The Eigenstructure of an Arbitrary Polynomial Matrix: Computational Aspects

Paul Van Dooren
Philips Research Laboratory
Av. Van Becelaere 2, Box 8
B-1170 Brussels, Belgium

and

Patrick Dewilde
Delft University of Technology
Department of Electrical Engineering
Mekelweg 4
2628 CD Delft, The Netherlands

Submitted by Paul A. Fuhrmann

ABSTRACT

We give a new numerical method to compute the eigenstructure—i.e. the zero structure, the polar structure, and the left and right null space structure—of a polynomial matrix $P(\lambda)$. These structural elements are of fundamental importance in several systems and control problems involving polynomial matrices. The approach is more general than previous numerical methods because it can be applied to an arbitrary $m \times n$ polynomial matrix $P(\lambda)$ with normal rank r smaller than m and/or n. The algorithm is then shown to compute the structure of the left and right null spaces of $P(\lambda)$ as well. The speed and accuracy of this new approach are also discussed.

1. BACKGROUND

In the numerical literature the problem of computing the "generalized eigenvalues" (or zeros) of a *polynomial matrix*

$$P(\lambda) = P_0 + P_1\lambda + \cdots + P_d\lambda^d, \tag{1}$$

where the P_i's are real or complex coefficient matrices, has always been restricted to the *regular* case, i.e. where $P(\lambda)$ is square and $\det P(\lambda) \not\equiv 0$ [11, 1, 7, 14, 10]. The finite zeros of $P(\lambda)$ are then the zeros of the polynomial $\det P(\lambda)$. When $\operatorname{rank} P_d = n$, this is a polynomial of degree nd and all the

0024-3795/83/020545-35$3.00

zeros of $P(\lambda)$ are finite. When rank $P_d < n$, the degree of $\det P(\lambda)$, say k, is less than nd. Numerical analysts then say that $P(\lambda)$ has k *finite eigenvalues* (zeros)—namely, the zeros of $\det P(\lambda)$—and $(nd - k)$ *infinite eigenvalues* (zeros).

On the other hand, a precise definition of the zeros of an arbitrary polynomial matrix can be given through the Smith form [15, 4] and is currently used in system theory [12]. Every $m \times n$ polynomial matrix $P(\lambda)$ can indeed be transformed to the canonical form:

$$M_{mm}(\lambda)P_{mn}(\lambda)N_{nn}(\lambda) = \left[\begin{array}{ccc|c} l_1(\lambda) & & & \\ & \ddots & & 0_{r,n-r} \\ & & l_r(\lambda) & \\ \hline & 0_{m-r,r} & & 0_{m-r,n-r} \end{array}\right], \quad (2)$$

where

(i) $M(\lambda)$ and $N(\lambda)$ are unimodular (i.e. regular polynomial matrices with *constant* determinant),

(ii) $l_i(\lambda)$ are monic polynomials and $l_i(\lambda)$ divides $l_{i+1}(\lambda)$.

This form is unique and is used to define the following concepts. The *normal rank* r of the polynomial matrix $P(\lambda)$ is the number of *invariant polynomials* $l_i(\lambda)$ in the form (2). The *zeros* of $P(\lambda)$ are the zeros of any of the $l_i(\lambda)$. Let α be a zero of $P(\lambda)$; then each $l_i(\lambda)$ can be factorized as

$$l_i(\lambda) = (\lambda - \alpha)^{\sigma_i} g_i(\lambda), \quad (3)$$

where $g_i(\lambda)$ is regular at α. The exponents $\{\sigma_1, \ldots, \sigma_r\}$ of the factors $(\lambda - \alpha)^{\sigma_i}$ are called the *structural indices at* α [16] of $P(\lambda)$, and the nontrivial factors $(\lambda - \alpha)^{\sigma_i}$ (i.e. with $\sigma_i \neq 0$) are called its *elementary divisors at* α.

Note that in the regular case we have $m = n = r$ and

$$\det P(\lambda) = \prod_{i=1}^{r} l_i(\lambda). \quad (4)$$

We thus retrieve the original definition for the finite zeros of a regular polynomial matrix. The form (2), though, does not immediately suggest any definition for an infinite zero. In circuit theory McMillan [9] studied so-called "infinite frequencies" which may occur in passive electrical networks (see also [21], [2]). His definition of the eigenstructure at infinity implicitly relies

on a transformation of variables $\lambda = 1/\mu$ which reduces the polynomial matrix $P(\lambda)$ to a *rational matrix* $R(\mu)$. McMillan's analysis was in fact performed for the larger class of rational matrices, but when specialized to the case of polynomial matrices it also yields appropriate definitions. We now develop the background for these definitions.

DEFINITION 1. A rational matrix $R(\lambda)$ is called *regular at* α if the constant matrix $R(\alpha)$ is square, bounded and invertible.

Unimodular matrices are clearly regular at any finite α. After extraction of the left factor $G(\lambda) = \operatorname{diag}\{g_1(\lambda), \ldots, g_r(\lambda), I_{m-r}\}$, which is also regular at α because of (3), we may thus rewrite (2) as follows:

$$M_\alpha(\lambda)P(\lambda)N_\alpha(\lambda) = \left[\begin{array}{ccc|c} (\lambda-\alpha)^{\sigma_1} & & & \\ & \ddots & & 0_{r,n-r} \\ & & (\lambda-\alpha)^{\sigma_r} & \\ \hline & 0_{m-r,r} & & 0_{m-r,n-r} \end{array}\right], \quad (5)$$

where $M_\alpha(\lambda)$ and $N_\alpha(\lambda)$ are regular at α and the $\{\sigma_i | i = 1, \ldots, r\}$ form a nondecreasing sequence. This decomposition is unique, just like the decomposition (2) [9]. We thus see that the structural indices at a finite point α can as well be defined through the canonical form (5) obtained under transformations that are regular at α. Moreover, the latter extends easily to the point at infinity. Indeed, $P(\lambda)$ can be decomposed as [9]

$$M_\infty(\lambda)P(\lambda)N_\infty(\lambda) = \left[\begin{array}{ccc|c} (1/\lambda)^{\sigma_1} & & & \\ & \ddots & & 0_{r,n-r} \\ & & (1/\lambda)^{\sigma_r} & \\ \hline & 0_{m-r,r} & & 0_{m-r,n-r} \end{array}\right] \quad (6)$$

where $M_\infty(\lambda)$ and $N_\infty(\lambda)$ are regular at infinity and the $\{\sigma_i | i = 1, \ldots, r\}$ form a nondecreasing sequence. Again, the form (6) is unique and the corresponding σ_i are called the *structural indices at* ∞ [16]. A degree theory for rational matrices, developed by McMillan [9], uses the forms (5) and (6) to define the *polar degree* $\delta_p(\alpha)$ and *zero degree* $\delta_z(\alpha)$ at any point α as

$$\delta_p(\alpha) = -\sum_{\sigma_i < 0} \sigma_i(\alpha), \qquad \delta_z(\alpha) = \sum_{\sigma_i > 0} \sigma_i(\alpha). \quad (7a,b)$$

Notice that for a polynomial matrix all $\sigma_i(\alpha)$ are nonnegative for any finite point α, while they may be negative for the point at infinity. The only poles of a polynomial matrix thus lie at the point at infinity, and zeros can be infinite as well as finite. For regular matrices, the definition (7) of degree (or multiplicity) is equivalent with the one generally used, namely the multiplicity of the zero α of $\det P(\lambda)$. But for the point at infinity this is *not* true. In the context of dynamical systems (see [23], [21]) it can be shown that the definition (7) for the point at infinity makes more sense and also holds for singular polynomial matrices (i.e. with normal rank $r < m$ or $r < n$), while this is impossible for the definition using $\det P(\lambda)$.

DEFINITION 2. The set of zeros (respectively poles) of $P(\lambda)$, together with its structural indices at these points, is called the *zero structure* (respectively *pole structure*) of $P(\lambda)$.

For singular polynomial matrices an additional structural element is usually of interest: the structure of the *right* and *left null space* of $P(\lambda)$. Let $\cdot^T$ denote the transpose of a vector or matrix. The sets

$$\mathcal{N}_r(P) \triangleq \{ v(\lambda) | P(\lambda)v(\lambda) = 0 \}, \tag{8a}$$

$$\mathcal{N}_l(P) \triangleq \{ u(\lambda) | u^T(\lambda)P(\lambda) = 0 \} \tag{8b}$$

are vector spaces over the field of *rational* functions in λ. Considering $P(\lambda)$ as a (albeit special) matrix with elements in this field, we immediately have that $\mathcal{N}_r$ and $\mathcal{N}_l$ have dimensions $n - r$ and $m - r$, respectively (see [4], [3]). Moreover, it is always possible to choose a polynomial basis

$$\{p_1(\lambda), \ldots, p_k(\lambda)\} \tag{9}$$

for any such vector space $\mathcal{S}$ [26]. Let us define the *index* d_i of a polynomial vector $p_i(\lambda)$ as the maximum polynomial degree in its components; then (9) is called a *minimal polynomial basis* for $\mathcal{S}$ if the sum of the indices d_i is minimal over all polynomial bases for $\mathcal{S}$. These indices are invariant for a given space $\mathcal{S}$, except for their ordering [26]. When corresponding to the spaces $\mathcal{N}_r(P)$ and $\mathcal{N}_l(P)$, they are called the *right and left minimal indices* of $P(\lambda)$ (see [3] for an extensive discussion).

DEFINITION 3. The sets of right and left minimal indices of $P(\lambda)$ are called the *right* and *left null space structures* of $P(\lambda)$.

A simple example will clarify the above definitions.

EXAMPLE 1. The polynomial matrix

$$P(\lambda) \triangleq \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 0 \\ 1 & 4 & 2 \\ 0 & -1 & -2 \end{bmatrix} \lambda + \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 1 & 4 & 2 \end{bmatrix} \lambda^2 \quad (10)$$

has the Smith form (2)

$$D(\lambda) = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & \lambda - 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc} 1 & -1 & -1 \\ -\lambda & 1+\lambda & \lambda \\ \hline 0 & -\lambda & 1 \end{array}\right] P(\lambda) \left[\begin{array}{cc|c} 1 & -3 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right]. \quad (11)$$

Hence $\lambda - 1$ is the only finite elementary divisor, and the finite zero structure is thus given by $\lambda = 1$ with structural indices $\{0, 1\}$. Since the normal rank r equals 2, $\mathcal{N}_r$ and $\mathcal{N}_l$ both have dimension 1. From (11) they are clearly spanned by

$$\mathcal{N}_r = \left\langle \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix} \right\rangle, \qquad \mathcal{N}_l = \left\langle \begin{bmatrix} 0 \\ -\lambda \\ 1 \end{bmatrix} \right\rangle. \quad (12)$$

Moreover, these polynomial bases happen to be minimal in this example, as is easily checked [3]. For the pole and zero structure at $\lambda = \infty$ we perform the decomposition (6), which yields

$$\left[\begin{array}{ccc} 0 & 0 & 1 \\ 1 & -1 & -1 \\ \hline 0 & 1 & -\frac{1}{\lambda} \end{array}\right] P(\lambda) \left[\begin{array}{cc|c} -3 - \frac{3}{\lambda - 1} & 4 + \frac{3}{\lambda - 1} & 6 \\ 1 + \frac{1}{\lambda - 1} & 1 - \frac{1}{\lambda - 1} & -2 \\ 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} \left(\frac{1}{\lambda}\right)^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad (13)$$

where the transformation matrices are indeed regular at $\lambda = \infty$. The structural indices at $\lambda = \infty$ are then $\{-2, 0\}$, indicating a double pole at infinity.

