COMPUTATION OF ZERO DIRECTIONS OF TRANSFER FUNCTIONS

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ABSTRACT

In this note we describe a state space approach to compute so-called zero directions of a rational transfer function $H(\lambda)$. The method works on the coefficient matrices of a minimal state space realization of the transfer function $H(\lambda)$ and does not require its Taylor expansion around each zero. Moreover, it uses only unitary transformations to find the structure at each zero.

1. Background and definitions

Finding zeros and their structural indices is a problem that occurs naturally when one wants to describe the solution set of particular matrix equations involving rational matrices [4, 5, 7] and has recently received attention in the context of interpolation problems [1]. The structure at a point λ_0 which is a pole or a zero of a transfer function $H(\lambda)$ is defined via the Smith McMillan form of the $m \times n$ rational matrix $H(\lambda)$ at the point λ_0:

$$M(\lambda) \cdot H(\lambda) \cdot N(\lambda) =$$

$$\left[\begin{array}{ccc|c} (\lambda-\lambda_0)^{\sigma_1} & & 0 & \\ & \ddots & & \\ 0 & & (\lambda-\lambda_0)^{\sigma_r} & \\ \hline & & & 0_{m-r,n-r} \end{array}\right] \quad (1)$$

where $M(\lambda)$ and $N(\lambda)$ are rational and non singular at λ_0, r is the normal rank of $H(\lambda)$ and the *structural indices* σ_i of $H(\lambda)$ at the point λ_0 satisfy $\sigma_1 \geq \ldots \geq \sigma_r$. Notice that positive indices refer to zeros of the transfer function and negative indices refer to poles. The computation of the above decomposition is based on the Euclidean algorithm and Gaussian elimination over the ring of polynomials, which is intractable and numerically unreliable [11]. For this reason it is often replaced by the following technique, based on the expansion around the point λ_0 [11]. Assume that $H(\lambda)$ does not have a pole at λ_0 but only a zero. Then it has a Taylor expansion :

$$H(\lambda) \doteq$$

$$H_0 + (\lambda-\lambda_0)H_1 + (\lambda-\lambda_0)^2 H_2 + (\lambda-\lambda_0)^3 H_3 + \ldots \quad (2)$$

From this we define the Toeplitz matrices

$$T_{\lambda_0,k} \doteq \begin{bmatrix} H_0 & & & \\ H_1 & H_0 & & \\ \vdots & \ddots & \ddots & \\ H_{k-1} & \ldots & H_1 & H_0 \end{bmatrix} \quad (3)$$

One shows that the rank increments

$$\rho_k \doteq rankT_{\lambda_0,k+1} - rankT_{\lambda_0,k}$$

are in fact in one to one correspondence with the index set $\sigma_i, i = 1, \ldots, r$ [11].

The connection between (1) and (3) follows from the following observation. Let $x(\lambda)$ and $y^T(\lambda)$ be the vectors

$$x(\lambda) \doteq$$

$$x_0 + (\lambda-\lambda_0)x_1 + (\lambda-\lambda_0)^2 x_2 + \ldots + (\lambda-\lambda_0)^{k-1} x_{k-1} \quad (4)$$

$y(\lambda) \doteq$

$$y_0+(\lambda-\lambda_0)y_1+(\lambda-\lambda_0)^2y_2+\ldots+(\lambda-\lambda_0)^{k-1}y_{k-1} \quad (5)$$

then we call these *zero vectors of order* k if

$$H(\lambda)x(\lambda) = O(\lambda-\lambda_0)^k, \quad (6)$$

$$y^T(\lambda)H(\lambda) = O(\lambda-\lambda_0)^k, \quad (7)$$

and if the respective vectors x_0 and y_0 are nonzero (the latter condition avoids trivial solutions) [1] [5]. Using the expansion (2) we can rewrite (6) (7) as

$$T_{\lambda_0,k}\begin{bmatrix} x_0 \\ \vdots \\ x_{k-1}\end{bmatrix} = 0, \quad (8)$$

$$\begin{bmatrix} y_{k-1}^T & \ldots & y_0^T \end{bmatrix} T_{\lambda_0,k} = 0. \quad (9)$$

So the problem of finding these zero directions is apparently solved by computing the null spaces of these Toeplitz matrices, for which recursive algorithms have been proposed in [11]. Using this approach one constructs exactly r such zero vectors of orders $\sigma_i,\ i=1,\ldots,r$, respectively. Yet, all this still requires the computation of the Taylor expansions (2) in each zero of the transfer function $H(\lambda)$ and the complexity of the subsequent rank search is still quite high [11].

