A unitary method for deadbeat control.

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Abstract

In this paper we give a numerical method to perform deadbeat control on a multi input system. The method constructs, in a recursive manner, a unitary state-space transformation yielding a coordinate system in which the feedback matrix is computed by merely solving a set of linear equations. Moreover, in this coordinate system one easily constructs the minimum norm solution to the problem. Along the way, the algorithm also computes orthonormal bases for the controllable subspaces of an (A,B)-pair, independently of the invertibility of the matrix A. Partial results of numerical stability are also obtained.

1. Introduction

We consider the linear system

$$x_{i+1} = Ax_i + Bu_i \qquad (1)$$

where n is the state dimension and m the input dimension. The problem of deadbeat control is to find a state feedback $u_i = Fx_i + v_i$ such that the resulting system:

$$x_{i+1} = (A+BF)x_i + v_i \qquad (2)$$

has a nilpotent matrix (A+BF), i.e. $(A+BF)^k = 0$, for some minimal power k. The solution of the homogenous part of the system (2) then ´dies out´ after k steps [7]. This problem has been considered by several authors and several efforts have been undertaken recently to come up with numerically reliable methods to solve the problem [3][4][5][9][10][19]. The method presented in this paper is very similar to the one developed in [9][10]. In our special case though a simplified algorithm can be obtained which allows for an analysis of the numerical behavior of the method and permits the construction of the minimum norm solution to the problem.

2. Problem formulation

The method described in this paper is based on the use of unitary transformations only. These transformations are chosen because of their property of norm invariance with respect to certain norms:

$$\|U.A.V\| = \|A\| \qquad \text{for } U,V \text{ unitary, i.e. } U'U=UU'=I \quad V'V=VV'=I$$

where $\|.\|$ stands for both the spectral and Frobenius norms [14], and .´ denotes the conjugate transpose of a matrix. As shown in the next section, this guarantees that the errors performed by the algorithm do not blow up -therefore resulting in a numerically stable algorithm- and also that the feedback matrix in the transformed coordinate system has still the same norm.

To start with, the (A,B)-pair is transformed via a unitary state-space transformation V to the block form (see e.g. [2][3][6][13][15][17][18]):

$$\left[\begin{array}{c|c} V'B & V'AV \end{array}\right] = \left[\begin{array}{c|cc} B_c & A_c & X \\ 0 & 0 & A_{\bar{c}} \end{array}\right] = \qquad (3a)$$

$$\begin{array}{c} \left[\begin{array}{c|cccccc} B_1 & A_{1,1} & A_{1,2} & \cdots & \cdots & A_{1,k} & A_{1,k+1} \\ & A_{2,1} & A_{2,2} & \cdots & \cdots & A_{2,k} & A_{2,k+1} \\ & & \ddots & \ddots & & \vdots & \vdots \\ 0 & & 0 & \ddots & \ddots & \vdots & \vdots \\ & & & & A_{k,k-1} & A_{k,k} & A_{k,k+1} \\ 0 & & & 0 & & & A_{k+1,k+1} \end{array}\right] \begin{array}{l} \}r_1 \\ \}r_2 \\ \\ \\ \}r_k \\ \}n_{k+1} \end{array} \\ \begin{array}{ccccccc} \underbrace{}_{m} & \underbrace{}_{r_1} & \underbrace{}_{r_2} & & \underbrace{}_{r_{k-1}} & \underbrace{}_{r_k} & \underbrace{}_{n_{k+1}} \end{array} \end{array} \qquad (3b)$$

Here B_1 and the $A_{i,i-1}$ off-diagonal blocks have full row rank r_i by construction. The modes of $\underline{A}_c$ are clearly uncontrollable and no feedback will affect their location. Let us define the i-th reachable subspace as:

$$R_i(A,B) = \langle B\rangle + A\langle B\rangle + A^2\langle B\rangle + \ldots + A^{i-1}\langle B\rangle \qquad (4)$$

where $\langle . \rangle$ denotes the range of a matrix. One easily checks that (with $r_0 = m$, $r_i = 0$ for $i>k$) [16][17]):

$$\dim.R_i(A,B) = \dim.R_i(A_c,B_c) = \sum_{j=1}^{i} r_j \qquad i=0,\ldots \qquad (5)$$