Notice that in Definitions 2 and 3 of structural elements, we only considered *indices* (such as structural indices and minimal indices) and *points* (such as poles and zeros), but not *vectors* (such as vectors associated with poles and zeros [7, 14, 16] or minimal bases associated with minimal indices [3, 24]). The reason for this is that the structural elements of $P(\lambda)$ play a fundamental role in the *structure* of the solution of problems involving $P(\lambda)$. We give some examples to illustrate this.

Let us consider all possible (rational) solutions to the matrix equation

$$P_1(\lambda)X(\lambda) = P_2(\lambda),$$

where $P_i(\lambda)$ is an arbitrary $m \times n_i$ *polynomial* matrix, $i = 1, 2$. By e. g. checking the zero structure at a given point λ_0 of $P_1(\lambda)$ and of the compound matrix

$$P(\lambda) = \left[P_1(\lambda) | P_2(\lambda) \right]$$

and by comparing them, one can tell if there is a solution $X(\lambda)$ without a *pole* at λ_0 [33]. By checking this at $\lambda_0 = \infty$ one knows if there is a *proper* solution $X(\lambda)$; by checking it at all finite zeros of $P(\lambda)$ one knows if there is a *polynomial* solution $X(\lambda)$ [33]. Such information about the existence of a specific solution is considered to be "structural," whereas the actual solution depends on the bases corresponding to the structural elements of $P_1(\lambda)$ and $P(\lambda)$.

Consider further the *polynomial* matrix equation

$$P_1(\lambda)X_1(\lambda) = P_2(\lambda)X_2(\lambda),$$

where P_i, $i = 1, 2$, are known and X_i, $i = 1, 2$, unknown. Rewriting this as

$$P(\lambda)X(\lambda) = 0$$

with

$$P(\lambda) = \left[P_1(\lambda) | -P_2(\lambda) \right], \qquad X(\lambda) = \begin{bmatrix} X_1(\lambda) \\ \hline X_2(\lambda) \end{bmatrix},$$

we see that the columns of $X(\lambda)$ lie in the right null space of $P(\lambda)$. If $P_1(\lambda)$ is

square and intervertible and $X_1(\lambda)$ is supposed to be square, then the columns of $X(\lambda)$ *span* this null space [3]. The structural information of $\mathcal{N}_r(P)$ then allows one to describe all possible solutions to this equation and also carries information about any minimal realization of the rational matrix $T(\lambda) = P_1^{-1}(\lambda)P_2(\lambda)$ (see e.g. [3]).

The evaluation of the eigenstructure of $P(\lambda)$ is needed in several applications [30–32], which indicates the need for reliable algorithms to compute this structure. The standard Gaussian elimination for polynomial matrices, needed for the decompositions (2), (5), and (6), is known to be numerically unstable [29, 19]. In the sequel we give a stable method to compute the zeros of $P(\lambda)$. For this the minimal indices and the infinite structure are computed first, and $P(\lambda)$ is reduced to a regular matrix form, yielding a generalized eigenvalue problem. Since system theory is at the origin of the definitions for the structural elements we are interested in, it is natural that the techniques used also rely on some system theoretical concepts. In the next section we give a quick review of the background needed for the computational techniques developed here.

2. THE PENCIL APPROACH

It is easy to prove that the $m \times n$ polynomial matrix $P(\lambda)$ given in (1), and the pencil

$$\begin{aligned} S(\lambda) &= \lambda B_0 - A_0 \\ &\triangleq \lambda \begin{bmatrix} & -I_n & & \\ & & \ddots & \\ & & & -I_n \\ P_d & \cdots & P_2 & P_1 \end{bmatrix} - \begin{bmatrix} -I_n & & & \\ & \ddots & & \\ & & -I_n & \\ & & & -P_0 \end{bmatrix} \end{aligned} \tag{14}$$

have the same finite zeros [8]. Indeed, by performing unimodular row and column transformations on (14) one finds that $\lambda B_0 - A_0$ has the Smith form

$$M(\lambda)[\lambda B_0 - A_0]N(\lambda) = \operatorname{diag}\{I_n, \ldots, I_n, D(\lambda)\}, \tag{15}$$

where $D(\lambda)$ is the Smith form (2) of $P(\lambda)$. Other pencils can be found in the literature [10, 11] that share the same property (15). If $P(\lambda)$ is regular, the problem is thereby reduced to a regular "generalized eigenvalue problem," for which reliable software is now available [10, 25, 5]. Following the definition of "zero" commonly used in numerical analysis (see our introduc-

tion), $\lambda B_0 - A_0$ and $P(\lambda)$ are taken to have the same zero degree at infinity, namely $nd - k$, where k is the number of finite zeros. In that research area, the above approach thus solves completely the eigenstructure problem when $P(\lambda)$ is regular, and it has become a very popular approach (see [1], [11], [7], [14], [18] for a discussion of alternative methods).

In linear system theory, however, the above approach is not satisfactory, since the computed multiplicity of the zero at infinity generally does *not* coincide with the usual definition there (see Section 1). Indeed, the simple 2×2 polynomial matrix ($d = 1$)

$$P(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has no finite zeros and therefore two infinite zeros according to the definitions used in numerical analysis. On the other hand, only *one* infinite zero is found according to the system theoretical definitions recalled in our introduction. The latter is in agreement with the fact that the homogeneous differential system ($\lambda = d/dt$)

$$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad t \geqslant 0; \qquad \begin{bmatrix} X_1(0_-) \\ X_2(0_-) \end{bmatrix} = \begin{bmatrix} X_{10} \\ X_{20} \end{bmatrix}$$

has only *one* (impulsive) solution:

$$\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} -X_{20}\delta(t) \\ 0 \end{bmatrix}, \qquad t \geqslant 0,$$

where $\delta(t)$ denotes the Dirac impulse. The *degree* at a point λ_0 in system theory is indeed connected to the *number* of independent solutions with fundamental frequency λ_0, which happen to be "impulsive" solutions when $\lambda_0 = \infty$ (see [23] for a more extensive discussion).

Another reason why the definition using $\det P(\lambda)$ is unsatisfactory is that $P(\lambda)$ may be singular also. When $P(\lambda)$ is singular, $\lambda B_0 - A_0$ is also singular because of (15). For such pencils, a canonical form has been developed by Kronecker [6].

THEOREM 1 [4]. *Let $\lambda B - A$ be a pencil of matrices, and let*

(i) *the index sets $\{r_1,\dots,r_p\}$ and $\{l_1,\dots,l_q\}$ be its right and left minimal indices, respectively,*

(ii) *$\{(\lambda - \alpha_i)^{m_{ij}},\ j = 1,\dots,d_i\}$ be its elementary divisors at α_i,*

(iii) *$\{\mu^{n_j},\ j = 1,\dots,d\}$ be the elementary divisors of $B - \mu A$ at $\mu = 0$, also called the infinite elementary divisors of $\lambda B - A$.*

Then there exist constant row and column transformations S and T that reduce $\lambda B - A$ to the canonical form

$$S(\lambda B - A)T = \operatorname{diag}\{L_{r_1}, \ldots, L_{r_p}, L_{l_1}^T, \ldots, L_{l_q}^T, I - \lambda N, \lambda I - J\} \quad (16a)$$

where

(*i*) L_k *is the* $k \times (k+1)$ *bidiagonal pencil*

$$L_k \triangleq \underbrace{\begin{bmatrix} \lambda & -1 & & \\ & \ddots & \ddots & \\ & & \lambda & -1 \end{bmatrix}}_{k+1} \Bigg\} k \quad (16b)$$

(*ii*) J *is in Jordan canonical form with Jordan blocks*

$$\lambda I_{m_j} - J_{m_j}(\alpha_i) \triangleq \underbrace{\begin{bmatrix} \lambda - \alpha_i & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & \lambda - \alpha_i \end{bmatrix}}_{m_j} \Bigg\} m_j$$

$$j = 1, \ldots, d_i, \quad (16c)$$

at the eigenvalue α_i, *and*

(*iii*) N *is nilpotent and in Jordan canonical form with Jordan blocks*

$$I_{n_j} - \lambda J_{n_j}(0) \triangleq \underbrace{\begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & \ddots & -\lambda \\ & & & 1 \end{bmatrix}}_{n_j} \Bigg\} n_j$$

$$j = 1, \ldots, d. \quad (16d)$$

The canonical form (16) clearly reveals the null space structure and the finite zero structure of $\lambda B - A$ through Theorem 1(i) and (ii) respectively.

The structure of $\lambda B - A$ at ∞ can also be retrieved from this canonical form as shown by the following theorem:

THEOREM 2 [16, 24]. *The pole-zero structure of an arbitrary pencil* $\lambda B - A$ *at* ∞ *is determined by the canonical form* (16) *as follows*:

(i) *there are* $b = \operatorname{rank} B$ *negative structural indices and they are all* -1.

(ii) *the nonnegative structural indices are given by the set* $\{n_j - 1 | j = 1, \ldots, d\}$.

Proof. According to Theorem 1(iii), the elementary divisors of $B - \mu A$ are the sizes of the Jordan blocks (16d). Using the decomposition (5), there exist then matrices $M_0(\mu)$ and $N_0(\mu)$ that are regular at $\mu = 0$ and such that

$$M_0(\mu)[B - \mu A]N_0(\mu) = \left[\begin{array}{ccc|c} I_b & & & \\ & \mu^{n_1} & & \\ & & \ddots & 0 \\ & & & \mu^{n_d} \\ \hline & 0 & & 0 \end{array}\right], \tag{17}$$

where $0 < n_1 \leqslant n_2 \leqslant \cdots \leqslant n_d$. By putting $\mu = 0$ in (17), we find that $b = \operatorname{rank} B$, since $M_0(\mu)$ and $N_0(\mu)$ are regular at $\mu = 0$. Dividing (17) by μ, we obtain, as desired, a decomposition of the type (6):

$$M_0(\mu)\left[\frac{1}{\mu}B - A\right]N_0(\mu) = \left[\begin{array}{ccc|c} \frac{1}{\mu}I_b & & & \\ & \mu^{n_1 - 1} & & \\ & & \ddots & 0 \\ & & & \mu^{n_d - 1} \\ \hline & 0 & & 0 \end{array}\right]. \tag{18}$$

■

REMARKS.

