrix $P(\lambda)$ also.

The construction of the Smith–McMillan form and the reduction to row/column proper form are both based on elementary column and row operations on some polynomial matrix (see [5], [35], [42] for the actual algorithms) but these transformations are numerically unstable as shown in the following example. A typical row operation to be performed in the construction of the Smith canonical form of

$$P(\lambda) \triangleq \begin{bmatrix} \delta & \lambda \\ \lambda & \lambda \end{bmatrix}$$

(with δ small but nonzero) would be

$$\begin{bmatrix} 1 & \\ -\lambda/\delta & 1 \end{bmatrix} P(\lambda) = \begin{bmatrix} \delta & \lambda \\ 0 & \lambda-\lambda^2/\delta \end{bmatrix}.$$

The coefficients in the transformation and in the trans-

formed matrix— and therefore also the numerical errors in these matrices—blow up when δ becomes very small. Similar examples can be given for the reduction algorithm to row/column proper form. In each of these algorithms the choice of the pivot is determined by the powers of λ and not by the numerical coefficients, hence precluding any proof of numerical stability.

On the other hand, for the computation of the structure of the above example we could as well use the *QZ* algorithm since it is a regular pencil, thus avoiding the unstable construction of the Smith form of $P(\lambda)$. In the numerical literature it is shown [59] how to reduce a regular polynomial matrix of arbitrary degree to a pencil with the same zero structure, thus again avoiding the unstable construction of the Smith canonical form. It is in fact a special case of the results given in [25] and discussed in Section V: there a method is given to construct pencils with the same structural information as a given polynomial matrix or even as a rational transfer function described by a polynomial system $\{T(\lambda), U(\lambda), V(\lambda), W(\lambda)\}$. The main reason why these extensions were never tackled in the numerical literature is, we believe, the possible ill-posedness of these extensions. In system theoretic applications the restricted conditioning described in Section II allows us to circumvent this difficulty. A simple example for this is due to Wilkinson [43], [62]. The pencil

$$P(\lambda) \triangleq \begin{bmatrix} \lambda-2 & 0 \\ 0 & 0 \end{bmatrix}$$

has normal rank 1 but an arbitrary ϵ-perturbation of it ($|\epsilon_i|<\epsilon$),

$$P_\epsilon(\lambda) \triangleq \begin{bmatrix} \lambda(1+\epsilon_1)-(2+\epsilon_2) & \epsilon_3\lambda-\epsilon_4 \\ \epsilon_5\lambda-\epsilon_6 & \epsilon_7\lambda-\epsilon_8 \end{bmatrix}$$

will have normal rank 2 and can have very different zeros. Taking, e.g., $\epsilon_7=\epsilon_8=0$ we have that $\det \cdot P_\epsilon(\lambda)=(\epsilon_3\lambda-\epsilon_4)(\epsilon_5\lambda-\epsilon_6)$ and then $P_\epsilon(\lambda)$ has the rather arbitrary zeros $\bar{\lambda}_1=\epsilon_4/\epsilon_3$, $\bar{\lambda}_2=\epsilon_6/\epsilon_5$. On the other hand, $P_\epsilon(\lambda_0)$ has ϵ-rank 1 for almost any value λ_0 whence one is tempted to take $r=1$ as its "numerical" normal rank. Once making this choice the "numerical" right and left null space of $P_\epsilon(\lambda_0)$ are ϵ-close to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $[0 \quad 1]$, respectively, for almost any λ_0 whence it is again sound to assume these are constant. The only remaining structural element is then, according to the decomposition (2), a single zero which is indeed ϵ-close to 2 ($P_\epsilon(2)$ has indeed ϵ-rank equal to 0).

When restricting oneself thus to perturbations maintaining a left and right null space of index 0, all structural elements, i.e., the only zero and the direction of the left and right null vectors, are well conditioned. The "restricted condition number" for these elements is thus close to 1. For the differential system

$$y(t)=P_\epsilon\left(\frac{d}{dt}\right)\cdot u(t)$$

the answers obtained in such a way are acceptable. Indeed, *any* input signal parallel to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ will almost be blocked completely—whence $\mathbf{N}_r(P_\epsilon)=\text{im}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is logical —and inputs parallel to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ will almost be blocked completely only if multiplied by e^{2t}—whence $\lambda=2$ is a logical zero. Note finally that the algorithms for the decompositions (3) and (4) would indeed give these "logical" answers, although they are only well conditioned in a restricted sense. Even when running on the perturbed pencil the decomposition (3) gave ϵ-close results to the expected ones since the appropriate rank decisions are made in the algorithm (see [1] for numerical experiments).