The remaining subsystem (A_c,B_c) is thus controllable because of the above rank properties and its controllability indices are given by :

$$\begin{array}{l} \text{there are } (r_{i+1} - r_i) \text{ controllability indices } c_j \\ \qquad \text{equal to } i \text{ for } i=1,\ldots,k \end{array} \qquad (6)$$

Notice that a system with m inputs always has m controllability indices [21]. The number k of indices r_i, on the other hand, does not depend on m but is equal to the largest controllability index:

$$k = c_m \qquad (7)$$

In the sequel we assume $\underline{A}_c$ does not exist and we thus identify (A,B) with (A_c,B_c).

Consider now the spaces (called i-th controllable subspace in [8]):

$$S_i(A,B) = \{ x \mid A^i x \in A^{i-1}\langle B\rangle + \ldots + \langle B\rangle \} \qquad i=1,\ldots,k \qquad (8)$$

This linear subspace is the set of all initial conditions x_0 to (1), that can be driven to zero in time i. Let A^{-1} denote the functional inverse of a map. Applied to a subspace S this thus means :

$$A^{-1} S = \{ x \mid Ax \in S \} \qquad (9)$$

It is shown in [1][12] that the spaces S_i satisfy the recursion :

$$\begin{array}{c} S_{i+1} = A^{-1}(S_i + \langle B\rangle) \\ \{0\} = S_0 \subset S_1 \subset \ldots \subset S_l = S_{l+1} = \ldots \end{array} \qquad (10)$$

where

$$l = \min \{ i \mid S_i = S_{i+1} \} \qquad (11)$$

The spaces R_i and S_i are known to be invariant under feedback [1][12]

$$\begin{array}{c} R_i(A,B) = R_i(A+BF,B) \\ S_i(A,B) = S_i(A+BF,B) \end{array} \qquad (12)$$

One proves for a controllable system (A,B) that :

$$\begin{array}{c} l = k \\ \dim.S_i = \dim.R_i = d_i = \sum_{j=1}^{i} r_j \end{array} \qquad (13)$$

This immediately follows from the fact that $\bar{A} = A+BF$ can be chosen to be invertible [21] and that then :

$$R_i = \bar{A}^{-i} S_i \qquad (14)$$

This property will be used in the sequel. A feedback matrix F is now a solution to the deadbeat control problem iff [1][12]:

$$(A+BF)S_i \subset S_{i-1} \qquad i=1,\ldots,k \qquad (15)$$

Let U be a unitary transformation partitioned in k blocks of r_k columns :

$$U = \Big[\underbrace{\bar{U}_1}_{r_1} \;\Big|\; \ldots \;\Big|\; \underbrace{\bar{U}_k}_{r_k} \Big] \qquad (16)$$

such that :

$$S_i = \left\langle \left[\bar{U}_1 \;\vdots\; \ldots \;\vdots\; \bar{U}_i \right] \right\rangle \tag{17}$$

Let then F be a solution of (15) then one proves that :

$$U'(A+BF)U = \begin{array}{c} \left[\begin{array}{cccc} 0 & A_{1,2} & \cdots & A_{1,k} \\ & 0 & \ddots & \vdots \\ & & \ddots & A_{k-1,k} \\ 0 & & & 0 \end{array}\right] \begin{array}{l} \}r_1 \\ \}r_2 \\ \vdots \\ \}r_{k-1} \\ \}r_k \end{array} \\ \underbrace{\;}_{r_1}\;\underbrace{\;}_{r_2}\qquad\qquad\underbrace{\;}_{r_k} \end{array} \tag{18}$$

This follows from the fact that in the new coordinate system (i.e. after the similarity transformation U'.U) the spaces S_i are spanned by :

$$S_i = \left\langle \begin{bmatrix} I_{d_i} \\ 0 \end{bmatrix} \right\rangle \quad i=1,\ldots,k \tag{19}$$

