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Carl F. Kurth (SM'64–F'79), for a photograph and biography please see page 104 of the February issue of this TRANSACTIONS.

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George S. Moschytz (M'65–SM'77–F'78), for a photograph and biography please see page 104 of the February issue of this TRANSACTIONS.

On the Determination of the Smith–Macmillan Form of a Rational Matrix From Its Laurent Expansion

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***Abstract*—A novel method is presented to determine the Smith–Macmillan form of a rational $m \times n$ matrix $R(p)$ from Laurent expansions in its poles and zeros. Based on that method, a numerically stable algorithm is deduced, which uses only a minimal number of terms of the Laurent expansion, hence providing a shortcut with respect to cumbersome and unstable procedures based on elementary transformations with unimodular matrices.**

The method can be viewed as a generalization of Kublanovkaya's algorithm for the complete solution of the eigenstructure problem for $\lambda I - A$. From a system's point of view it provides a handy and numerically stable way to determine the degree of a zero of a transfer function and unifies a number of results from multivariable realization and invertibility theory. The paper presents a systematic treatment of the relation between the eigen-information of a transfer function and the information contained in partial fraction or Laurent expansions. Although a number of results are known, they are presented in a systematic way which considerably simplifies the total picture and introduces in a natural way a number of novel techniques.

I. INTRODUCTION

THE PROBLEM of efficient determination of the Smith–Macmillan form of a rational $m \times n$ matrix $R(p)$ does not seem to have received a great deal of attention in the past, although its importance as a key element in systems analysis and design can hardly be denied. The classical method of using unimodular matrix manipulations is cumbersome and not suited for numerical computations, because it results in an extraordinarily large number of polynomial manipulations. In all methods based hereon, numerical stability is lost because pivoting is not based on the coefficients of p but on its power [1].

On the other hand, a number of papers are devoted to the realization problem for system transfer functions and a host of algorithms have been devised [2]–[7]. Another set of algorithms were proposed for system inversion both in the case of systems over a finite field [8]–[10] and in the case of systems over $\mathbb{C}$ or $\mathbb{R}$ [11], and criteria for system invertibility where derived [12], [13]. Most of these methods require the handling of large size matrices and none devote any attention to the numerical stability problem.

An answer to the invertibility problem is needed, e.g., in coding theory where one is interested in deciding whether the transfer function has a unique zero at infinity (of large degree) and if so, in determining the degree of that zero and the inverse of the matrix. Also, in invertibility theory one wishes to know whether there is actually a zero at infinity in which case the system cannot be inverted. In

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both cases, the structure of the zero at a given point and its numerical determination is of crucial importance.

Problems of system factorization from the synthesis [14], [15] and from the analysis [16] point of view have also been studied for coprime factorization [17] and spectral factorization [18], [19]. Most of these techniques are based in some way or the other on properties of the Toeplitz matrices resulting from a Laurent expansion of the transfer function $R(p)$.

In this paper, we show that the Laurent expansion of $R(p)$ in one of its poles/zeros (in this case they may coincide) provides in a very simple way all the information needed to determine the Smith–Macmillan form of $R(p)$. The method as deduced is local and recursive, treats each pole/zero individually, and uses only the exact amount of information needed from a smallest possible partial expansion. As soon as all the relevant information is collected, the algorithm proceeds to a next point.

Moreover we present an implementable version of the algorithm which is *fast* (because of the work savings) and *numerically stable* (only unitary transformations are performed on the data). This fast version is also shown to boil down to Silverman's structure algorithm [11] when used on state-space descriptions and also to the eigenstructure algorithm of Kublanovskaya (for an excellent discussion of this algorithm, see [20]), when used on $\lambda I - A$ in one of the eigenvalues α of A. The present theory gives additional insights in both specific applications. The introduced concept of rank function is also closely related to Forney's use of valuations [25]. The algorithm presented is in fact a handy way for computing these valuations.