1. In principle, a pole is a point where an entry of the matrix becomes infinite, while a zero is a point where the rank of the matrix drops below its normal value (see Section 1). The occurrence of poles and zeros at a certain point is therefore easily checked by filling in the value of the point, except perhaps for checking zeros at $\lambda = \infty$. Coalescent poles and zeros indeed give

rise to problems, since conventions have to be made about when a matrix with infinite entries is considered to be singular also. The definition via (2), (5), (6) resolves that problem, since the diagonal matrix (6) can very well have infinite entries at $\lambda = \infty$ [all factors $(1/\lambda)^{\sigma_i}$ with $\sigma_i < 0$] and be singular at the same time [all factors $(1/\lambda)^{\sigma_i}$ with $\sigma_i > 0$ are zero]. For a pencil $\lambda B - A$, a sufficient condition for the absence of zeros at $\lambda = \infty$ is clearly rank $B =$ (normal rank), in which case $\lambda B - A$ has no infinite elementary divisors (all n_j are zero). Necessary and sufficient conditions are given e.g. in [24].

2. Poles and zeros may coincide. In the present (pencil) case, all the poles are at infinity. The blocks (16b), (16c), and (16d) all contribute to the pole at infinity. The block (16c) has a zero of degree m_j at α_i. The block (16d) has both a pole and a zero at infinity of degree $n_j - 1$.

3. Note that a "finite" elementary divisor $(\lambda - \alpha)^m$ indicates a zero at α of degree m, but an "infinite" elementary divisor μ^n only indicates a zero at $\lambda = \infty$ of degree $n - 1$. This is because the so-called "infinite elementary divisors" are defined on the polynomial matrix $B - \mu A$ instead of the rational matrix $(1/\mu)B - A$.

Using the above two theorems, we thus see that the complete eigenstructure (i.e. pole-zero and null space structure) of an arbitrary pencil can be recovered via the Kronecker canonical form. Recently, a numerical stable algorithm has been developed to compute the Kronecker canonical form of an arbitrary pencil [17] (see Section III), which thus solves the eigenstructure problem for pencils. One might expect now that the eigenstructure of the pencil (14) reflects, in some sense, the eigenstructure of the related polynomial matrix $P(\lambda)$, even when it is singular.

According to (15) this is indeed true for the finite zeros and their nontrivial (i.e. positive) structural indices. But for the other structural elements this does not hold in general. In order to clarify this, we first recall some results from linear system theory. Notice that when we partition the pencil (14) as follows:

$$S(\lambda) \triangleq \left[\begin{array}{c|c} T(\lambda) & -U(\lambda) \\ \hline V(\lambda) & W(\lambda) \end{array}\right]$$

$$\triangleq \left[\begin{array}{cccccc|c} I_n & -\lambda I_n & & & & & \\ & \ddots & \ddots & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & -\lambda I_n & & \\ & & & & I_n & -\lambda I_n & \\ \hline \lambda P_d & \lambda P_{d-1} & \cdots & \cdots & \lambda P_2 & \lambda P_1 + P_0 & \end{array}\right], \quad (19)$$

then we have

$$P(\lambda) = V(\lambda)T^{-1}(\lambda)U(\lambda) + W(\lambda). \tag{20}$$

Such quadruples of polynomial matrices $\{T(\lambda), U(\lambda), V(\lambda), W(\lambda)\}$, where $T(\lambda)$ is regular, have been widely studied in linear system theory [12]. The compound matrix $S(\lambda)$ of such a quadruple is called a *system matrix* with *transfer function* $P(\lambda)$ as given in (20). In general, the system matrix is polynomial and the transfer function rational [12], but in our specific case $S(\lambda)$ is a pencil and the expression (20) is polynomial, since $T(\lambda)$ is unimodular. In [24], [22] a one to one correspondence is established between the eigenstructure of a so-called strongly irreducible quadruple $\{T(\lambda), U(\lambda), V(\lambda), W(\lambda)\}$ and the eigenstructure of its transfer function (see also [16]). Strong irreducibility amounts to the absence of finite *and* infinite zeros in the polynomial matrices (21a, b) [22].

The theorem below is extracted from [22] and is specialized here to the case where $T(\lambda)$, $U(\lambda)$, $V(\lambda)$ and $W(\lambda)$ are pencils.

THEOREM 3a. *Let $T(\lambda)$ be a regular pencil and suppose that the pencils*

$$\begin{bmatrix} T(\lambda) & -U(\lambda) \\ V(\lambda) & W(\lambda) \\ 0 & I_n \end{bmatrix}, \quad \begin{bmatrix} T(\lambda) & -U(\lambda) & 0 \\ V(\lambda) & W(\lambda) & -I_m \end{bmatrix} \tag{21a, b}$$

have no finite or infinite zeros. Then the eigenstructure of the transfer function $P(\lambda) = V(\lambda)T^{-1}(\lambda)U(\lambda) + W(\lambda)$ is to be retrieved from the eigenstructure of the pencils

$$S_p(\lambda) \triangleq \begin{bmatrix} T(\lambda) & -U(\lambda) & 0 \\ V(\lambda) & W(\lambda) & -I_m \\ 0 & I_n & 0 \end{bmatrix}, \quad S_z(\lambda) = \begin{bmatrix} T(\lambda) & -U(\lambda) \\ V(\lambda) & W(\lambda) \end{bmatrix} \tag{22a, b}$$

as follows:

(i) *the poles of $P(\lambda)$ are the zeros of $S_p(\lambda)$, and their nontrivial structural indices have opposite signs;*

(ii) *the zeros of $P(\lambda)$ are the zeros of $S_z(\lambda)$, and their nontrivial structural indices are the same;*

(iii) *the left and right minimal indices of $P(\lambda)$ and $S_z(\lambda)$ are the same.*

Notice that only the *nontrivial* structural indices of $S_p(\lambda)$, $S_z(\lambda)$, and $P(\lambda)$ are related, since these matrices have different normal rank. The conditions (21a, b) can be simplified, in the case of pencils only, to [19]:

THEOREM 3b. *The pencils* (21a, b) *have no* (*finite or infinite*) *zeros if the pencils*

$$\begin{bmatrix} T(\lambda) \\ V(\lambda) \end{bmatrix}, \begin{bmatrix} T(\lambda) & -U(\lambda) \end{bmatrix} \tag{23a, b}$$

have no (*finite or infinite*) *elementary divisors.*

Proof. See [19, Section V A]. ■

For the specific system (19), (20) the pencils corresponding to (23a, b) are given by

$$\left[\begin{array}{ccccccc} I_n & -\lambda I_n & & & & & \\ & \ddots & \ddots & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & -\lambda I_n & \\ & & & & & I_n & \\ \hline \lambda P_d & \cdot & \cdot & \cdot & \cdot & \lambda P_2 & \end{array}\right], \tag{24a, b}$$

$$\left[\begin{array}{ccccc|c} I_n & -\lambda I_n & & & & & \\ & \ddots & \ddots & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & -\lambda I_n & & \\ & & & & I_n & -\lambda I_n & \end{array}\right].$$

Because of the special structure of these pencils, it is easy to see that (24a) has linearly independent columns and (24b) linearly independent rows, for all finite λ. Therefore (see Remark 1), these pencils have no finite elementary divisors. The coefficient of λ in (24b) also has linearly independent rows, which (again according to Remark 1) guarantees the absence of infinite elementary divisors in (24b). The remaining condition, namely the absence of infinite elementary divisors in (24a), is in general *not* satisfied. In the next section, we give a (fast) procedure to extract from the system (19) a *reduced*

system which satisfies the conditions of Theorem 3. The construction is based on a transformation of $S(\lambda)$ of the type

$$\begin{array}{c} \overbrace{}^{p_1} \quad \overbrace{}^{p_2} \\ \begin{matrix} p_1\{ \\ p_2\{ \end{matrix} \left[\begin{array}{c|c|c} T_{11}(\lambda) & T_{12}(\lambda) & -U_1(\lambda) \\ \hline 0 & T_{22}(\lambda) & -U_2(\lambda) \\ \hline 0 & V_2(\lambda) & W(\lambda) \end{array}\right] \end{array}$$

$$\triangleq \left[\begin{array}{c|c} T_{rc}(\lambda) & -U_r(\lambda) \\ \hline V_c(\lambda) & W(\lambda) \end{array}\right]$$

$$\triangleq \left[\begin{array}{c|c} R & \\ \hline & I_m \end{array}\right] \left[\begin{array}{c|c} T(\lambda) & -U(\lambda) \\ \hline V(\lambda) & W(\lambda) \end{array}\right] \left[\begin{array}{c|c} C & \\ \hline & I_n \end{array}\right], \tag{25}$$

where R and C are constant invertible row and column transformations. Since $T(\lambda)$ is regular, $T_{rc}(\lambda)$ is regular and so are $T_{11}(\lambda)$ and $T_{22}(\lambda)$. It is then easy to check (see e.g. [12]) that both $\{T_{rc}(\lambda), U_r(\lambda), V_c(\lambda), W(\lambda)\}$ and $\{T_{22}(\lambda), U_2(\lambda), V_2(\lambda), W(\lambda)\}$ are systems with the same transfer function as $\{T(\lambda), U(\lambda), V(\lambda), W(\lambda)\}$. The idea is now to construct R and C such that the reduced system $\{T_{22}(\lambda), U_2(\lambda), V_2(\lambda), W(\lambda)\}$ satisfies Theorem 3. A simple condition for this is now derived.

THEOREM 4. *Let* $\{T(\lambda), U(\lambda), V(\lambda), W(\lambda)\}$ *be as in* (19), *and let us transform this system as in* (25). *Then* $\{T_{22}(\lambda), U_2(\lambda), V_2(\lambda), W(\lambda)\}$ *satisfies Theorem* 3 *if the coefficient of* λ *in* $\begin{bmatrix} T_{22}(\lambda) \\ V_2(\lambda) \end{bmatrix}$ *has linearly independent columns.*

Proof. Because of (24b), $[T_{rc}(\lambda) | -U_r(\lambda)]$ has linearly independent rows for all λ, and therefore its submatrix $[T_{22}(\lambda) | -U_2(\lambda)]$ has the same property. Similarly, (24a) yields that

$$\begin{bmatrix} T_{rc}(\lambda) \\ V_c(\lambda) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} T_{22}(\lambda) \\ V_2(\lambda) \end{bmatrix}$$

have linearly independent columns for all λ. According to Remark 1, these

four pencils thus have no finite elementary divisors. Again from (24b), it follows that the coefficient of λ in $[T_{rc}(\lambda)| - U_r(\lambda)]$, and therefore also in $[T_{22}(\lambda)| - U_2(\lambda)]$, have linearly independent rows. If, by assumption, the columns of the coefficient of λ in $\begin{bmatrix} T_{22}(\lambda) \\ V_2(\lambda) \end{bmatrix}$ are linearly independent, then these last two pencils have no infinite elementary divisors according to Remark 1. All conditions of Theorem 3 are then satisfied. ■

The problem of extracting a reduced model satisfying Theorem 3 is thereby reduced to a simple rank condition of a constant matrix, which is exploited in the next section. Note that in Theorem 3b and Theorem 4 special features of pencils were used, and that they do not hold for reduced polynomial systems in general.