3. A recursive method

We now describe a method for constructing U and F such that U'(A+BF)U has the above form (18). The algorithm is recursive and consists of k steps, where k is as defined above in (3b). We will show that at the end of each step i the following form is obtained:

$$U_i'(A+BF_i)U_i = \begin{bmatrix} A_d^i & X \\ 0 & A_s^i \end{bmatrix} \begin{array}{l} \}\, d_i = \sum_{j=1}^{i} r_j \\ \}\, n-d_i \end{array} \tag{20}$$

$$U_i'B = \begin{bmatrix} B_d^i \\ B_s^i \end{bmatrix} \begin{array}{l} \}\, d_i \\ \}\, n-d_i \end{array} \tag{21}$$

Here the subsystem (A_d^i, B_d^i) is already "beaten to death":

$$\left[B_d^i \;\vdots\; A_d^i \right] = \left[\begin{array}{c:cccc} B_1^i & 0 & A_{1,2}^i & \cdots & A_{1,i}^i \\ \vdots & & \ddots & \ddots & \vdots \\ & & & \ddots & A_{i-1,i}^i \\ B_i^i & & & & 0 \end{array}\right] \begin{array}{l} \}r_1 \\ \\ \}r_{i-1} \\ \}r_i \end{array} \tag{22}$$

and the subsystem (A_s^i, B_s^i) is in staircase form :

$$\left[B_s^i \;\vdots\; A_s^i \right] = \left[\begin{array}{c:cccc} B_{i+1}^i & A_{i+1,i+1}^i & \cdots & \cdots & A_{i+1,k}^i \\ & A_{i+2,i+1}^i & \ddots & & A_{i+2,k}^i \\ 0 & & \ddots & \ddots & \vdots \\ & & & A_{k,k-1}^i & A_{k,k}^i \end{array}\right] \begin{array}{l} \}r_{i+1} \\ \}r_{i+2} \\ \\ \}r_k \end{array} \tag{23}$$

where the leading blocks B^i_{i+1} and $A^i_{j,j-1}$ $j=i+2,..,k$ have full row rank. At the beginning of step 1 (i.e. at the end of step 0) this is indeed satisfied since this is merely the staircase form (3). We now derive step i of the recursive algorithm. At the beginning of this step we thus have the configuration (20)-(23) with i decremented by 1. We then construct transformation and feedback matrices

$$\hat{U}_i = \begin{bmatrix} 1_{d_{i-1}} & 0 \\ 0 & U^i_s \end{bmatrix}, \qquad \hat{F}_i = \begin{bmatrix} 0 & F^i_s \end{bmatrix} \tag{24}$$

that only affect the subsystem (A^{i-1}_s, B^{i-1}_s). We are thus trying to find matrices F^i_s and U^i_s such that:

$$U^{i'}_s(A^{i-1}_s + B^{i-1}_s F^i_s)U^i_s = \begin{array}{c} \left[\begin{array}{c|c} 0 & X \\ \hline 0 & X \end{array}\right] \begin{array}{l} \}r_i \\ \}n-d_i \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{r_i} & \underbrace{\qquad}_{n-d_i} \end{array} \end{array} \tag{25}$$

or equivalently:

$$(A^{i-1}_s + B^{i-1}_s F^i_s)U^i_s = \begin{array}{c} \left[\begin{array}{c|c} 0 & X \end{array}\right] \\ \begin{array}{cc} \underbrace{\quad}_{r_i} & \underbrace{\qquad}_{n-d_i} \end{array} \end{array} \tag{26}$$

This is obtained as follows: let U^i_s be a unitary transformation triangularizing A^{i-1}_s (with r_j ,j=i,...,k as in (3)):

$$A^{i-1}_s U^i_s = R^i = \left[\begin{array}{ccccccc} R^i_{i,i} & . & . & . & . & . & R^i_{i,k} \\ & . & & & & & . \\ & & . & & & & . \\ & & & . & & & . \\ & & & & . & & . \\ & & & & & & R^i_{k,k} \end{array}\right] \begin{array}{l} \}r_i \\ . \\ . \\ . \\ . \\ \}r_k \end{array} \tag{27}$$

Here again the result was paritioned conformably with A . Then solve the equation

$$R^i_{i,i} = -B^{i-1}_i .X \tag{28}$$

This equation has a solution since B^{i-1}_i has full row rank and thus has a left inverse [14]. A minimum norm solution G_i is given by

$$G_i = -B^{i-1^+}_i .R^i_{i,i} \tag{29}$$

where $.^+$ denotes the Moore Penrose inverse of a matrix. Notice that this is a minimum norm solution in the spectral norm and moreover the unique minimum norm solution in the Frobenius norm [14].