II. Preliminaries

Suppose $R(p)$ is a general rational $m \times n$ matrix, then it has a unique Smith–Macmillan form $\Lambda(p)$ [21], given by

$$R(p) = M(p)\Lambda(p)N(p) \tag{1}$$

where M and N are $m \times m$, respectively, $n \times n$ unimodular matrices and

$$\Lambda(p) = \left[\begin{array}{c|c} e_1/f_1 & 0_{r,n-r} \\ \diagdown & \\ e_r/f_r & \\ \hline 0_{m-r,r} & 0_{m-r,n-r} \end{array}\right]. \tag{2}$$

The e_i and f_i are monic polynomials, whereby e_i divides e_{i+1}, f_j divides f_{j-1}, and e_i is mutually prime with f_i [21]. If a *finite* point $\alpha \in \mathbb{C}$ is a zero of any e_i, then it is called a zero of $R(p)$. If it is a zero of any f_j, then it is called a pole of $R(p)$. (These definitions are not exactly standard, but common and logical; for an extensive discussion see [22].) It should be stressed at this point that a single point α can be at the same time both a pole and a zero of $R(p)$, each with specific order and degree.

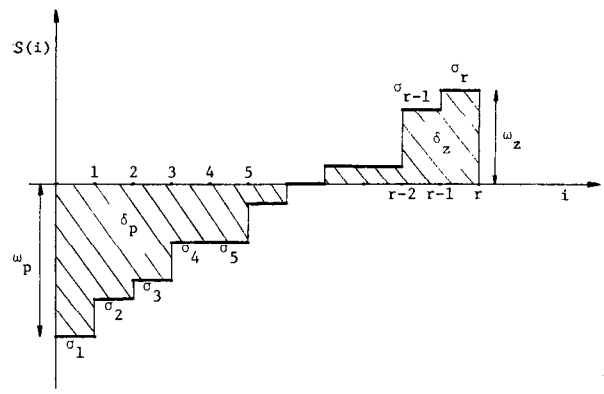

Fig. 1.

With respect to a given $R(p)$ and one of its poles or zeros α, we define a function $\mathcal{S}(i)$ which characterizes the Smith–Macmillan form completely as far as the occurrence of the pole/zero α in its entries is concerned. Let $\Lambda(p)$ be factorized as $\Lambda(p) = \Lambda_\alpha(p) \cdot G(p)$ with

$$\Lambda_\alpha = \left[\begin{array}{c|c} (p-\alpha)^{\sigma_1} & 0_{r,n-r} \\ \diagdown & \\ (p-\alpha)^{\sigma_r} & \\ \hline 0_{m-r,r} & 0_{m-r,n-r} \end{array}\right]$$

$$\text{and } G = \operatorname{diag}\{g_1, \cdots, g_n\} \tag{3}$$

where the g_i have no pole or zero at α (put $g_i = 1$ for $r < i \leqslant n$). Let, for $0 < i \leqslant r$:

$$\mathcal{S}(i) = \sigma_i \text{ for } i \text{ integer}$$

$$\mathcal{S}(i) = \sigma_{i+} \text{ for } i \text{ noninteger and where } i+ \text{ is the upwards rounded version of } i. \tag{4}$$

The picture of $\mathcal{S}(i)$ is thus a step function as shown in Fig. 1. Because of the divisibility properties of e_i and f_j, $\mathcal{S}(i)$ is a nondecreasing staircase. This function also gives the order $\omega_p(\omega_z)$ and the degree $\delta_p(\delta_z)$ of the pole (zero) α [7], [23]:

$$\begin{cases} \omega_p = -\sigma_1 & (\text{if } \sigma_1 \leqslant 0) \\ \omega_z = \sigma_r & (\text{if } \sigma_r \geqslant 0) \end{cases} \qquad \begin{cases} \delta_p = -\sum\limits_{\sigma_i < 0} \sigma_i \\ \delta_z = \sum\limits_{\sigma_i > 0} \sigma_i. \end{cases} \tag{5}$$

The degrees can also be defined as areas delimited by the graph of $\mathcal{S}(i)$ (shaded areas in Fig. 1).

Example 1: Let $\Lambda_\alpha(p) = \operatorname{diag}\{(p-\alpha)^{-2}, 1, (p-\alpha)^3, (p-\alpha)^3\}$. Then $\mathcal{S}(i)$ is given as shown in Fig. 2. ■

Suppose

$$R(p) = \sum_{i=-l}^{\infty} R_i (p-\alpha)^i \tag{6}$$

is a Laurent expansion of $R(p)$ at α. We will denote by

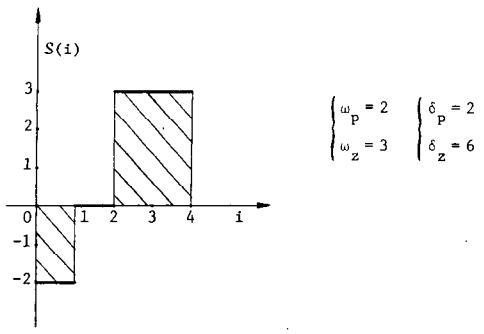

Fig. 2.