Finally, we want to make some remarks about the stability of the algorithms applied to the computation of the structure of polynomial and rational matrices. Once a GSSM for the given transfer function is given, say, e.g., in the form (58), the reduction process to the form (67) and the decompositions (3) and (4) of the pencils (72) are backward stable since they use only unitary transformations. But in fact, the backward errors induced in the original model (58) do not respect the special structure, i.e., the 0 and I blocks, in this GSSM. Yet, one can prove numerical stability in a stricter sense, namely, that the computed structure corresponds *exactly* to a slightly perturbed polynomial system $\{\bar{T}(\lambda), \bar{U}(\lambda), \bar{V}(\lambda), \bar{W}(\lambda)\}$. For this perturbed system the coefficients of $\bar{X}(\lambda)$ and $X(\lambda)$ are ϵ-close where X stands for each of the polynomial matrices T, U, V, and W (see, e.g., [1], [60]).

When the transfer function is originally not given by a polynomial system model then numerical stability in such a strict sense cannot be proved in general, because some parameter representations of rational matrices are not flexible enough (see [1] for a discussion). Some of these parameterizations of rational matrices even allow one to determine the structural elements through different techniques (see, e.g., [44]) but this is beyond the scope of this paper.

VII. Conclusion

In this paper we have treated a number of system theoretical problems from a numerical point of view. The given algorithms were shown to be "reliable" because of their controlled numerical behavior in the presence of rounding errors. The mere use of unitary transformations indeed allows us to give backward bounds for the rounding errors performed during these algorithms. We also commented on the possible ill-posedness of some of the tackled problems and showed how to deal numerically with this delicate aspect of the problems.

The given algorithms for state-space models can be viewed as appropriate adaptations of the "pencil algorithm" given in [6] to the specific "pencil problems" occurring in linear system theory. The kind of pencils encountered in that area have a special structure which unfortunately is not respected by the general algorithm. The adapted algorithms given in this paper are therefore

strongly inspired by other approaches from the system theoretic literature.

The basic idea of Algorithm 1 can be recognized in an algorithm given by Rosenbrock [19] and since then in several other algorithms [45]–[47], [21], [22], [48], [63]. We believe that a backward stable version of this algorithm was first given in [21]. The application to the Kalman decomposition is extensively treated in [22].

Algorithms 2 and 3 are very close to Silverman's structure algorithm (compare [27], [28], [49], [50]) and to Wonham's recursive definitions (see [17], [48], [51], [52]). These approaches use special types of transformations precluding any proof of numerical stability, while here we restrict ourselves to unitary transformations (see also [21] and [48, Remark 14]). Some authors [40], [53], [54], [20] have started to use numerically reliable software to tackle several of the previous problems but their approach is rather different from the one described here. Their methods yet do not always respect the specific structure of the processed pencils and are less elegant in some cases.

Special attention was also paid here to generalized state-space models because of their importance in numerical computations. The numerical tools needed for handling these models are indeed basically the same as those needed for state-space systems but these GSSM's allow us to tackle several additional problems (pole/zero structure and null space structure of polynomial and rational transfer functions, inverse problem) in a rather elegant way. The algorithms we use for their computation [6], [7] have been proved to be numerically stable in contrast to some alternative algorithms from system theory [41], [42], [55], [56].

In this paper we have put the emphasis on the use of stable transformations in the solution of our problems but we have not detailed the actual implementation of the algorithms. In our discussion on row and columns compressions—which are a basic tool in all the algorithms—we suggested the use of the SVD because it is the only reliable tool for the determination of the ϵ-rank [3]. Unfortunately, it is more expensive in computing time than other methods such as, e.g., Householder reductions with pivoting [8]. The latter method is an acceptable alternative for the SVD since the examples where it fails in its numerical rank determination, are rather pathological [13]. In the case of large sparse systems—which, e.g., often occur in network theory—it might be interesting to switch to elementary transformation with pivoting because of computing time and possible preservation of sparsity. Other schemes exploiting sparsity of the models, such as the Lanczos procedure, can also be appropriate in some cases [22]. In each of these alternatives there is a tradeoff between computing time and reliability of the rank determination.

We feel that the rather "theoretical" approach chosen here helps to develop better insights in the numerical problems occurring in the class of problems treated here or even in other related problems of system theory. Numerical experiments would, of course, be appropriate for illustrating this new approach. But for this we refer to separate work since we want to keep this (already long) paper within an acceptable length. Algorithm 1 is implemented in [22] and illustrated by numerous examples. The same is done for Algorithms 3 and 3* in [58], but specifically aimed at the computation of zeros. A comparison with two other methods [40], [53] is also given there. The implementation of Algorithms 2 and 3 for the computation of $\hat{V}$ and $\hat{R}$ has not been completed yet. The number of operations performed by the given algorithms depend on the type of transformations used for the compressions, but they normally grow cubically with the size of the processed pencil (see [21], [58] for more precise figures). For some additional numerical experiments about the computational method described in Section V, we refer to [1].

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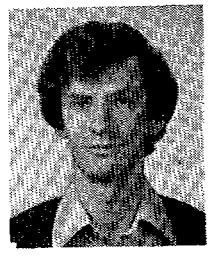

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