2. A state-space approach

Any transfer function $H(\lambda)$ can be realized by a *generalized state-space realization* $\{A,B,C,D,E\}$ such that

$$H(\lambda) = C(\lambda E-A)^{-1}B+D. \quad (10)$$

If $H(\lambda)$ happens to be proper – $H(\infty)$ bounded – then E can be chosen the identity and one refers to the system $\{A,B,C,D\}$ as a *state-space realization* of $H(\lambda)$. One often associates with the above realization (10) the pencil

$$S(\lambda) \doteq \left[\begin{array}{c|c} A-\lambda E & B \\ \hline C & D\end{array}\right] \quad (11)$$

called the *system matrix* of the realization. (Notice that $H(\lambda)$ is its Schur complement.) Techniques for constructing a realization (10) are described in e.g. [10]. It is also shown there how to obtain an *irreducible generalized state space realization* of the transfer function $H(\lambda)$. These are realizations (10) for which

$$\left[\begin{array}{c} A-\lambda E \\ \hline C\end{array}\right] \quad (12)$$

and

$$\left[\begin{array}{c|c} A-\lambda E & B\end{array}\right] \quad (13)$$

have no finite or infinite zeros (see also [12]). In [10] realizations with stronger conditions are actually derived, namely for which

$$inf\{\mathcal{X}|dim.A\mathcal{X}+E\mathcal{X} = dim.\mathcal{X}, Im.B\subset\mathcal{X}\} = \mathcal{R}^n$$

$$sup\{\mathcal{X}|dim.A\mathcal{X}+E\mathcal{X} = dim.\mathcal{X}, \mathcal{X}\subset Ker.C\} = \emptyset$$

where n is the order of the matrices E and A and $dim.A\mathcal{X}+E\mathcal{X} = dim.\mathcal{X}$ defines a deflating subpace of the pencil $\lambda E - A$ in $\mathcal{R}^n$ (this is a generalization of the concept of invariant subspace [8]).

3. Zero vectors of $S(\lambda)$ and $H(\lambda)$

As was shown in [12], much of the structure of $H(\lambda)$ can be retrieved in that of $S(\lambda)$, when the underlying generalized state-space realization is irreducible. It is shown there that in that case

- the pole structure of $H(\lambda)$ at its finite and infinite poles is identical to the zero structure of $\lambda E - A$ at its finite and infinite zeros.
- the zero structure of $H(\lambda)$ at its finite and infinite zeros is identical to the zero structure of $S(\lambda)$ at its finite and infinite zeros.
- the left and right null-space structures of $H(\lambda)$ and $S(\lambda)$ are the same.

In the above relations, *structure* refers to the index sets of the finite and infinite elementary devisors and of the left and right minimal bases. In the rest of this section we show there is also a relation between the zero *directions* as well as the index sets.

Theorem 1

If $H(\lambda)$ has no poles at $\lambda=\lambda_0$ and if $x(\lambda)$ is a right zero vector of order k at λ_0 of $S(\lambda)$ then partitioning it as $x^T(\lambda) \doteq [x_1^T(\lambda), x_2^T(\lambda)]$ conformably with the block columns of $S(\lambda)$, we have that $x_2(\lambda)$ is a right zero vector of order k at λ_0 of $H(\lambda)$. The dual result holds for left zero vectors of order k.

Proof :

The proof is similar for left and right vectors. Let $x(\lambda)$ be a right zero vector of order k of $S(\lambda)$ then we have that :

$$S(\lambda)x(\lambda) = O(\lambda - \lambda_0)^k. \tag{14}$$

Partitioning this conformably, we get :

$S(\lambda)x(\lambda) =$

$$\left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right] \cdot \left[\begin{array}{c} x_1(\lambda) \\ \hline x_2(\lambda) \end{array}\right] = O(\lambda - \lambda_0)^k. \tag{15}$$

Since λ_0 is not a pole of $H(\lambda)$ we have that it is not an eigenvalue of $\lambda E - A$ and hence the matrix transformations

$$T(\lambda) \doteq \left[\begin{array}{c|c} I & 0 \\ \hline -C(A - \lambda E)^{-1} & I \end{array}\right]$$

is regular at λ_0, i.e. $T(\lambda)$ and $T^{-1}(\lambda)$ both have a Taylor expansion around λ_0. As a consequence, applying this to the left of (15), we have :

$T(\lambda)S(\lambda)x(\lambda) =$

$$\left[\begin{array}{c|c} A - \lambda E & B \\ \hline 0 & H(\lambda) \end{array}\right] \cdot \left[\begin{array}{c} x_1(\lambda) \\ \hline x_2(\lambda) \end{array}\right] = O(\lambda - \lambda_0)^k. \tag{16}$$

and hence we clearly have the requested property $H(\lambda)x_2(\lambda) = O(\lambda - \lambda_0)^k$. □

If a set of linear independent vectors of order k of $S(\lambda)$ are grouped in a matrix $X(\lambda)$, then again the bottom matrix $X_2(\lambda)$ has the same property for $H(\lambda)$, as shown now below.