$T_i(R,\alpha)$ the Toeplitz matrix ($i \geq -l$)

$$T_i(R,\alpha)=\begin{bmatrix} R_{-l} & \cdots & R_{i-1} & R_i \\ & \ddots & & R_{i-1} \\ & & & \vdots \\ & & & R_{-l} \end{bmatrix}. \tag{7}$$

In the sequel we show how to obtain $\mathcal{S}(i)$ at any point α by working on $T_i(R,\alpha)$. Our main theorem retrieves this information from a smallest possible number of terms in the Laurent expansion of $R(p)$ at the point α.

III. Smith–Macmillan Information

We demonstrate that the rank information of the Toeplitz matrices $T_i(R,\alpha)$ completely determines $\Lambda_\alpha(p)$. We introduce first some terminology.

Definition 3.1

Starting with (6) and using the notation defined by (7) we define the *rank index* $\rho_i(R,\alpha)$ as (we drop R and α when superfluous because of the context)

$$\rho_i \triangleq \text{rank } T_i - \text{rank } T_{i-1}. \tag{8}$$

Hereby, we suppose the nonexisting T_i ($i<-l$) to have rank zero. It is easy to check then that

$$\text{rank } T_i = \sum_{j=-l}^{i} \rho_j. \tag{9}$$

Definition 3.2

Let $R(p)$ and $Q(p)$ be two $m\times n$ rational matrices related by

$$R(p)=E(p)\cdot Q(p)\cdot F(p) \tag{10}$$

where E and F are $m\times m$, respectively, $n\times n$ rational matrices. We then say that R and Q are *similar at* α if E and F are regular at α ($E(\alpha)$ and $F(\alpha)$ invertible).

Proposition 3.3

If $R(p)$ and $Q(p)$ are similar at α then they have the same rank indices at α.

Proof: Since E and F are regular at α they both have a Taylor expansion at α:

$$\begin{aligned} E(p)&=E_0+E_1(p-\alpha)+E_2(p-\alpha)^2+\cdots \\ F(p)&=F_0+F_1(p-\alpha)+F_2(p-\alpha)^2+\cdots \end{aligned} \tag{11}$$

where $E_0=E(\alpha)$ and $F_0=F(\alpha)$.

Expanding $R(p)$ and $Q(p)$ in Laurent series at α, it is easy to see from (10) that they have the same order l for the pole α and that their coefficients satisfy

$$\begin{bmatrix} R_{-l} & \cdots & R_i \\ & \ddots & \\ & & R_{-l} \end{bmatrix} = \begin{bmatrix} E_0 & \cdots & E_{l+i} \\ & \ddots & \\ & & E_0 \end{bmatrix} \cdot \begin{bmatrix} Q_{-l} & \cdots & Q_i \\ & \ddots & \\ & & Q_{-l} \end{bmatrix} \cdot \begin{bmatrix} F_0 & \cdots & F_{l+i} \\ & \ddots & \\ & & F_0 \end{bmatrix} \tag{12}$$

for $i \geq -l$. Since E_0 and F_0 are invertible, the Toeplitz matrices built on $E(p)$ and $F(p)$ are also. From (12) it follows then that

$$\text{rank } T_i(R,\alpha)=\text{rank } T_i(Q,\alpha) \tag{13}$$

for $i \geq -l$. ■

Corollary 3.4

A rational matrix R has the same rank indices as its Smith–Macmillan form in any finite point.

Proof: Unimodular matrices have no finite poles or zeros and are thus regular in any finite point $\alpha\in\mathbb{C}$. A rational matrix is thus similar to its Smith–Macmillan form at any finite point α. The proof is then completed by Proposition 3.3. ■

Corollary 3.5

A rational matrix R has the same rank indices at α as $\Lambda_\alpha(p)$ (where Λ_α is given by (3)).