Using the feedback:

$$F^i_s = \begin{bmatrix} G_i & 0 \ldots & \ldots 0 \end{bmatrix} U^{i'}_s \tag{30}$$

it is easily seen that $(A^{i-1}_s + B^{i-1}_s F^i_s)U^i_s$ is equal to

$$(A^{i-1}_s + B^{i-1}_s F^i_s)U^i_s = \begin{array}{c} \left[\begin{array}{ccccc} 0 & R^i_{i,i+1} & . & . & R^i_{i,k} \\ & R^i_{i+1,i+1} & . & . & R^i_{i+1,k} \\ & & . & & . \\ & & & . & . \\ & & & & R^i_{k,k} \end{array}\right] \begin{array}{l} \}r_i \\ \}r_{i+1} \\ . \\ . \\ \}r_k \end{array} \\ \begin{array}{ccccc} \underbrace{\quad}_{r_i} & \underbrace{\quad}_{r_{i+1}} & . & . & \underbrace{\quad}_{r_k} \end{array} \end{array} \tag{31}$$

and thus satisfies (26).

We now prove that $U_s^{i'}$ has the same block structure as A_s^{i-1}, i.e.:

$$U_s^{i'} = \begin{array}{c} \left[\begin{array}{ccccc} U_{i,i} & U_{i,i+1} & \cdots & & U_{i,k} \\ U_{i+1,i} & U_{i+1,i+1} & \cdots & & U_{i+1,k} \\ & \ddots & \ddots & & \vdots \\ & 0 & \ddots & & \vdots \\ & & & U_{k,k-1} & U_{k,k} \end{array}\right] \begin{array}{l} \}r_i \\ \}r_{i+1} \\ \\ \\ \}r_k \end{array} \\ \underbrace{}_{r_i} \quad \underbrace{}_{r_{i+1}} \quad \quad \underbrace{}_{r_k} \end{array} \tag{32}$$

Indeed, when writing (27) as :

$$A_s^{i-1} = R^i . U_s^{i'} \tag{33}$$

it follows that the bottom $n-d_i$ rows of R^i are linearly independent since those of A_s^{i-1} are, and since the invertible column transformation $U_s^{i'}$ does not affect the independence of the rows of the matrix R^i. Thus the blocks $R^i_{j,j}$, $j=i+1,\ldots,k$ in (27) are invertible. Let S be the upper triangular inverse of the $(n-d_i)x(n-d_i)$ bottom part of R^i, then multiplying the bottom $n-d_i$ rows of (33) with S from the left, we obtain:

$$S. \left[\begin{array}{ccccc} A^{i-1}_{i+1,i} & A^{i-1}_{i+1,i+1} & \cdots & & A^{i-1}_{i+1,k} \\ & \ddots & & & \vdots \\ & 0 & \ddots & & \vdots \\ & & & A^{i-1}_{k,k-1} & A^{i-1}_{k,k} \end{array}\right] \begin{array}{l} \}r_{i+1} \\ \\ \\ \}r_k \end{array} = \tag{34a}$$

$$\left[\begin{array}{ccccc} U_{i+1,i} & U_{i+1,i+1} & \cdots & & U_{i+1,k} \\ \vdots & \ddots & & & \vdots \\ & & \ddots & & \vdots \\ U_{k,i} & \cdots & & U_{k,k-1} & U_{k,k} \end{array}\right] \begin{array}{l} \}r_{i+1} \\ \\ \\ \}r_k \end{array} \tag{34b}$$

Since S is an upper triangular invertible matrix we indeed find that the blocks $U_{j,i}$ with $j>i+1$ are zero and that the blocks $U_{j,j-1}$ have linearly independent rows, which completes the proof.