3. ALGORITHMS

The method of reduction to a quadruple satisfying Theorem 3, as well as the computation of the Kronecker structure of the obtained pencils $S_z(\lambda)$ and $S_p(\lambda)$, is based on the quasi-Schur form for singular pencils [17]. An arbitrary pencil $\lambda B - A$ can indeed be reduced by unitary transformations U and V to the form

$$U(\lambda B - A)V \triangleq \begin{bmatrix} \lambda B_r - A_r & * & * & * \\ 0 & \lambda B_i - A_i & * & * \\ 0 & 0 & \lambda B_f - A_f & * \\ 0 & 0 & 0 & \lambda B_l - A_l \end{bmatrix}, \quad (26)$$

where

(i) $\lambda B_r - A_r$ and $\lambda B_l - A_l$ are nonsquare pencils with only *right* and *left null spaces*, respectively, and containing the corresponding minimal indices of $\lambda B - A$;

(ii) $\lambda B_i - A_i$ and $\lambda B_f - A_f$ are regular pencils which have only *infinite* and *finite elementary divisors*, respectively, which are those of $\lambda B - A$.

Furthermore, the form (26) can be obtained with a backward-stable algorithm which at the same time determines the Kronecker indices and the infinite elementary divisors of $\lambda B - A$ through the fine structure obtained in $\lambda B_l - A_l$, $\lambda B_r - A_r$, and $\lambda B_i - A_i$. This algorithm, described in [17], is referred to as

the *pencil algorithm*. The eigenstructure of $\lambda B_f - A_f$ can be computed in a stable way using the *QZ algorithm* [10], which constructs unitary transformations Q and Z that reduce a regular pencil—in this case $\lambda B_f - A_f$—to an upper triangular form (see [10], [25] for more details):

$$Q(\lambda B_f - A_f)Z \triangleq \lambda \begin{bmatrix} b_{11} & & * \\ & \ddots & \\ 0 & & b_{kk} \end{bmatrix} - \begin{bmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{kk} \end{bmatrix}. \tag{27}$$

The ratios a_{ii}/b_{ii} are then the finite zeros of $\lambda B - A$.

We quickly review the numerical algorithm that obtains the form (26). A modified (and fast) version of it will be developed later on. For a more detailed discussion, we refer to [17]. Let U and V be unitary matrices transforming the rows and columns, respectively, of an arbitrary $m \times n$ matrix A as follows:

$$\begin{matrix} \rho \{ \\ \mu \{ \end{matrix} \left[\begin{array}{c} A_r \\ \hline 0 \end{array}\right] \triangleq UA, \qquad \underbrace{[\,0\,}_{\nu} \underbrace{|\,A_c\,]}_{\rho} \triangleq AV, \tag{28a, b}$$

where A_r and A_c have ρ linearly independent rows and columns respectively (ρ is then clearly the rank of A). We call the transformations (28a) and (28b) a *row compression* and a *column compression* of A, respectively, and say that A_r and A_c have *full row rank* and *full column rank*. Such transformations are used recursively in the following algorithm, acting on the $m \times n$ pencil $\lambda B - A$.

Algorithm 1.

comment initialization
$j := 1$, $B_{1,1}^{(0)} := B$; $A_{1,1}^{(0)} := A$; $m_1 := m$; $n_1 := n$;

step_j: *comment* compress the columns of the $m_j \times n_j$ matrix $B_{j,j}^{(j-1)}$ with V_j and partition $B_{j,j}^{(j-1)} V_j$ and $A_{j,j}^{(j-1)} V_j$ analogously;

$$\underbrace{[\,0\,}_{\nu_j} \underbrace{|\,B_{j+1}\,]}_{\rho_j} := B_{j,j}^{(j-1)} V_j; \quad \underbrace{[\,A_j\,}_{\nu_j} \underbrace{|\,A_{j+1}\,]}_{\rho_j} := A_{j,j}^{(j-1)} V_j,$$

if $\nu_j = 0$ *then go to* exit;

comment also update and partition blocks with column index j;
for $i = 1$ *step* 1 *until* $j-1$ *do*

begin $\underbrace{[B_{i,j}^{(j)}}_{\nu_j} \underbrace{|B_{i,j+1}^{(j)}]}_{\rho_j} := B_{i,j}^{(j-1)} V_j$;

$[A_{i,j}^{(j)} | A_{i,j+1}^{(j)}] := A_{i,j}^{(j-1)} V_j$ *end*;

comment compress the rows of the $m_j \times \nu_j$ matrix A_j with U_j and partition $U_j A_j$, $U_j A_{j+1}$, and $U_j B_{j+1}$ analogously;

$$\begin{matrix}\mu_j\{\\ \sigma_j\{\end{matrix}\left[\begin{array}{c} A_{j,j}^{(j)} \\ \hline 0 \end{array}\right] := U_j A_j; \quad \begin{matrix}\mu_j\{\\ \sigma_j\{\end{matrix}\left[\begin{array}{c} A_{j,j+1}^{(j)} \\ \hline A_{j+1,j+1}^{(j)} \end{array}\right] := U_j A_{j+1};$$

$$\begin{matrix}\mu_j\{\\ \sigma_j\{\end{matrix}\left[\begin{array}{c} B_{j,j+1}^{(j)} \\ \hline B_{j+1,j+1}^{(j)} \end{array}\right] := U_j B_{j+1};$$

comment update;

$m_{j+1} := m_j - \mu_j$; $n_{j+1} := n_j - \nu_j$; $j := j+1$; *go to* step _j;

exit: *comment* k is the number of steps performed;

$k := j-1$; *stop*

At the end of this algorithm we obtain something of the form

$$Q(\lambda B - A)Z \triangleq \left[\begin{array}{c|c} \lambda B_{ri} - A_{ri} & * \\ \hline 0 & \lambda B_{fl} - A_{fl} \end{array}\right]$$

$$\triangleq \lambda \left[\begin{array}{cccc|c} 0 & B_{1,2}^{(2)} & \cdots & B_{1,k}^{(k)} & B_{1,k+1}^{(k)} \\ & 0 & \cdots & B_{2,k}^{(k)} & B_{2,k+1}^{(k)} \\ & & \ddots & \vdots & \vdots \\ & & & 0 & B_{k,k+1}^{(k)} \\ \hline 0 & 0 & \cdots & 0 & B_{k+1,k+1}^{(k)} \end{array}\right]$$

$$\begin{matrix} \underbrace{}_{\nu_1} & \underbrace{}_{\nu_2} & & \underbrace{}_{\nu_k} & \underbrace{}_{n_{k+1}} \end{matrix}$$

$$- \left[\begin{array}{cccc|c} A_{1,1}^{(1)} & A_{1,2}^{(2)} & \cdots & A_{1,k}^{(k)} & A_{1,k+1}^{(k)} \\ & A_{2,2}^{(2)} & \cdots & A_{2,k}^{(k)} & A_{2,k+1}^{(k)} \\ & & \ddots & \vdots & \vdots \\ & & & A_{k,k}^{(k)} & A_{k,k+1}^{(k)} \\ \hline 0 & 0 & \cdots & 0 & A_{k+1,k+1}^{(k)} \end{array}\right] \begin{matrix} \}\mu_1 \\ \}\mu_2 \\ \\ \}\mu_k \\ \}m_{k+1} \end{matrix} \qquad (29)$$

$$\begin{matrix} \underbrace{}_{\nu_1} & \underbrace{}_{\nu_2} & & \underbrace{}_{\nu_k} & \underbrace{}_{n_{k+1}} \end{matrix}$$

where

(i) Q and Z represent the accumulated row and column compressions respectively, performed by the algorithm;

(ii) $B^{(i)}_{i-1,i}$ have full column rank ν_i;

(iii) $A^{(i)}_{i,i}$ have full row rank μ_i;

(iv) $B^{(k)}_{k+1,k+1}$ has full column rank n_{k+1}.

The idea of the algorithm can easily be followed with the above decomposition (29). At the beginning of each step j a decomposition of the type (29) is at hand with $j = k+1$. An additional step is then performed if the above rank condition (iv) is not satisfied. This step j consists of a pair of compressions performed on $\lambda B^{(k)}_{k+1,k+1} - A^{(k)}_{k+1,k+1}$, which adds an additional "stair" to $\lambda B_{ri} - A_{ri}$ and reduces the size of $\lambda B_{fl} - A_{fl}$. This is repeated until rank condition (iv) is met. From rank conditions (ii), (iii) one also derives the inequalities

$$\nu_1 \geqslant \mu_1 \geqslant \nu_2 \geqslant \mu_2 \geqslant \cdots \geqslant \nu_k \geqslant \mu_k \geqslant \nu_{k+1} \triangleq 0. \tag{30}$$

The following observation can now be made about the above decomposition (see [17] for proofs and more details):

(a) Since B_{fl} has full column rank, the pencil $\lambda B_{fl} - A_{fl}$ has no infinite elementary divisors and no right null space.

(b) The rank properties of the $A^{(i)}_{i,i}$ and $B^{(i)}_{i-1,i}$ blocks ensure that the pencil $\lambda B_{ri} - A_{ri}$ has no left null space and no finite elementary divisors ($\alpha B_{ri} - A_{ri}$ indeed has full row rank for all finite α)

Hence, in (29) we have obtained the separation between the blocks $\lambda B_r - A_r$ and $\lambda B_i - A_i$ of (26) on the one hand (in $\lambda B_{ri} - A_{ri}$), and between the blocks $\lambda B_f - A_f$ and $\lambda B_l - A_l$ on the other hand (in $\lambda B_{fl} - A_{fl}$). Moreover, the indices μ_i and ν_i in (29) determine completely the right minimal indices and the infinite elementary divisors of $\lambda B - A$ as follows (see [17]):

(i) there are

$$\nu_i - \mu_i \tag{31a}$$

Kronecker blocks L_{i-1} of size $i-1$ for $i = 1, \ldots, k$;

(ii) there are

$$\mu_i - \nu_{i+1} \tag{31b}$$

infinite elementary divisors $(1/\lambda)^i$ of degree i for $i = 1, \ldots, k$.