Proof: R and Λ have the same rank indices at any finite point, and Λ and Λ_α are obviously similar at α (see (3)) since G is regular at α. ■

Since $R(p)$ and $\Lambda_\alpha(p)$ have the same rank indices at α we can deduce their properties from the Toeplitz matrices $T_i(\Lambda_\alpha,\alpha)$. Those have special properties because of the specific form of $\Lambda_\alpha(p)$: 1) all rows of $T_i(\Lambda_\alpha,\alpha)$ are either zero or have only one nonzero entry, and 2) the nonzero rows of $T_i(\Lambda_\alpha,\alpha)$ are linearly independent. Because of this last property the following holds:

$$\begin{aligned} \rho_i &= \text{rank } T_i(\Lambda_\alpha,\alpha) - \text{rank } T_{i-1}(\Lambda_\alpha,\alpha) \\ &= \text{rank } [\Lambda_{-l},\cdots,\Lambda_i] \end{aligned} \tag{14}$$

where Λ_j is the jth coefficient in the Laurent expansion of $\Lambda_\alpha(p)$ at α. From the higher mentioned properties of $T_i(\Lambda_\alpha,\alpha)$ it follows then that this rank equals the number of 1's in $[\Lambda_{-l},\cdots,\Lambda_i]$ or also the number of powers σ_j smaller than i in $\Lambda_\alpha(p)$. This can immediately be read off from $\mathcal{S}(i)$ of $R(p)$ at α.

Example 2: Resuming example 1: $\Lambda_\alpha(p)=\text{diag}\,\{(p-\alpha)^{-2}, 1, (p-\alpha)^3, (p-\alpha)^3\}$ (see Fig. 3). ■

If we associate a rank function $\mathcal{R}(i)$ to the rank indices ρ_i as follows (staircase function):

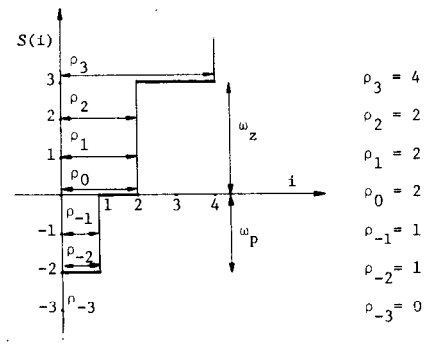

Fig. 3.

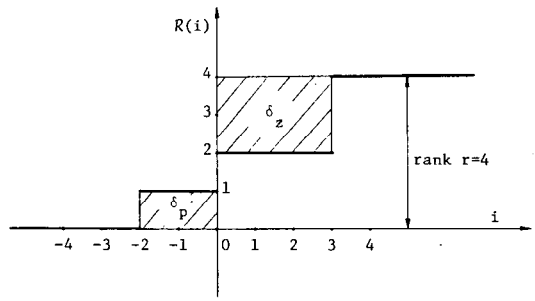

Fig. 4.

$\mathcal{R}(i)=\rho_i$ for i integer

$\mathcal{R}(i)=\rho_{i-}$ for i noninteger where $i-$ is the downwards rounded version of i

then obviously the $\mathcal{R}$ staircase is the reflection of the $\mathcal{S}$ staircase with respect to the bisectrice (see examples 1 and 2) except that $\mathcal{R}$ is also defined for $i<-\omega_p$ and $i>\omega_z$. Remark that $\mathcal{R}$ is also nondecreasing.

Example 3: Using example 2 we obtain $\mathcal{R}(i)$ as shown in Fig. 4. ■

From this similarity between $\mathcal{R}$ and $\mathcal{S}$ the following results immediately follow trivially.

Corollary 3.6

$$\boxed{\omega_p=-\min\{i|\rho_i\neq 0\}} \quad (\text{if} \geqslant 0)$$

$$\boxed{\omega_z=\min\{i|\rho_i=r\}} \quad (\text{if} \geqslant 0)\ (\text{where } r \text{ is the rank of } R(p), \text{ and can be } <m,n). \ ■$$

Corollary 3.7

$$\boxed{\delta_p=\sum_{i=-\omega_p}^{-1}\rho_i}$$

$$\boxed{\delta_z=\sum_{i=1}^{\omega_z}(r-\rho_i)}. \ ■$$

These easy results were derived earlier (e.g., [23]) but are put here in a form convenient for the sequel.

A rank search of the Toeplitz matrices $T_i(R,\alpha)$ gives all information about the occurrence of the pole/zero α in the Smith–Macmillan form of $R(p)$. As soon as a rank index ρ_k equals the normal rank r of $R(p)$ the search can be terminated. By then, a minimal number of coefficients of the Laurent expansion at α has been employed in the computation. In the next section we develop a fast recursive algorithm that performs this rank search in a numerically stable way.