Because of the above structure of the transformation $U_s^{i'}$, we have that, after this step i, the following form is obtained:

$$\left[\begin{array}{c:c} U_s^{i'} B_s^{i-1} & U_s^{i'} (A_s^{i-1} + B_s^{i-1} F_s^i) U_s^i \end{array}\right] =$$

$$\left[\begin{array}{c:c:cccc} B^i_i & 0 & A^i_{i,i+1} & \cdots & & A^i_{i,k} \\ \hdashline B^i_{i+1} & & A^i_{i+1,i+1} & \cdots & & A^i_{i+1,k} \\ & & A^i_{i+2,i+1} & \ddots & & A^i_{i+2,k} \\ 0 & 0 & & \ddots & \ddots & \vdots \\ & & & & \ddots & \vdots \\ & & & & A^i_{k,k-1} & A^i_{k,k} \end{array}\right] \tag{35}$$

$$F^i_s U^i_s = \left[\begin{array}{c:c} G_i & 0 \end{array}\right]$$

Moreover the identities:

$$B^i_{i+1} = U_{i+1,i} .B^{i-1}_i \; ; A^i_{j+1,j} = U_{j+1,j} .R^i_{j,j} \; ,j=i+1,..,k-1 \tag{36}$$

follow from (31)(32) and these imply that the blocks in (36) have full column rank.

Embedding this in (20)-(23) for i decremented by 1, we clearly retrieve the form (20)-(23) at the end of step i. The updating of the transformation and feedback matrices are easily checked to be :

$$U_i = U_{i-1} . \hat{U}_i \tag{37a}$$

$$F_i U_i = F_{i-1} U_i + \hat{F}_i \hat{U}_i = F_{i-1} U_{i-1} + \hat{F}_i \hat{U}_i \tag{37b}$$

The last equality follows from the fact that the feedback matrices $F_{i-1} U_{i-1}$ have their last $(n-d_i)$ columns equal to zero and are therefore unaffected by the subsequent transformations $\hat{U}_j$, for $j>i-1$.
After k steps of this recursion, one finally obtains:

$$U'_k(A+BF_k)U_k = \left[\begin{array}{c|ccccc} B^k_1 & 0 & A^k_{1,2} & \dots & \dots & A^k_{1,k} \\ B^k_2 & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & A^k_{k-1,k} \\ B^k_k & & & & & 0 \end{array}\right] \tag{38}$$

where

$$U_k = \hat{U}_1 \hat{U}_2 \dots \hat{U}_k \tag{39a}$$

$$\hat{U}_i = \begin{bmatrix} I_{d_{i-1}} & 0 \\ 0 & U^i_s \end{bmatrix} \tag{39b}$$

$$F_k = \hat{F}_1 + \hat{F}_2 + \dots + \hat{F}_k \tag{39c}$$

$$\hat{F}_i U_k = \begin{bmatrix} 0 \dots 0 \; G_i \; 0 \dots 0 \end{bmatrix} \tag{39d}$$

This is now clearly in the form (18) as requested in section 2.

4. Numerical Considerations

In this section we discuss the numerical stability of the above method and show that it yields a minimum norm solution to the problem when solutions are not unique.

To analyze the stability of the algorithm, we first remark that the transformation matrix U obtained by the algorithm is independent of the feedback matrix F and that it satisfies (16)(17), i.e. the first d_i columns of U span the i-th controllable subspace S_i in the coordinate system of (3). This can be checked by induction on the algorithm but it also follows from (15)(16)(17) and the fact that the spaces S_i are defined independently of the feedback matrix F (see e.g. in (8) or (10)).
Let us now look at the problem in the coordinate system of (A_u,B_u), partitioned conformably with the original (A,B)-pair :

$$A_u = U'AU = \begin{bmatrix} A^u_{11} & \dots & A^u_{1k} \\ \vdots & & \vdots \\ A^u_{k1} & \dots & A^u_{kk} \end{bmatrix} ; B_u = U'B = \begin{bmatrix} B^u_1 \\ \vdots \\ B^u_k \end{bmatrix} = \begin{bmatrix} B^k_1 \\ \vdots \\ B^k_k \end{bmatrix} \tag{40}$$