A "dual" algorithm is now defined as follows [17]. Let us perform the above algorithm on the pertranspose (i.e. transpose over the antidiagonal, denoted by $\cdot^P$) $\lambda S^P - T^P$ of the pencil $\lambda S - T$:

$$Q(\lambda S^P - T^P)Z = \left[\begin{array}{c|c} \lambda S_{ri} - T_{ri} & * \\ \hline 0 & \lambda S_{fl} - T_{fl} \end{array}\right]. \tag{32}$$

Pertransposing this again, we obtain

$$Z^P(\lambda S - T)Q^P = \left[\begin{array}{c|c} \lambda S_{fl}^P - T_{fl}^P & * \\ \hline 0 & \lambda S_{ri}^P - T_{ri}^P \end{array}\right]$$

$$\triangleq \left[\begin{array}{c|c} \lambda S_{rf} - T_{rf} & * \\ \hline 0 & \lambda S_{il} - T_{il} \end{array}\right], \tag{33}$$

where we have used the identity $(XY)^P = Y^P X^P$. Using this rule on the Kronecker decomposition (16) also, it is easily seen that pertransposition does not alter the elementary divisors of a pencil, but that it interchanges the left and right null spaces and their indices. Therefore $\lambda S_{rf} - T_{rf}$ (being the pertranspose of $\lambda S_{fl} - T_{fl}$) contains the right null space structure and finite elementary divisors of $\lambda S - T$, and $\lambda S_{il} - T_{il}$ (being the pertranspose of $\lambda S_{ri} - T_{ri}$) contains the left null space structure and infinite elementary divisors of $\lambda S - T$.

This dual algorithm thus separates the right null space structure and the finite elementary divisors on the one hand, from the left null space structure and infinite elementary divisors on the other hand. Therefore, applying this dual algorithm to the two diagonal blocks $\lambda B_{ri} - A_{ri}$ and $\lambda B_{fl} - A_{fl}$ of (29) finally yields the desired decomposition (26). More details about this decomposition and how the fine structure of the diagonal pencils $\lambda B_r - A_r$, $\lambda B_i - A_i$, and $\lambda B_l - A_l$ reveal their eigenstructure can be found in [17]. In the sequel we denote the dual algorithm by Algorithm 1^P.

When applying now Algorithm 1 to the pencil $\lambda B_0 - A_0$ in (14), we can efficiently exploit the sparsity of that pencil. The first transformation that Algorithm 1 applies to the pencil (14) is a column compression of the sparse matrix B_0, which clearly is obtained by merely compressing the columns of P_d with a unitary transformation W_d. Let us perform this column transformation

on the blocks P_1, $i = 2,\ldots,d$, and define

$$\underbrace{[\ 0\ |}_{\sigma_d}\ \underbrace{P_d^-\ ]}_{\rho_d} \triangleq \bar{P}_d \triangleq P_d W_d$$

$$\underbrace{[\ P_i^+\ |}_{\sigma_d}\ \underbrace{P_i^-\ ]}_{\rho_d} \triangleq \bar{P}_i \triangleq P_i W_d, \qquad i = 2,\ldots,d-1. \tag{34}$$

Then, by multiplying $\lambda B_0 - A_0$ on the left by $U_d \triangleq \operatorname{diag}\{W_d^*,\ldots,W_d^*, I_m\}$ and on the right by $V_d \triangleq \operatorname{diag}\{W_d,\ldots,W_d, I_n\}$, we obtain

$$U_d(\lambda B_0 - A_0)V_d \triangleq \lambda \left[\begin{array}{ccc|c} -I_n & & & \\ & \ddots & & \\ & & -I_n & \\ & & & -W_d^* \\ \hline \bar{P}_d & \cdots & \bar{P}_2 & P_1 \end{array}\right] - \left[\begin{array}{ccc|c} -I_n & & & \\ & \ddots & & \\ & & -I_n & \\ \hline & & & -P_0 \end{array}\right]. \tag{35}$$

By defining the following matrices:

$$\tilde{P}_i \triangleq \left[P_{i+1}^- | P_i^+\right], \qquad i = 2,\ldots,d-1,$$

$$B_{ri} \triangleq 0_{\sigma_d} \qquad X \triangleq \left[0_{\sigma_d\rho_d} | -I_{\sigma_d}\right],$$

$$\begin{matrix}\sigma_d\{ \\ \rho_d\{\end{matrix}\left[\begin{array}{c} W_d^+ \\ \hline W_d^- \end{array}\right] \triangleq W_d^*, \tag{36}$$

$$[Y_1|Y_0] \triangleq \left[\begin{array}{c|c} -I_{\rho_d} & 0_{\rho_d n} \\ 0_{\sigma_d\rho_d} & -W_d^+ \end{array}\right],$$

$$\left[\begin{array}{c|c} S_1^+ & S_0^+ \\ \hline S_1^- & S_0^- \end{array}\right] \triangleq \left[\begin{array}{c|c} 0_{\rho_d} & -W_d^- \\ \hline P_2^- & P_1 \end{array}\right],$$

the pencil (35) can also be written as

$$\lambda \left[\begin{array}{c|ccc|cc} B_{ri} & X & & & & \\ \hline & 0 & -I_n & & & \\ & & \ddots & & & \\ & & & -I_n & & \\ & 0 & & 0 & Y_1 & Y_0 \\ \hline & 0 & 0 \cdots & 0 & S_1^+ & S_0^+ \\ & \tilde{P}_{d-1} & \tilde{P}_{d-2} \cdots & \tilde{P}_2 & S_1^- & S_0^- \end{array}\right] - \left[\begin{array}{c|ccc|cc} -I_{\sigma_d} & & & & & \\ \hline & -I_n & & & & \\ & & \ddots & & & \\ & & & -I_n & & \\ \hline & & & & -I_{\rho_d} & \\ & & & & & -P_0 \end{array}\right]. \tag{37}$$

At this stage, step 1 of Algorithm 1 has been performed on the pencil (14), since (37) is indeed in the form (see (29) for $k = 1$)

$$\lambda \left[\begin{array}{c|c} B_{1,1}^{(1)} & B_{1,2}^{(1)} \\ \hline 0 & B_{2,2}^{(1)} \end{array}\right] - \left[\begin{array}{c|c} A_{1,1}^{(1)} & A_{1,2}^{(1)} \\ \hline 0 & A_{2,2}^{(1)} \end{array}\right] \tag{38}$$

with $B_{1,1}^{(1)} = B_{ri} = 0_{\sigma_d}$ and $A_{1,1}^{(1)} = I_{\sigma_d}$. We now prove that at each step k (for $k < d-1$) of Algorithm 1 applied to $\lambda B_0 - A_0$ a similar form can be obtained:

$$U_j \cdots U_d(\lambda B_0 - A_0)V_d \cdots V_j \triangleq \left[\begin{array}{c|c} \lambda B_{ri} - A_{ri} & * \\ \hline 0 & \lambda B_{j+1,j+1}^{(j)} - A_{j+1,j+1}^{(j)} \end{array}\right]$$

$$\triangleq \lambda \left[\begin{array}{c|ccc|cc} B_{ri} & X & & & & \\ \hline & 0 & -I_n & & & \\ & & \ddots & & & \\ & & & -I_n & & \\ & 0 & & 0 & Y_1 & Y_0 \\ \hline & 0 & & 0 & S_1^+ & S_0^+ \\ & \tilde{P}_j & & \tilde{P}_2 & S_1^- & S_0^- \end{array}\right]$$

$$-\left[\begin{array}{c|c|c} -I_{s_j} & & \\ \hline & -I_n & \\ & \ddots & \\ & -I_n & \\ \hline & & -I_{r_j} \\ & & P_0 \end{array}\right], \tag{39}$$

where the matrix

$$\left[\begin{array}{c|c} Y_1 & Y_0 \\ \hline S_1^+ & S_0^+ \end{array}\right] \tag{40}$$

has full column rank and B_{ri} is nilpotent as in (29). For $k=1$, (37) is of the form (39) with $j=d-1$, $s_j=\sigma_d$, and $r_j=\rho_d$. The rank condition (40) and the nilpotency of B_{ri} are satisfied because of (36). The proof for larger k goes by induction. Suppose we have the form (39) at the beginning of step k. It follows then from the rank condition (40) and from the special structure of $B_{k+1,k+1}^{(k)}$ in (39) that its columns are linearly independent iff those of $\tilde{P}_j$ are linearly independent. If this is not the case, a column compression is performed on $B_{k+1,k+1}^{(k)}$ which is obtained by merely compressing the columns of $\tilde{P}_j$ with a unitary transformation W_j. Therefore we multiply (39) on the left by $U_j \triangleq \operatorname{diag}\{I_{s_j}, W_j^*, \ldots, W_j^*, I_{r_j}, I_m\}$ and on the right by $V_j \triangleq \operatorname{diag}\{I_{s_j}, W_j, \ldots, W_j, I_{r_j}, I_n\}$. The matrices in (39), modified by this, are transformed as

$$\underbrace{[\,0\,|\,}_{\sigma_j}\underbrace{P_j^-\,]}_{\rho_j} \triangleq \tilde{P}_j W_j, \qquad \underbrace{[\,P_i^+\,|\,}_{\sigma_j}\underbrace{P_i^-\,]}_{\rho_j} \triangleq \tilde{P}_i W_j \quad \text{for } i=2,\ldots,j-1,$$

$$\underbrace{[\,X^+\,|\,}_{\sigma_j}\underbrace{X^-\,]}_{\rho_j} \triangleq XW_j, \qquad \begin{matrix}\sigma_j\{\\ \rho_j\{\end{matrix}\left[\begin{array}{c|c} Y_1^+ & Y_0^+ \\ \hline Y_1^- & Y_0^- \end{array}\right] \triangleq W_j^*[Y_1|Y_0]. \tag{41}$$

With the following updating definitions:

$$\tilde{P}_i \triangleq \underbrace{[P_{i+1}^-|\,}_{\rho_j}\underbrace{P_i^+\,]}_{\sigma_j} \qquad \text{for} \quad i=2,\ldots,j-1, \tag{42a}$$

$$[B_{ri}|X] \triangleq \left[\begin{array}{cc|cc} B_{ri} & X^+ & X^- & 0 \\ 0 & 0_{\sigma_j} & 0 & -I_{\sigma_j} \end{array}\right], \tag{42b}$$

$$\left[\begin{array}{c|c} Y_1 & Y_0 \\ \hline S_1^+ & S_0^+ \\ \hline S_1^- & S_0^- \end{array}\right] \triangleq \left[\begin{array}{ccc} -I_{\rho_j} & 0 & 0 \\ \hline 0 & Y_1^+ & Y_0^+ \\ \hline 0_{\rho_j} & Y_1^- & Y_0^- \\ \hline 0 & S_1^+ & S_0^+ \\ \hline P_2^- & S_1^- & S_0^- \end{array}\right]. \tag{42c}$$

the transformed pencil (39) can again be written in the form (39). By induction, the rank condition (40) is clearly satisfied because of (42c), as well as the uilpotency of B_{ri} because of (42b). Notice also that by induction we have that $j = d - k$, $r_j = \rho_d + \cdots + \rho_{j+1}$, and $s_j = \sigma_d + \cdots + \sigma_{j+1}$.