IV. Toeplitz Rank Search

In this section we describe a recursive algorithm to transform a Toeplitz matrix T_i to a matrix whose nonzero rows are linearly independent. We use, therefore, row transformations to reduce an arbitrary matrix A to the form

$$U\cdot A=\left[\begin{array}{c}A^-\\ \hline 0\end{array}\right]\}\rho$$

whereby A^- has ρ linearly independent rows. In the sequel we call such a transformation a row compression of the matrix A.

Since each Toeplitz matrix $T_i(R,\alpha)$ is an embedding of any lower order Toeplitz matrix $T_j(R,\alpha)$ (for $j<i$), it can be expected that the compression of these smaller Toeplitz matrices can be used to transform T_i. This idea of making full use of the Toeplitz structure of T_i is applied in the following recursive Toeplitz rank search.

The algorithm, acting on the sequence $R_{-l},\cdots,R_i$, defines a set of indices $\{\rho_{-l},\cdots,\rho_i\}$ which will be proven to be the rank indices of T_i.

Algorithm 4.1

1) $j=-l$; $\tilde{R}_k=R_k$ for $k=-l,\cdots,i$.

2) Construct a row transformation U_j that computes the rank ρ_j of $\tilde{R}_j$ and puts it in compressed form

$$U_j\tilde{R}_j=\left[\begin{array}{c}R_j^-\\ \hline 0\end{array}\right]\begin{array}{l}\}\rho_j\\ \}\nu_j\end{array}.$$

3) Multiply $\tilde{R}_k$ (for $k=j+1,\cdots,i$) left with U_j and partition analogously to 2)

$$U_j\tilde{R}_k=\left[\begin{array}{c}R_k^-\\ \hline R_k^+\end{array}\right]\begin{array}{l}\}\rho_j\\ \}\nu_j\end{array}.$$

4) Update the $\tilde{R}_k$ as

$$\tilde{R}_k=\left[\begin{array}{c}R_{k-1}^-\\ \hline R_k^+\end{array}\right], \qquad \text{for } k=j+1,\cdots,i.$$

5) If $j<i$ then $j=j+1$; go to 2
else stop. ■

Remark that the symbols $\tilde{R}_k$, R_k^-, and R_k^+ denote in each step a different object. In a more rigorous notation we should use an additional index referring to the current step j, giving $\tilde{R}_k^{(j)}$, $R_k^{-(j)}$, $R_k^{+(j)}$. For sake of conciseness, we only add this index when confusion is possible.

For better understanding we draw the actions of step j on the matrix

$$\tilde{T} \triangleq \left[\tilde{R}_j^{(j)} \mid \tilde{R}_{j+1}^{(j)} \mid \cdots \mid \tilde{R}_{i-1}^{(j)} \mid \tilde{R}_i^{(j)} \right]. \quad (15)$$

Steps 2 and 3 then multiply $\tilde{T}$ left with U_j such that

$$U_j \cdot \tilde{T} = \left[\begin{array}{c|c|c|c|c} \frac{R_j^{-(j)}}{0} & \frac{R_{j+1}^{-(j)}}{R_{j+1}^{+(j)}} & \cdots & \frac{R_{i-1}^{-(j)}}{R_{i-1}^{+(j)}} & \frac{R_i^{-(j)}}{R_i^{+(j)}} \end{array} \right] \begin{array}{l} \}\rho_j \\ \}\nu_j \end{array} \quad (16)$$

with $R_j^{-(j)}$ having full rank ρ_j. The new $\tilde{T}$ matrix is then (by steps 4 and 5) one block shorter and looks like

$$\tilde{T} = \left[\begin{array}{c|c|c|c|c} \frac{R_j^{-(j)}}{R_{j+1}^{+(j)}} & \frac{R_{j+1}^{-(j)}}{R_{j+2}^{+(j)}} & \cdots & \frac{R_{i-2}^{-(j)}}{R_{i-1}^{+(j)}} & \frac{R_{i-1}^{-(j)}}{R_i^{+(j)}} \end{array} \right] \begin{array}{l} \}\rho_j \\ \}\nu_j \end{array} \quad (17)$$

which is (16) with the ν_j bottom rows shifted to the left and the last block then chopped off. After the definition of the $\tilde{R}_k^{(j+1)}$ blocks this is also equal to

$$\tilde{T} = \left[\tilde{R}_{j+1}^{(j+1)} \mid \tilde{R}_{j+2}^{(j+1)} \mid \cdots \mid \tilde{R}_{i-1}^{(j+1)} \mid \tilde{R}_i^{(j+1)} \right]. \quad (18)$$

This is repeated until $\tilde{T}$ is vanishing (at $j=i$). In the algorithm the old version of $\tilde{T}$ is deleted and replaced by its new version (18), which also justifies the dropping of the step indices.