Since in this coordinate system there exists a feedback matrix F_u such that $A_u+B_uF_u$ has the form (18), we have that:

$$< \begin{bmatrix} A^u_{11} \\ \vdots \\ A^u_{k1} \end{bmatrix} > \subset < \begin{bmatrix} B^u_1 \\ \vdots \\ B^u_k \end{bmatrix} > \tag{41}$$

Because of the special structure of the transformation matrices $\hat{U}_i$ (39b) it follows that:

$$\begin{bmatrix} B^u_i \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ B^u_k \end{bmatrix} \begin{bmatrix} A^u_{ii} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ A^u_{ki} \end{bmatrix} = \hat{U}'_k \dots \hat{U}'_i . \begin{bmatrix} B^{i-1}_i \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \begin{bmatrix} R^i_{ii} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \qquad (42)$$

with B^{i-1}_i of full row rank. The minimum norm solution G_i in (29) of the system (28) is thus also the minimum norm solution of

$$\begin{bmatrix} A^u_{ii} \\ \cdot \\ \cdot \\ \cdot \\ A^u_{ki} \end{bmatrix} + \begin{bmatrix} B^u_i \\ \cdot \\ \cdot \\ \cdot \\ B^u_k \end{bmatrix} . G_i = 0 \qquad (43)$$

and this is thus the corresponding i-th submatrix of F_u. Let us now write the analagous perturbed equations, where computed and therefore perturbed quantities are denoted with an upper bar. Unitary transformations can be performed in a backward stable manner [14][20] and thus equation (40) yields :

$$\bar{A}_u = \bar{U}'A\bar{U} \quad , \quad \bar{B}_u = \bar{U}'B \qquad (44a)$$

with

$$\bar{U} = U+\Delta U, \quad U'U=I_n \quad , \ ||\Delta U|| = \epsilon_u \qquad (44b)$$

and with ϵ_u of the order of the relative precision ϵ of the computer. Notice that in fact G_i is not computed in this coordinate system but rather by solving the equivalent equation (29). Each separate column of the computed solutions $\bar{G}_i$ i=1,...,k is then obtained in a backward stable manner but one can not guarantee [14] that there exists single ϵ perturbations $\bar{A}_e$ and $\bar{B}_e$ of $\bar{A}_u$ and $\bar{B}_u$, respectively, such that

$$(\bar{A}_e + \bar{B}_e . \bar{F}_u) \qquad (45)$$

has exactly the zero structure described by (18). Yet one can show (the proof is omitted here in the interests of brevity) that there exist perturbations $\bar{A}_e$ and $\bar{B}_e$ satisfying (45) with the weaker bounds :

$$\bar{A} = \bar{U}.\bar{A}_e.\bar{U}', \ \bar{A} = A+\Delta A \ , \quad ||\Delta A|| = \epsilon_a ||A||+ \epsilon_{bf} ||B||.||F|| \qquad (46a)$$

$$\bar{B} = \bar{U}.\bar{B}_e \ , \ \bar{B} = B+\Delta B \ , \quad ||\Delta B|| = \epsilon_b ||B|| \qquad (46b)$$

$$F = \bar{F}_u \bar{U}' \qquad (46c)$$

where ϵ_a, ϵ_b and ϵ_{bf} are of the order of ϵ, and where $\bar{U}$ and $\bar{F}_u$ are the exact matrices stored in computer (notice that we do not have F in computer). Moreover,

$$[(A+\Delta A)+(B+\Delta B)\bar{F}_u \bar{U}']^k = 0 \qquad (47)$$

Although we can not prove backward stability for this part of the algorithm, one obtains for the norm of $[A+B.\bar{F}_u \bar{U}']^k$ bounds that are of the same order than those that would be obtained for a stable algorithm. This method does therefore not behave worse than a stable method in this sense.

In order to prove that the obtained feedback matrix F is the unique minimum (Frobenius) norm solution to the problem we observe that this is the case by construction for each of the submatrices G_i of F_u in (43). Since for the Frobenius norm we have

$$||F||^2 = ||F_u||^2 = \sum_i ||G_i||^2 \qquad (48)$$

this then also holds for F_u and F in their respective coordinate systems.