The above recursion can now be repeated $d - 1$ times as outlined in the following algorithm. We drop the $\tilde{}$, since the same storage can be used for the updated P_i, and we drop the indices of W, ρ, σ, r, and S, since they are irrelevant.

ALGORITHM 2.

comment initialization;
$B_{ri} := X := S_0^+ := S_1^- := S_1^+ := Y_1 :=$ void;
$Y_0 := -I_n$; $S_0^- := P_1$; $r := 0$; $s := 0$;
for $j = d$ *step* -1 *until* 2 *do*
begin main loop;
comment compress the columns of P_j;
$\underbrace{[\ 0\ }_{\sigma} \mid \underbrace{P_j^-\ ]}_{\rho} := P_j W$; $r := r + \rho$; $s := s + \sigma$;
if $\sigma = 0$ *then go to exit* _2;
comment multiply P_i, Y_0, Y_1, X with W and partition analogously;
for $i = j - 1$ *step* -1 *until* 2 *do* $\underbrace{[\ P_i^+}_{\sigma} \mid \underbrace{P_i^-\ ]}_{\rho} := P_i W$;

$$\begin{matrix}\sigma\{ \\ \rho\{ \end{matrix}\left[\begin{array}{c} Y_0^+ \\ \hline Y_0^- \end{array}\right] := W^* Y_0; \quad \begin{matrix}\sigma\{ \\ \rho\{ \end{matrix}\left[\begin{array}{c} Y_1^+ \\ \hline Y_1^- \end{array}\right] := W^* Y_1; \quad \underbrace{[\ X^+}_{\sigma}\ \underbrace{X^-\ ]}_{\rho} := XW;$$

comment update;
for $i = j-1$ *step* -1 *until* 2 *do* $P_i := [P_{i-1}^- | P_i^+]$;

$$B_{ri} := \begin{bmatrix} B_{ri} & X^+ \\ 0 & 0_\sigma \end{bmatrix}; \; X := \begin{bmatrix} X^- & 0 \\ 0 & -I_\sigma \end{bmatrix}; \; S_1^- := \left[P_2^- \mid S_1^- \right];$$

$$S_1^+ := \begin{bmatrix} 0_\rho & Y_1^- \\ 0 & S_1^+ \end{bmatrix}; \; S_0^+ := \begin{bmatrix} Y_0^- \\ S_0^+ \end{bmatrix}; \; Y_1 := \begin{bmatrix} -I_\rho & 0 \\ 0 & Y_1^+ \end{bmatrix}; \; Y_0 := \begin{bmatrix} 0 \\ Y_0^+ \end{bmatrix};$$

end main loop

exit _1: *comment* normal exit of algorithm: none of the P_j had full column rank;
$T_{22}(\lambda) := \lambda S_1^+ + I_r$; $U_2(\lambda) := -\lambda S_0^+$; $V_2(\lambda) := \lambda S_1^-$; $W(\lambda) := \lambda P_1 + P_0$;
stop;

exit _2: *comment* a P_j was met with full column rank at step j;

$$T_{22}(\lambda) := \left[\begin{array}{cccccc|c} I_n & -\lambda I_n & & & & & 0 \\ & \ddots & \ddots & & & & \vdots \\ & & \ddots & \ddots & & & \vdots \\ & & & \ddots & \ddots & & \vdots \\ & & & & \ddots & -\lambda I_n & 0 \\ & & & & & I_n & \lambda Y_1 \\ \hline 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda S_1^+ + I_r \end{array}\right];$$

$$U_2(\lambda) := \left[\begin{array}{c} 0 \\ \vdots \\ \vdots \\ 0 \\ -\lambda Y_0 \\ \hline -\lambda S_0^+ \end{array}\right]$$

$$V_2(\lambda) := \left[\lambda P_j \cdots \lambda P_2 \mid \lambda S_1^-\right]; \; W(\lambda) := [\lambda P_1 + P_0]; \; \textit{stop}$$

We now prove that the pencil

$$\lambda B_{\text{red}} - A_{\text{red}} \triangleq \left[\begin{array}{c|c} T_{22}(\lambda) & -U_2(\lambda) \\ \hline V_2(\lambda) & W(\lambda) \end{array}\right] \tag{43}$$

satisfies the conditions of Theorems 3 and 4. For this notice that Algorithm 2

indeed performs a system reduction of the type (25). The transformations U_i and V_i in (39) indeed have the block form required in (25), and the matrices corresponding to $T_{21}(\lambda)$ and $V_1(\lambda)$ in (39) are indeed zero as required in (25). Because of Theorem 4 it thus only remains to prove that the coefficient or λ in $\begin{bmatrix} T_{22}(\lambda) \\ V_2(\lambda) \end{bmatrix}$ has full column rank. For the result at exit _1 this is the matrix

$$\left[\begin{array}{cccccc|c} 0 & -I_n & & & & & \\ & \ddots & \ddots & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & -I_n & \\ & & & & & 0 & Y_1 \\ \hline 0 & \cdot & \cdot & \cdot & \cdot & 0 & S_1^+ \\ P_j & \cdot & \cdot & \cdot & \cdot & P_2 & S_1^- \end{array}\right], \tag{44}$$

which indeed has full column rank because

$$P_j \text{ and } \begin{bmatrix} Y_1 \\ S_1^+ \end{bmatrix}$$

have full column rank [the first because of the stopping rule of the algorithm, the latter because of the rank condition (40)]. For the result at exit _2 we rewrite the matrix in terms of the matrices before the update [see (42c)]:

$$\left[\frac{S_1^+}{S_1^-}\right] := \left[\begin{array}{cc} 0_{\rho_2} & Y_1^- \\ 0 & S_1^+ \\ \hline P_2^- & S_1^- \end{array}\right] \tag{45}$$

Since at this exit of the algorithm P_2^- has full column rank, the matrix (45) has full column rank if the submatrix $\left[\frac{Y_1^-}{S_1^+}\right]$ has full column rank. We now prove inductively that this is satisfied at every step of the algorithm. Indeed, partitioning W_j as (here we add superscripts to indicate the step j)

$$W_j = \underbrace{\begin{bmatrix} W_{11}^{(j)} & W_{12}^{(j)} \\ W_{21}^{(j)} & W_{22}^{(j)} \end{bmatrix}}_{\substack{\sigma_j \quad \rho_j}} \begin{matrix} \}\rho_{j+1} \\ \}\sigma_{j+1} \end{matrix} \tag{46}$$

we have from (36), (41), (42) that the matrix $\left[\begin{array}{c} Y_1^- \\ \hline S_1^+ \end{array}\right]$ obeys the recursion

$$\left[\begin{array}{c} Y_1^- \\ \hline S_1^+ \end{array}\right]^{(d-1)} := \left[\begin{array}{c} -W_{12}^{(d-1)*} \\ \hline 0_{\rho_d} \end{array}\right];$$

$$\left[\begin{array}{c} Y_1^- \\ \hline S_1^+ \end{array}\right]^{(j)} := \left[\begin{array}{c|c} -W_{12}^{(j)*} & * \\ \hline 0 & \left[\begin{array}{c} Y_1^- \\ \hline S_1^+ \end{array}\right]^{(j+1)} \end{array}\right] \qquad \text{for} \quad j = d-2, \ldots, 2. \tag{47}$$

These matrices have full column rank if all the $W_{12}^{(j)*}$ have full column rank. From (41), (42) we now have

$$\underbrace{\left[P_{j+1}^{(j+1)-}\right.}_{\rho_{j+1}} \underbrace{\left|P_j^{(j+1)+}\right]}_{\sigma_{j+1}} = \underbrace{\left[\,0\,\right.}_{\sigma_j} \underbrace{\left|P_j^{(j)-}\right]}_{\rho_j} W_j^* \tag{48a}$$

and thus

$$P_{j+1}^{(j+1)-} = P_j^{(j)-} W_{12}^{(j)*}. \tag{48b}$$

Since in this last equation both $P_{\cdot}^{(\cdot)-}$ matrices have full column rank, so does $W_{12}^{(j)*}$. This thus completes the proof of the following theorem.

THEOREM 5. *The pencil* $\lambda B_{\text{red}} - A_{\text{red}}$ *resulting form Algorithm* 2 *(see* (43)*) has the same zero structure and null space structure as the polynomial matrix* $P(\lambda)$.

Meanwhile we can also say something about the original pencil $\lambda B_0 - A_0$ in (14). The transformations performed by Algorithm 2 are of the type

$$U(\lambda B_0 - A_0)V = \left[\begin{array}{c|c} \lambda B_{ri} - A_{ri} & * \\ \hline 0 & \lambda B_{\text{red}} - A_{\text{red}} \end{array}\right], \tag{49}$$

where U and V are unitary. This decomposition was shown to be equivalent to $d-1$ steps of Algorithm 1 performed on $\lambda B_0 - A_0$. In a second stage, Algorithm 1 should be continued on $\lambda B_{\text{red}} - A_{\text{red}}$ in order to obtain a decomposition of the type (29) for $\lambda B_{\text{red}} - A_{\text{red}}$. Inserting this in (49) yields

again a similar decomposition for $\lambda B_0 - A_0$, but with a larger $\lambda B_{ri} - A_{ri}$ part:

$$\left[\begin{array}{c|c} I & \\ \hline & U_{\text{red}} \end{array}\right] U \cdot (\lambda B_0 - A_0) \cdot V \left[\begin{array}{c|c} I & \\ \hline & V_{\text{red}} \end{array}\right]$$

$$\triangleq \left[\begin{array}{c|c} \lambda B_{ri} - A_{ri} & * \\ \hline 0 & \begin{array}{c|c} \lambda B_{ri}^{\text{red}} - A_{ri}^{\text{red}} & * \\ \hline 0 & \lambda B_{fl}^{\text{red}} - A_{fl}^{\text{red}} \end{array} \end{array}\right]. \tag{50}$$

Let $\{\mu_i^+ \mid i = 1,\ldots,k\}$ and $\{\nu_i^+ \mid i = 1,\ldots,k\}$ be the rank indices in the decomposition (29) for $\lambda B_0 - A_0$. Then those of $\lambda B_{\text{red}} - A_{\text{red}}$ are given by

$$\left.\begin{array}{l} \mu_i = \mu_{i+d-1}^+ \\ \nu_i = \nu_{i+d-1}^+ \end{array}\right\} \quad \text{for} \quad i = 1,\ldots,k-d+1. \tag{51}$$

It follows then from (31) that the right minimal indices and the infinite elementary divisors of $\lambda B_{\text{red}} - A_{\text{red}}$ are $d-1$ smaller than those of $\lambda B_0 - A_0$, and that those in $\lambda B_0 - A_0$ that were smaller than $d-1$ thus do not reappear in $\lambda B_{\text{red}} - A_{\text{red}}$. If $k < d$, then Algorithm 2 terminates at step k through exit _2, and $\lambda B_{\text{red}} - A_{\text{red}}$ has then no right minimal indices and infinite elementary divisors.