Theorem 4.2

Algorithm 4.1 produces the rank indices of the Toeplitz matrix T_i defined on the sequence $R_{-l}, \cdots, R_i$.

Proof: We prove this inductively. In the first step ($j=-l$) the required rank index $\rho_{-l}=\text{rank}\ (R_{-l})$ is indeed computed by compressing the rows of R_{-l}. We then multiply left each block row of T_i with the transformation U_{-l} (let $U \triangleq \text{diag}\,[U_{-l}, \cdots, U_{-l}]$):

$$U \cdot T_i = \left[\begin{array}{c|c|c|c|c|c} R_{-l}^- & R_{-l+1}^- & R_{-l+2}^- & & R_{i-1}^- & R_i^- \\ 0 & R_{-l+1}^+ & R_{-l+2}^+ & \cdots & R_{i-1}^+ & R_i^+ \\ & R_{-l}^- & R_{-l+1}^- & & R_{i-2}^- & R_{i-1}^- \\ & 0 & R_{-l+1}^+ & \cdots & R_{i-2}^+ & R_{i-1}^+ \\ & & \ddots & & \vdots & \vdots \\ & & & \ddots & \vdots & \vdots \\ & & & & R_{-l}^- & R_{-l+1}^- \\ & & & & 0 & R_{-l+1}^+ \\ & & & & & R_{-l}^- \\ & & & & & 0 \end{array} \right] \begin{array}{l} \}\rho_{-l} \\ \\ \\ \\ \\ \\ \\ \\ \}\nu_{-l} \end{array} \quad (19)$$

After permuting block rows of equal block length (the + blocks are permuted with the − blocks as indicated by the arrows), we obtain, using the indexed notation of (17) and (18):

$$P \cdot U \cdot T_i = \left[\begin{array}{c|c|c|c} R_{-l}^{(-l)} & |R_{-l+1}^{(-l)}| & \cdots & |R_i^{(-l)}| \\ & \tilde{R}_{-l+1}^{(-l+1)} & & \tilde{R}_i^{(-l+1)} \\ 0 & & \ddots & \\ 0 & & 0 & \tilde{R}_{-l+1}^{(-l+1)} \end{array} \right]$$

$$\triangleq \left[\begin{array}{c|c} V_{-l} & Y \\ \hline 0 & \hat{T}_i^{(-l+1)} \\ \hline 0 & 0 \end{array} \right] \begin{array}{l} \}\rho_{-l} \\ \\ \}\nu_{-l} \end{array}. \quad (20)$$

By premultiplying T_i with the transformation P·U we have compressed the Toeplitz matrix T_{-l} to the full rank matrix V_{-l} and separated the remaining rows in the form of a new Toeplitz matrix (without any additional computations). We show by induction that this is maintained in each step.

Induction step j: Suppose that at the beginning of step j, T_i is already transformed to

$$\left[\begin{array}{c|c} V_{j-1} & Y \\ \hline 0 & \hat{T}_i^{(j)} \\ \hline 0 & 0 \end{array} \right] \begin{array}{l} \}\delta_{j-1} = \sum_{k=-l}^{j-1} \rho_k \\ \\ \}\sum_{k=-l}^{j-1} \nu_k \end{array}$$

$$\text{with} \quad \hat{T}_i^{(j)} = \left[\begin{array}{c|c} \tilde{R}_j^{(j)} & \tilde{R}_i^{(j)} \\ 0 & \tilde{R}_j^{(j)} \end{array} \right] \quad (21)$$

and with V_{j-1} of full rank δ_{j-1} which is the rank of T_{j-1}. Step j acts on $\hat{T}_i^{(j)}$ analogously to (19) and (20) and transforms it to:

$$\begin{bmatrix} R_j^{-(j)} & |R_{j+1}^{-(j)}| & \cdots & |R_i^{-(j)}| \\ 0 & \begin{bmatrix} & \hat{T}_i^{(j+1)} & \end{bmatrix} & & \\ 0 & & 0 & \end{bmatrix} \begin{matrix} \}\rho_j \\ \\ \}\nu_j \end{matrix} \quad . \tag{22}$$