Theorem 2

If $H(\lambda)$ has no poles at $\lambda = \lambda_0$ and if $X(\lambda)$ is a basis of right zero vectors of order k at λ_0 of $S(\lambda)$ then the bottom block $X_2(\lambda)$ is a basis of right zero vectors of order k at λ_0 of $H(\lambda)$. The dual result holds for left zero vectors of order k.

Proof :

The proof is again similar for left and right vectors. Let $X(\lambda)$ be a basis of right zero vectors of order k of $S(\lambda)$ then we have that

$$S(\lambda) \cdot \left[\begin{array}{c} X_1(\lambda) \\ \hline X_2(\lambda) \end{array}\right] = O(\lambda - \lambda_0)^k. \tag{17}$$

and

$T(\lambda)S(\lambda)X(\lambda) =$

$$\left[\begin{array}{c|c} A - \lambda E & B \\ \hline 0 & H(\lambda) \end{array}\right] \cdot \left[\begin{array}{c} X_1(\lambda) \\ \hline X_2(\lambda) \end{array}\right] = O(\lambda - \lambda_0)^k. \tag{18}$$

Hence, $H(\lambda)X_2(\lambda) = O(\lambda - \lambda_0)^k$.

Now we have to prove that if the vectors of $X(\lambda)$ are linear independent then so must be those of $X_2(\lambda)$. In [1] [5] [6] it is shown that the columns of a matrix $X(\lambda)$ are a basis iff $X(\lambda_0)$ has linear independent columns. But writing down the constant term of (18) we have :

$$\left[\begin{array}{c|c} A - \lambda_0 E & B \\ \hline 0 & H(\lambda_0) \end{array}\right] \cdot \left[\begin{array}{c} X_1(\lambda_0) \\ \hline X_2(\lambda_0) \end{array}\right] = 0 \tag{19}$$

and this implies that $X_2(\lambda_0)$ has linear independent columns. Indeed, if this would not be the case, then there exists a vector z such that $X_2(\lambda_0)z = 0$. Then it would also follow from the full rank property of $X(\lambda_0)$ that $y \doteq X_1(\lambda_0)z \neq 0$ and from (19) that $(A - \lambda_0 E)y = 0$, which contradicts the assumptions that $H(\lambda)$ has no poles at λ_0. □

This thus shows that the problem of finding the zero directions of a transfer function is reduced to that of a pencil

$$S(\lambda) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] - \lambda \left[\begin{array}{c|c} E & 0 \\ \hline 0 & 0 \end{array}\right] \tag{20}$$

derived from an irreducible generalized state-space realization.

4. Zero directions of pencils

Finding the zero directions of pencils simplifies very much the above problem. First of all, we have immediately a generalized eigenvalue problem $S(\lambda)$ to find the zeros of the transfer function [2] [3]. These methods are based on the generalized Schur form of an arbitrary pencil. Secondly, once a particular zero λ_0 is found, one can update the generalized Schur form to find the Kronecker structure of the pencil at this eigenvalue [9] and as we show now below, also the zero directions of this eigenvalue.

Consider the expansion of the pencil around $\hat{\lambda} \doteq (\lambda - \lambda_0)$:

$$S(\lambda) = \hat{S}_0 + (\lambda - \lambda_0)\hat{S}_1 = \hat{S}_0 + \hat{\lambda}\hat{S}_1 \tag{21}$$

One could use again the above Toeplitz matrices to define kernels and from this zero directions, but this is very inefficient. The staircase algorithm [9] applied to $\hat{S}_0$ and $\hat{S}_1$ in fact gives all the information to find these directions. It is shown in [9] that there always exist unitary transformations U and V (these will be orthogonal when the system and λ_0 are real) such that :

$$U(\hat{S}_0 + \hat{\lambda}\hat{S}_1)V = \hat{A} - \hat{\lambda}\hat{E} = \tag{22}$$