Notice also that the other structural elements of $\lambda B_0 - A_0$ and $\lambda B_{\text{red}} - A_{\text{red}}$ are contained in $\lambda B_{fl}^{\text{red}} - A_{fl}^{\text{red}}$ and are thus the same.

This thus illustrates the importance of the reduction step performed by Algorithm 2: not only do we obtain a reduced pencil to work on, but some "fake" structural elements are meanwhile deflated. Notice also that the "fake" structural elements are deflated with a "fast" method, while the actual structure of $P(\lambda)$, which is present in $\lambda B_{\text{red}} - A_{\text{red}}$, has to be computed with the comparatively slow Algorithm 1, followed by its dual form and the QZ algorithm. The following simple example illustrates the above ideas.

EXAMPLE 2. For the polynomial matrix of Example 1, the pencil (14) looks like

$$\lambda \left[\begin{array}{c|c} 0 & -I \\ \hline P_2 & P_1 \end{array}\right] - \left[\begin{array}{c|c} -I & 0 \\ \hline 0 & -P_0 \end{array}\right]$$

$$= \lambda \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 1 & 4 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 1 & 4 & 2 & 0 & -1 & -2 \end{array}\right]$$

$$- \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{52}$$

For illustrative simplicity we will use elementary instead of orthogonal transformations for all compressions.

Since $d = 2$, Algorithm 2 consists of only one step, which compresses the columns of P_2 and yields $\lambda B_{\text{red}} - A_{\text{red}}$:

$$\left[\begin{array}{c|c} \lambda B_{ri} - A_{ri} & * \\ \hline 0 & \lambda B_{\text{red}} - A_{\text{red}} \end{array}\right]$$

$$\triangleq \lambda \left[\begin{array}{cc|ccc} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & -\frac{1}{2} & -2 & -1 \\ 0 & 0 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 2 & 0 & -1 & -2 \end{array}\right]$$

$$\underbrace{}_{\nu_1^+}$$

$$- \left[\begin{array}{cc|ccc} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} \\ \\ \}\mu_1^+ \\ \\ \\ \\ \end{matrix} \tag{53}$$

$$\underbrace{}_{\nu_1^+}$$

The first step of Algorithm 1 performed on $\lambda B_{\text{red}} - A_{\text{red}}$ then reads:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} (\lambda B_{\text{red}} - A_{\text{red}}) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ \frac{4}{3} & \frac{1}{6} & -2 & 1 \end{bmatrix}$$

$$= \lambda \left[\begin{array}{cc:cc} 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -2 \end{array}\right] \begin{matrix} \\ \\ \\ \\ \end{matrix} \begin{matrix} \\ \\ \} \mu_1 = \mu_2^+ \\ \\ \end{matrix} - \left[\begin{array}{cc:cc} -1 & 0 & 0 & 0 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} \\ \\ \} \mu_1 = \mu_2^+ \\ \\ \end{matrix}$$

$$\underbrace{\qquad}_{\nu_1 = \nu_2^+} \qquad\qquad \underbrace{\qquad}_{\nu_1 = \nu_2^+}$$

(54)

Here Algorithm 1 terminates, since $B_{2,2}^{(1)}$ has full column rank ($\nu_2 = \nu_3^+ = 0$). Using (31), we now find that $\lambda B_{\text{red}} - A_{\text{red}}$—and thus $P(\lambda)$—has

$$\nu_1 - \mu_1 = 1 \qquad \text{(right minimal index equal to 0)},$$

$$\mu_1 - \nu_2 = 1 \qquad \text{(infinite elementary divisor of degree 1)}.$$

On the other hand, $\lambda B_0 - A_0$ has

$$\nu_2^+ - \mu_2^+ = 1 \qquad \text{(right minimal index equal to 1)},$$

$$\mu_2^+ - \nu_3^+ = 1 \qquad \text{(infinite elementary divisor of degree 2)}$$

and no others (since $\nu_1^+ - \mu_1^+ = 0$ and $\mu_1^+ - \nu_2^+ = 0$). This shows indeed that the corresponding structural elements of $\lambda B_{\text{red}} - A_{\text{red}}$ are $d - 1 = 1$ smaller than those of $\lambda B_0 - A_0$. For the remaining structural elements the dual Algorithm 1^P is applied to

$$\lambda B_{fl}^{\text{red}} - A_{fl}^{\text{red}} \triangleq \lambda \begin{bmatrix} 3 & -4 \\ 0 & 0 \\ 3 & -2 \end{bmatrix} - \begin{bmatrix} -6 & 2 \\ -3 & 2 \\ 0 & 0 \end{bmatrix}, \tag{55}$$

yielding after two steps

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \left(\lambda B_{fl}^{\text{red}} - A_{fl}^{\text{red}}\right) \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix} = \lambda \left[\begin{array}{cc|cc} -3 & & -4 & \\ & & & \\ \hline 0 & & -2 & \\ & & & \\ \hline & 0 & & 0 \end{array}\right] - \left[\begin{array}{cc|cc} -3 & & 2 & \\ & & & \\ \hline 0 & & 0 & \\ & & & \\ \hline & 0 & & 2 \end{array}\right] \begin{matrix} \} \nu_2^P \\ \\ \} \nu_1^P \end{matrix}$$

$\underbrace{\qquad\qquad}_{\mu_1^P}$ $\underbrace{\qquad\qquad}_{\mu_1^P}$

(56)

The algorithms give the "dual" indices $\mu_1^P = 1$, $\mu_2^P = 0$, $\nu_1^P = 1$, $\nu_2^P = 1$, and $\nu_3^P = 0$. This reveals, in a way dual to (31), that there is $\nu_2^P - \mu_2^P = 1$ left minimal index equal to 1 and isolates the finite structure part

$$\lambda B_f^{\text{red}} - A_f^{\text{red}} = -3\lambda + 3, \tag{57}$$

revealing a finite elementary divisor $(\lambda - 1)$. The QZ algorithm is superfluous here, since the dimension of $\lambda B_f^{\text{red}} - A_f^{\text{red}}$ is only 1. Through Theorem 2, one now retrieves the zero and null space structure of $\lambda B_{\text{red}} - A_{\text{red}}$ and thus also $P(\lambda)$.

Notice that Algorithm 1^P was not applied to $\lambda B_{ri}^{\text{red}} \mathrm{i} - A_{ri}^{\text{red}}$ in order to separate the infinite elementary divisors from the right null space structure, since the sequence of μ_i's and ν_i's reveals both these structural elements of $\lambda B_{ri}^{\text{red}} - A_{ri}^{\text{red}}$.

Let us now focus on the pole structure of $P(\lambda)$. According to Theorem 3 this structure can be identified through the pencil $S_p(\lambda)$, here given by

$$S_p(\lambda) \triangleq \lambda \left[\begin{array}{c|ccc|ccc} 0 & -\frac{1}{2} & -2 & -1 & 0 & 0 & 0 \\ \hline 2 & 1 & 3 & 0 & & & \\ 0 & 1 & 4 & 2 & & 0 & \\ 2 & 0 & -1 & -2 & & & \\ \hline 0 & & & & & & \\ 0 & & 0 & & & 0 & \\ 0 & & & & & & \end{array}\right] - \left[\begin{array}{c|ccc|ccc} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & -2 & 2 & & & \\ 0 & 0 & 1 & 2 & & -I & \\ 0 & 0 & 0 & 0 & & & \\ \hline 0 & & & & & & \\ 0 & & I & & & 0 & \\ 0 & & & & & & \end{array}\right]. \tag{58}$$

This is a regular pencil with only infinite zeros, since, according to Theorem 3, these are the poles of $P(\lambda)$. They could thus be computed with Algorithm 1 running on $S_p(\lambda)$. A faster method is discussed in [20] (see also references therein) and only uses the matrices $P_d,\dots,P_1$. It is shown in [16], [20] that this algorithm is in fact nothing but Algorithm 2 and that the pole structure of $P(\lambda)$ can be identified through the indices ρ_i as in the following theorem.

THEOREM 6. *Let the ρ_i be given as in Algorithm 2, and let $\rho_{d+1} \triangleq 0$, $\rho_1 \triangleq n - \nu_1 = n - \nu_d^+$. Then $P(\lambda)$ has $\rho_i - \rho_{i+1}$ structural indices equal to $-i$ at $\lambda = 0$ for $i = 1,\dots,d$.*

Using Theorems 2–6, we thus see that the complete eigenstructure of $P(\lambda)$ can be recovered from the pencil (14) via the reduction Algorithm 2 and via Algorithms 1, 1^P and the QZ algorithm running on the reduced pencil (43).

4. CONCLUDING REMARK

All numerical algorithms using floating point arithmetic on matrix problems are subject to a progressive buildup of rounding errors. One would hope that for some algorithms a certain type of "controlled numerical behavior" of rounding errors could be guaranteed. The notion of "backward stability" [27], introduced for this purpose, certifies that the numerical implementation $\bar{f}(A)$ of an exact algorithmic result $f(A)$ of the matrix A satisfies

$$\bar{f}(A) = f(A + E_a) \qquad \text{for} \quad \|E_a\| < k\varepsilon\|A\|$$

where $\|\cdot\|$ is some matrix norm, ε is the relative precision of the computer, and k is some constant close to 1. This in fact says that the *computed result* $\bar{f}(A)$ is the *exact answer* $f(A + E_a)$ to the *slightly perturbed* initial data $A + E_a$ (see [27] for an extensive treatment of the subject).