Embedding this in (21), we again obtain (21) for j incremented by 1. Thereby, V_j has the form (for some X)

$$V_j = \left[\begin{array}{c|c} V_{j-1} & X \\ \hline 0 & R_j^{-(j)} \end{array}\right] \begin{matrix} \}\delta_{j-1} \\ \}\rho_j \end{matrix} .$$

Since both $R_j^{-(j)}$ and V_{j-1} have full rank, V_j also has full rank equal to $\rho_j + \delta_{j-1}$. The row operations applied on T_i do not affect the original column ordering of T_i nor of any of its embedded Toeplitz submatrices (e.g., T_j). This means that T_j and V_j have the same row space and hence the same rank. Therefore, ρ_j is indeed the required rank index and V_j is a row-compression of T_j. ■

Corollary 4.3

Algorithm 4.1 implicitly computes a row transformation U which transforms T_i to the following compressed form where the $R_j^{-(j)}$ blocks have full rank ρ_j.

$$U \cdot T_i = \begin{bmatrix} \boxed{R_{-l}^{-(-l)}} & \cdots & R_i^{-(-l)} \\ & \ddots & \vdots \\ & \boxed{R_j^{-(j)}} & \cdots \quad R_i^{-(j)} \\ & & \ddots \quad \vdots \\ & & \boxed{R_i^{-(i)}} \\ & 0 & \end{bmatrix} \begin{matrix} \}\rho_{-l} \\ \vdots \\ \}\rho_j \\ \vdots \\ \}\rho_i \\ \} \sum_{k=-l}^{i} \nu_k \end{matrix} \tag{23}$$

■

It is useful at this point to discuss several aspects related to the algorithm.

1) For efficient rank determination one is forced to use the singular value decomposition (SVD) [20]. The U_j matrices are then derived from the SVD of $\tilde{R}_j$ (* denoting Hermitian conjugate):

$$\tilde{R}_j = U_j \Sigma_j V_j^* \quad \text{and} \quad U_j^* \tilde{R}_j = \Sigma_j V_j^* . \tag{24}$$

On putting the singular values smaller than ϵ, equal to zero one gets

$$U_j^* \tilde{R}_j = \left[\begin{array}{c} R_j^- \\ \hline 0 \end{array}\right] \begin{matrix} \}\rho_j \\ \}\nu_j \end{matrix} \tag{25}$$

where the rows of R_j^- are orthogonal to each other and have norms greater than ϵ. By using the row transformations U_j^* the overall transformation matrix U in (23) will also be unitary, which guarantees the numerical stability of the Toeplitz rank search.

2) An SVD of T_i would require $0[m^2 n(l+i+1)^3]$ operations while the fast version using the Toeplitz structure requires only $0[m^2 n(l+i+1)^2]$ operations (moreover it yields the ranks of all lower order Toeplitz matrices).

3) Stop criterion:

a) When only the rank indices $\rho_{-l}, \cdots, \rho_i$ are derived, then Algorithm 4.1 will compute them, when acting on the sequence $R_{-l}, \cdots, R_i$. This is for instance the case when one is only interested in the polar structure at α [2].

b) If the total Smith–Macmillan information is required there are different possibilities.

i) The order d of the zero α is known. Then Algorithm 4.1 acting on $R_{-l}, \cdots, R_d$ will give all required information.

ii) The order d of the zero α is not known but $R(p)$ has a finite expansion at α. This is, e.g., always the case with polynomial matrices (see Section V). Then the algorithm will be performed on this finite sequence and can be continued an arbitrary number of steps, if one cares to shift in zeros at the end. It should be remarked that the Toeplitz rank search does not stop when reaching the end of the sequence. Example 4 (Section V) will show an instance where Toeplitz ranks keep increasing even after the length of the sequence has been reached. Yet, the algorithm will only process the significant blocks while the transformations on zero blocks remain trivially zero.

iii) The order d is unknown and $R(p)$ has an infinite expansion. Then we start with an arbitrary sequence $R_{-l}, \cdots, R_i$ (i can also be equal to $-l$) and perform Algorithm 4.1. If at the end of this, ρ_i is not equal to the normal rank r, the sequence should be enlarged to, e.g., $R_{-l}, \cdots, R_i, R_{i+1}$. Algorithm 4.1 can be adapted for such growing sections but if one wants to avoid double work, some intermediary results and transformations must be kept in memory. This can be done efficiently with a minimal memory content by following algorithm (using indexed notation).