$$\left[\begin{array}{cccc|c} 0 & A_{1,2} & & * & \\ & \ddots & \ddots & & * \\ & & \ddots & A_{\ell-1,\ell} & \\ & & & 0 & \\ \hline & & & & A_r \end{array}\right] - \hat{\lambda} \left[\begin{array}{cccc|c} E_{1,1} & * & & * & \\ & \ddots & \ddots & & * \\ & & \ddots & * & \\ & & & E_{\ell,\ell} & \\ \hline & & & & E_r \end{array}\right]$$

where
(i) the $E_{i,i}$ matrices are of dimension $s_i \times t_i$ and of full row rank s_i,
(ii) the $A_{i,i+1}$ matrices are of dimension $s_i \times t_{i+1}$ and of full column rank t_{i+1},
(iii) A_r is of full column rank.
From this it follows immediately that

$$t_1 \geq s_1 \geq t_2 \geq s_2 \geq \ldots \geq t_\ell \geq s_\ell \geq 0 (\doteq t_{\ell+1}) \tag{23}$$

The transformations U and V are chosen unitary (or orthogonal in the real case) for reasons of numerical stability. Moreover, since these are constant transformations, they only transform the coordinate system in which we have to construct the zero directions $x(\hat{\lambda})$ and $y(\hat{\lambda})$. The above form (22) is appropriate for constructing $x(\hat{\lambda})$, but there exists a dual form where the role of columns and rows is interchanged, and which can thus be used to construct $y(\hat{\lambda})$. Below we focus on finding solutions $x_V(\hat{\lambda})$ in the coordinate system of (22) but the corresponding zero directions of $S(\hat{\lambda})$ are easily seen to be $x(\hat{\lambda}) = V x_V(\hat{\lambda})$.

For (22) it is easy to see that a zero vector x_0 of order 0 must be in the null space of $\hat{A}$, i.e. $x_0 \in Im \begin{bmatrix} I_{t_1} \\ 0 \end{bmatrix}$. A basis for these zero directions of order 0 is thus the matrix

$$X_0^0 \doteq \begin{bmatrix} I_{t_1} \\ 0 \end{bmatrix}. \tag{24}$$

For zero directions $x_0 + \hat{\lambda}x_1$ of order 1, we consider the equation

$$\hat{A}x_0 = 0, \; \hat{E}x_0 = \hat{A}x_1. \tag{25}$$

Clearly x_0 must again be in $Im \begin{bmatrix} I_{t_1} \\ 0 \end{bmatrix}$ and the second equation thus implies $\hat{E}x_0 \in Im \begin{bmatrix} I_{s_1} \\ 0 \end{bmatrix}$ and also $\hat{A}x_1 \in Im \begin{bmatrix} I_{s_1} \\ 0 \end{bmatrix}$. Because of the rank property of $\hat{A}$ we then have $x_1 \in Im \begin{bmatrix} I_{t_1+t_2} \\ 0 \end{bmatrix}$. The vectors $x_1 \in Im \begin{bmatrix} I_{t_1} \\ 0 \end{bmatrix}$ lead to trivial solutions with $x_0 = 0$ and must thus be excluded. So take $x_1 \in \begin{bmatrix} 0 \\ I_{t_2} \\ 0 \end{bmatrix}$ and call x_{12} the subvector in the range of the submatrix I_{t_2} above. Then it is easy to see that

$$x_0 = \begin{bmatrix} E_{11}^{+}A_{12}x_{12} \\ 0 \end{bmatrix} \in Im \begin{bmatrix} I_{t_1} \\ 0 \end{bmatrix},$$

where M^+ denotes the pseudoinverse of a matrix M. Both equations are satisfied with $x_0 \neq 0$ since E_{11}^{+} and A_{12} are both of full column rank. A basis for the nontrivial zero directions of order 1 is thus

$$X_1^1 \doteq \begin{bmatrix} 0 \\ I_{t_2} \\ 0 \end{bmatrix}, \; X_0^1 \doteq \hat{E}^+\hat{A}X_1^1 = \begin{bmatrix} E_{11}^{+}A_{12} \\ 0 \end{bmatrix},$$

and X_0^1 has full column rank t_2 since E_{11}^{+} and A_{12} have full column rank.