The backward numerical stability of Algorithms 1, 1^P, 2 and of the QZ algorithm is easily proved because of the use of unitary transformations (see [19], [16], [17]). The final decomposition

$$U(\lambda B - A)V = \left[\begin{array}{c|ccc} \lambda B_{ri} - A_{ri} & \multicolumn{3}{c}{*} \\ \hline 0 & \lambda B_r^{\text{red}} - A_r^{\text{red}} & * & * \\ & 0 & \lambda B_i^{\text{red}} - A_i^{\text{red}} & * \\ & 0 & 0 & \lambda B_f^{\text{red}} - A_f^{\text{red}} \\ & 0 & 0 & 0 \end{array} \right. \left. \begin{array}{c} * \\ * \\ * \\ \lambda B_l^{\text{red}} - A_l^{\text{red}} \end{array} \right] \tag{59}$$

with $\lambda B_f^{\text{red}} - A_f^{\text{red}}$ in upper triangular form, yields all the required information and can be proved to correspond *exactly* to a slightly perturbed version of the pencil (14) (we denote perturbed quantities with an overbar):

$$\lambda \bar{B} - \bar{A} \triangleq \lambda(B + E_b) - (A + E_a),$$

$$\|\bar{B} - B\| < \Pi_b \varepsilon \|B\|, \qquad \|\bar{A} - A\| < \Pi_a \varepsilon \|A\|, \tag{60}$$

where $\|\cdot\|$ is the Frobenius norm, ε is the machine accuracy of the computer, and the Π are some polynomial expressions in m, n, and d (see [16], [17], [27]). This is misleading in a sense, since (14) corresponds to a polynomial matrix $P(\lambda)$ only because of its specific structure (0 and I blocks), and the perturbations E_b and E_a almost always destroy this property. Yet, when $P(\lambda)$ has been previously scaled so that

$$\|C\| = 1, \qquad C \triangleq \begin{bmatrix} P_0 & P_1 & \cdots & P_d \end{bmatrix}, \tag{61}$$

then there exists a slightly perturbed version $\bar{P}(\lambda)$ of $P(\lambda)$ with

$$\|\bar{C} - C\| < \Pi_c \varepsilon \|C\|, \qquad \bar{C} \triangleq \begin{bmatrix} \bar{P}_0 & \bar{P}_1 & \cdots & \bar{P}_d \end{bmatrix} \tag{62}$$

and whose eigenstructure (i.e. null space structure, finite zero structure, and "true" infinite and "fake" infinite structure) is *exactly* the one computed by the above procedure. This can be proved as follows. We can always construct transformations of the type $I + E_u$ and $I + E_v$ with

$$\|E_u\| < \Pi_u \varepsilon, \qquad \|E_v\| < \Pi_v \varepsilon \tag{63}$$

such that

$$(I + E_u)(\lambda \bar{B} - \bar{A})(I + E_v) \tag{64}$$

is again of the form (14), but now with slightly perturbed matrices $\bar{P}_i$. This can be obtained by block elimination of Gaussian type and by block scaling of the pivots. The order in which the 0 and $\pm I$ blocks are "restored" is given by their indices in the following example ($d = 4$):

$$\lambda \begin{bmatrix} 0_1 & -I_3 & 0_{11} & 0_9 \\ 0_1 & 0_3 & -I_5 & 0_9 \\ 0_1 & 0_3 & 0_5 & -I_7 \\ X & X & X & X \end{bmatrix} - \begin{bmatrix} I_2 & 0_{12} & 0_{10} & 0_8 \\ 0_2 & I_4 & 0_{10} & 0_8 \\ 0_2 & 0_4 & I_6 & 0_8 \\ 0_2 & 0_4 & 0_6 & X \end{bmatrix} \tag{65}$$

The transformations 1, 3, 5, 7, 8, 10, and 12 are column operations; the others are row operations. From (60), (63), (64) we now easily obtain (62), because (14) and (65) have the same shape and are ε-close to each other. We thus have obtained that the present approach is backward stable in a "strict" sense, namely that the computed eigenstructure corresponds exactly to a slightly perturbed polynomial matrix $\bar{P}(\lambda)$.

For the regular case [i.e. $\det P(\lambda) \equiv 0$] several "numerical" methods have been derived [1, 7, 10, 11, 14]. The methods computing *all* the zeros [10, 11] require $O(n^3d^3)$ operations, i.e. $O(n^2d^2)$ operations per computed zero (1 operation standing for 1 addition and 1 multiplication). As a comparison with our preliminary reduction scheme for the deflation of the "fake" zeros at ∞, the following operation count is obtained for Algorithm 2. We use ρ_j Householder reflections for the rank compression of $\tilde{P}_j$ [19]. Then, taking into account that one Householder reflection on (the rows or columns of) an $s \times t$ matrix requires $2st$ operations [27], we have that step j of Algorithm 2 requires less than

$$a_j = 2\rho_j mn + (j-2)2\rho_j mn + 2\rho_j n^2 + 2\rho_j r_j n + 2\rho_j s_j n \tag{66}$$

operations. Since $s_j + r_j = (d-j)n$ and $m = n$ (for the regular case), we have

$$a_j = 2\rho_j dn^2. \tag{67}$$

In step j, $\sigma_j = n - \rho_j$ zeros at ∞ are deflated. When $\sigma_j = 1$ we thus need $2dn^3$ operations for the deflation of one zero. When $\sigma_j = n-1$ this figure reduces to approximately $2dn$ operations per deflated zero (notice that $1 \leqslant \sigma_j \leqslant n-1$ if a reduction is performed), for an average of $\sigma_j \approx \rho_j \approx n/2$. Algorithm 2 requires $2dn^2$ operation per deflated zero at ∞, which compares rather favorably with the $O(d^2n^2)$ figure of the previous algorithms. According to [11] such polynomial matrices with many zeros at ∞ often occur in practice. After this preliminary deflation of "fake" zeros at ∞ we recommend, for the regular case, the direct use of the QZ algorithm instead of Algorithm 1, which would deflate the "true" remaining zeros at infinity. Algorithm 1 is indeed much slower than the QZ algorithm, but has the advantage of good recognition of infinite zeros, even when they are multiple and thus ill-conditioned. Another advantage is that polynomial matrices which were presumed to be regular but in fact are not or are "almost" not regular will be detected by Algorithm 1 but not always by the QZ algorithm (see [17], [28], [29] for more details).

For the singular case (i.e. when the normal rank $r \leqslant m$ and/or n) one had recourse to the computation of the Smith canonical form or to related forms

based on elementary column and row operations on $P(\lambda)$ (see [19] and references therein), but these methods are known to be numerically unstable [19, 29]. It was only recently that other algorithms based on concepts of linear system theory (such as generalized state space systems [13]) were developed for tackling the singular case. Unfortunately these first attempts were unstable (see [19] and references therein). The present paper fills the resulting gap in a numerically sound manner and tries to make the connections with the system theoretic literature, because most of its ideas originated there and most of its applications also are to be found there [19].

The main reason why the singular case was never tackled in the numerical literature is, we believe, the possible ill conditioning of this extension. Although the algorithms proposed in this paper are numerically stable, one has then to redefine appropriately the conditioning of the computed eigenstructure (see [19]), since arbitrary perturbations may alter the computed results completely (see [17]); but such a discussion is beyond the scope of this paper.

REFERENCES

1 P. Anselone and L. Rall, The solution of characteristic value-vector problems by Newton's method, *Numer. Math.* 11:38–45 (1968).

2 O. Bosgra and A. van der Weiden, Realization in generalized state-space form for polynomial system matrices and the definitions of poles zeros and decoupling zeros at infinity, *Internat. J. Control* 33:393–411 (1981).

3 G. D. Forney, Jr., Minimal bases of rational vector spaces with applications to multivariable linear systems, *SIAM J. Control* 13:493–520 (1975).

4 F. R. Gantmacher, *Theory of Matrices* Vols. I, II, Chelsea, New York, 1959.

5 B. S. Garbow et al., *Matrix Eigensystem Routines*—EISPACK *Guide Extension*, Springer, New York, 1977.

6 L. Kronecker, Algebraische Reduction der Schaaren Bilinearer Formen, *Sitzungsber. Akad.*, 1890, pp. 763–776.

7 V. Kublanovskaya, On an approach to the solution of the generalized latent value problem for λ-matrices, *SIAM J. Numer. Anal.* 7:532–537 (1970).

8 P. Lancaster, *Lambda-Matrices and Vibrating Systems*, Pergamon, Oxford, 1966.

9 B. McMillan, Introduction to formal realizability theory I, II, *Bell System Tech. J.* 31:217–279, 541–600 (1952).

10 C. Moler and G. Stewart, An algorithm for the generalized matrix eigenvalue problem, *SIAM J. Numer. Anal.* 10:241–256 (1973).

11 G. Peters and J. Wilkinson, $Ax = \lambda Bx$ and the generalized eigenproblem, *SIAM J. Numer. Anal.* 7:479–492 (1970).

12 H. H. Rosenbrock, *State-space and Multivariable Theory*, Wiley, New York, 1970.

13 H. H. Rosenbrock, Structural properties of linear dynamical systems, *Internat. J. Control* 20:191–202 (1974).

14 A. Ruhe, Algorithms for the nonlinear eigenvalue problem, *SIAM J. Numer. Anal.* 10:674–689 (1973)
15 H. J. S. Smith, On systems of linear congruences, *Proc. London Math. Soc.* IV:241–249 (1873).
16 P. Van Dooren, The generalized eigenstructure problem—Applications in linear system theory, Doctoral Thesis, Univ. of Louvain, 1979.
17 P. Van Dooren, The computation of Kronecker's canonical form of a singular pencil, *Linear Algebra Appl.* 27:103–140 (1979).
18 P. Van Dooren, Computing the eigenvalues of a polynomial matrix, in *Proceedings of the IBM-NFWO Symposium*, Brussels, 1979, pp. 213–223.
19 P. Van Dooren, The generalized eigenstructure problem in linear system theory, *IEEE Trans. Automat. Control* 26:111–129 (1981).
20 P. Van Dooren, P. Dewilde, and J. Vandewalle, On the determination of the Smith-McMillan form of a rational matrix from its Laurent expansion, *IEEE Trans. Circuits and Systems* 26:180–189 (1979).
21 G. Verghese, Infinite frequency behaviour in generalized dynamical systems, Ph.D. Thesis, Stanford Univ., 1978.
22 G. Verghese, Comments on "Properties of the system matrix of a generalized state-space system, *Internat. J. Control* 31:1007–1009 (1980).
23 G. Verghese, B. Levy, and T. Kailath, Generalized state-space systems, *IEEE Trans. Automat. Control* 26:811–831 (1981).
24 G. Verghese, P. Van Dooren, and T. Kailath, Properties of the system matrix of a generalized state-space system, *Internat. J. Control* 30:235–243 (1979).
25 R. C. Ward, The combination shift QZ algorithm, *SIAM J. Numer. Anal.* 12:835–853 (1975).
26 J. Wedderburn, *Lectures on Matrices*, Dover, New York, 1964.
27 J. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon, Oxford, 1965.
28 J. Wilkinson, Linear differential equations and Kronecker's canonical form, in *Recent Advances in Numerical Analysis* (C. de Boor and G. Golub, Eds.), Academic, New York, 1978.
29 J. Wilkinson, Kronecker's canonical form and the QZ algorithm, *Linear Algebra Appl.* 28:285–303 (1979).
30 T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, N.J., 1980.
31 V. Kučera, *Discrete Linear Control: The Polynomial Approach*, Wiley, New York, 1979.
32 L. Pernebo, An algebraic theory for the design of controllers for linear multivariable systems. Parts I, II, *IEEE Trans. Automat. Control* AC-26:171–182, 183–194 (1981).
33 G. Verghese and T. Kailath, Rational matrix structure, *IEEE Trans. Automat. Control* AC-26:434–439 (1981).

Received 23 March 1982; revised 16 August 1982