We terminate this section with an example.

Example

Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This is already in staircase form ($r_1=2$, $r_2=1$) .We then find

$$F_u = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$A+BF = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_u+B_uF_u = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Notice that the general solution F_u^g to the deadbeat problem is

$$F_u^g = \begin{bmatrix} -1 & -1 & d \\ 0 & -1 & 1 \end{bmatrix}, \quad A_u+B_uF_u^g = \begin{bmatrix} 0 & 0 & d \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly F_u is the minimal Frobenius norm solution, but this can be checked not to hold for the 2-norm on this example (take e.g. $d=1/2$).

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References

[1] H. Akashi, H. Imai, M. Adachi,"On the optimal control of linear discrete-time systems by geometric approach", Int. J. Contr., Vol. 28, pp.853-867, 1978.

[2] D. Boley,"Computing the controllability/observability decomposition of a linear time-invariant dynamic system, a numerical approach", Ph.D. Thesis, Comp. Sc. Dept., Stanford Univ., June 1981.

[3] R. Eising,"Pole assignment, a new proof and algorithms", Memo COSOR 81-10, T.H. Eindhoven, The Netherlands, Aug. 1981.

[4] A. Emami-Naeini, G. Franklin,"Deadbeat control and tracking of discrete time systems", IEEE Trans. Aut. Contr., Vol. AC-27, pp.176-180, Febr. 1982.

[5] M. Konstantinov, P. Petkov, N. Christov,"Control of linear systems via the serial canonical form", Proceedings 2nd IFAC/IFIP Symp. Software for Computer Control, Vol. II, June 1979.

[6] M. Konstantinov, P. Petkov, N. Christov,"Synthesis of linear systems with desired equivalent form", J. Comp. Appl. Math., Vol. 6, pp.27-35, 1980.

[7] H. Kwakernaak, R. Sivan,"Linear Optimal Control Systems", New York, Wiley, 1972.

[8] F. L. Lewis,"A general Riccati equation solution to the deadbeat control problem", IEEE Trans. Aut. Contr., Vol. AC-27, pp.186-188, Feb. 1982.

[9] G. Minimis, C. Paige,"An algorithm for pole assignment of time invariant linear systems", submitted to Int. J. Control.

[10] G. S. Minimis, C. C. Paige, "An algorithm for pole assignment of time invariant multi-input linear systems", in Proceedings IEEE Conf. Dec. & Contr.

[11] C. Moler, "MATLAB User's Guide", Technical Report CS81-1, Dept. of Computer Science, University of New Mexico, June 1981.

[12] C. T. Mullis, "Time optimal discrete regulator gains", IEEE Trans. Aut. Contr., Vol. AC-17, pp.265-266, 1972.

[13] R. Patel,"Computation of minimal-order state-space realizations and observability indices using orthogonal transformations", Int. J. Contr., Vol. 33, pp.227-246, 1981.

[14] G.W. Stewart,"Introduction to Matrix Computations",New York, Academic, 1973.

[15] P. Van Dooren, A. Emami-Naeini, L. Silverman,"Stable extraction of the Kronecker structure of pencils", Proceedings 17th IEEE Conf. Decision and Control, pp.521-524, Jan. 1979.

[16] P. Van Dooren,"On the computation of the Kronecker canonical form of a singular pencil", Linear Algebra & Applications, Vol. 27, pp.103-141 , Oct. 1979.

[17] P. Van Dooren,"The generalized eigenstructure problem in linear system theory", IEEE Trans. Aut. Contr., Vol. AC-26, pp.111-129, Feb. 1981.

[18] A. Varga,"Numerically stable algorithm for standard controllability form determination", Electronic Letters, Vol. 17, pp.74-75, Jan. 1981.

[19] A. Varga,"A Schur method for pole assignment", IEEE Trans. Aut. Contr., Vol. AC-26, pp.517-519, April 1981.

[20] J. H. Wilkinson,"The algebraic eigenvalue problem", Oxford, Clarendon, 1965.

[21] W. M. Wonham,"Linear Multivariable Control :A Geometric Approach", 2nd Ed., New York, Springer, 1979.