For the general case of k-th order zero directions

$$x_0 + \hat{\lambda}x_1 + \ldots + \hat{\lambda}^{k-1}x_{k-1}$$

we must satisfy

$$\hat{A}x_0 = 0,\ \hat{E}x_0 = \hat{A}x_1,\ \ldots,\ \hat{E}x_{k-2} = \hat{A}x_{k-1}. \tag{26}$$

Here again, we have to exclude trivial directions, i.e. those contained in just $\hat{\lambda}$ times a lower order zero directions, and clearly these are vectors with $x_0 = 0$. By induction, one shows that a (non-trivial) basis for these vectors can be found from the starting matrix

$$X_{k-1}^{k-1} \doteq \begin{bmatrix} 0 \\ I_{t_k} \\ 0 \end{bmatrix} \tag{27}$$

and that the other matrices $X_j^{k-1}, j = k-2, \ldots, 0$ are found recursively from (26) using the identity $X_{j-1}^{k-1} = \hat{E}^+\hat{A}X_j^{k-1}$. One checks that these matrices are of the form

$$X_j^{k-1} \doteq \begin{bmatrix} * \\ X_{j,j+1}^{k-1} \\ 0 \end{bmatrix}, \tag{28}$$

where

$$X_{j,j+1}^{k-1} = E_{j+1,j+1}^{+}A_{j+1,j+2}\cdots E_{k-1,k-1}^{+}A_{k-1,k}.$$

The constant matrix in this basis is thus

$$X_0^{k-1} = \begin{bmatrix} E_{1,1}^{+}A_{1,2}\cdots E_{k-1,k-1}^{+}A_{k-1,k} \\ 0 \end{bmatrix}, \tag{29}$$

and is again of full column rank since all matrices $E_{i,i}^{+}$ and $A_{i,i+1}$ are of full column rank t_k.

Since the number of non-trivial vectors of order k is t_k it follows from (23) that k must be smaller than ℓ, the number of stairs in the staircase form (22). Using results of [9] we know from (22) that the pencil $S(\lambda)$ has
(i) $t_i - s_i$ Kronecker column indices equal to $i-1$,
(ii) $s_i - t_{i+1}$ elementary divisors $\hat{\lambda}^i$.
So the bases of zero directions of order k we constructed above actually consist of a mixture of vectors in the null space of $S(\lambda)$ and "true" zero vectors corresponding to elementary divisors $\hat{\lambda}^i$. In the work of [6] only the latter are constructed but an assumption of right regularity is assumed, which consist of saying there are no right Kronecker indices. In this case our approach would yield the same result. The bases we compute actually are of the same type as those in [6] but the algorithm to construct them are different.

In the general case we are treating here, we can also modify the above algorithm such as to avoid constructing any vectors in the null space of $S(\lambda)$ (or, equivalently, $H(\lambda)$). In order to do this, we again use a decomposition of [9]. Instead of finding (22) we first split off the left Kronecker indices in the top left corner of that decompostion and then proceed further with essentially the same form. We then have something of the type

$$U(\hat{S}_0 + \hat{\lambda}\hat{S}_1)V = \hat{A} - \hat{\lambda}\hat{E} = \tag{30}$$

$$\left[\begin{array}{c|cccc|c} A_l & & & * & & \\ \hline & 0 & A_{1,2} & & * & \\ & & \ddots & \ddots & & * \\ & & & \ddots & A_{\ell-1,\ell} & \\ & & & & 0 & \\ \hline & & & & & A_r \end{array}\right]$$

$$-\hat{\lambda}\left[\begin{array}{c|cccc|c} E_l & & & * & & \\ \hline & E_{1,1} & * & & * & \\ & & \ddots & \ddots & & * \\ & & & \ddots & * & \\ & & & & E_{\ell,\ell} & \\ \hline & & & & & E_r \end{array}\right]$$

where now in addition to the earlier conditions, A_l is of full row rank and the $E_{i,i}$ matrices are now square invertible. The pencil $A_l - \hat{\lambda}E_l$ has only column Kronecker indices and no elementary divisors $\hat{\lambda}^i$. From here on, essentially the same procedure as above holds, but starting from the diagonal blocks in the middle section of this pencil. The bases constructed in that manner will be linear independent from the right null space of the matrix.

After all these operations, we of course have to transform back to the coordinate system of $S(\lambda)$ by just multiplying these vectors by V, and then find the corresponding vectors of $H(\lambda)$ by just taking the bottom part of these transformed vectors. The overall process is then repeated for another $\hat{\lambda} = (\lambda - \lambda_i)$ by merely a shift in $S(\lambda)$.

5. Conclusions

The advantages of the new proposed method are considerable :

1. the pencil (8) has to be constructed anyway since it is one of the most reliable ways to evaluate the zeros of $H(\lambda)$
2. the expansions (21) around each zero λ_0 are trivial once (20) is constructed
3. the staircase algorithm works on matrices of *decreasing dimension* at each step [9] rather than on Toeplitz matrices (3) *increasing dimension* [11].

For these different reasons the newly proposed method is hence preferable both from the point of view of numerical robustness as numerical efficiency.

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6. References

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