Algorithm 4.4

1) $j = -l$;
2) $\tilde{R}_j^{(-l)} = R_j$;
3) if $j > -l$ then for $i = -l$ until $j-1$ do:
   a) multiply $\tilde{R}_j^{(i)}$ with U_i and partition rows

$$U_i \tilde{R}_j^{(i)} = \left[\begin{array}{c} R_j^{-(i)} \\ \hline R_j^{+(i)} \end{array}\right] \begin{matrix} \}\rho_i \\ \}\nu_i \end{matrix}$$

   b) define $\tilde{R}_j^{(i+1)}$ as

$$\tilde{R}_j^{(i+1)} \triangleq \left[\begin{array}{c} R_{j-1}^{-(i)} \\ \hline R_j^{+(i)} \end{array}\right] \begin{matrix} \}\rho_i \\ \}\nu_i \end{matrix}$$

4) construct a row transformation U_j that computes the rank ρ_j of $\tilde{R}_j^{(j)}$ and puts it in compressed form:

$$U_j\tilde{R}_j^{(j)}=\begin{bmatrix} R_j^{-(j)} \\ \hline 0 \end{bmatrix} \begin{matrix} \}\rho_j \\ \}\nu_j \end{matrix}$$

5) if $\rho_j=r$ then stop
else $j=j+1$; go to 2. ■

The step indices used in the algorithm refer to earlier notations and help the reader to check the procedure. While Algorithm 4.1 computes the form (23) row by row, the algorithm above does it column by column. Therefore, the procedure does not need to know a priori how many blocks to process.

4) Extension for $p=\infty$

The Smith–Macmillan form of a rational matrix does not reveal the pole/zero structure of the point $p=\infty$ since the transformations performed are not regular at infinity. Our approach with Laurent expansions however can still go through. If $R(p)$ has an expansion at infinity of the type

$$R(p)=D_l p^l+D_{l-1}p^{l-1}+\cdots D_0+D_{-1}p^{-1}+\cdots \quad (26)$$

then the rank search of Toeplitz matrices of the type

$$T_i=\begin{bmatrix} D_l & \cdots & D_{-i} \\ & \ddots & \vdots \\ & & D_l \end{bmatrix} \quad (27)$$

will give the pole/zero structure at $p=\infty$. It is easy to prove that this corresponds to the Smith–Macmillan structure of the point $\lambda=0$ of the transformed matrix $R(\lambda^{-1})$ (this way of coping with the point at infinity is not standard, but seems logical in present context).

V. Eigenstructure Algorithm

Suppose α is an eigenvalue of the $r\times r$ matrix A and we want to compute the Jordan structure of $\lambda I-A$ at α. It is known [1] that this is the same as the behavior at the point α of the Smith-form $\Lambda(\lambda)$ of that polynomial matrix. The Laurent expansion at α of that matrix is

$$(\alpha I-A)+(\lambda-\alpha)I \triangleq B+(\lambda-\alpha)I. \quad (28)$$

By computing the rank indices of the sequence $B,I,0,\cdots,0$ we thus obtain the required information. The rank function $\mathcal{R}(i)$ will give us the function $\mathcal{S}(i)$ of $\lambda I-A$ at α which also determines the Jordan structure of A at α since each elementary divisor $(\lambda-\alpha)^{\sigma_j}$ corresponds to a Jordan block of size σ_j [1].

Example 4: In order to illustrate the basic idea, let us compute the eigenstructure for

$$A=\begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

which has a unique eigenvalue at $\lambda=2$. Also we use elementary rather than orthogonal transformations for illustrative simplicity. In numerical practice, orthogonal transformations would be used. The Laurent expansion of

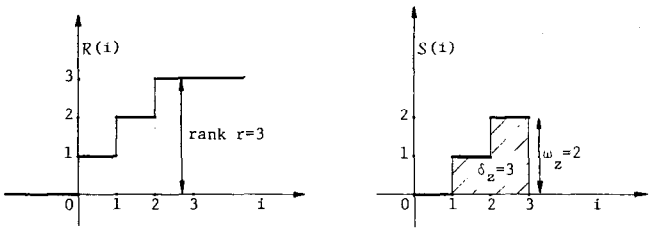

Fig. 5.

$\lambda I-A$ at $\lambda=2$ is

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix}+I\cdot(\lambda-2).$$

Since the only eigenvalue of A is $\lambda=2$, we can construct the Smith form by looking only at the rank indices of $\lambda=2$. As remarked previously, we can work on $\tilde{T}=[B|I]$ only, keeping in mind that null blocks are following (refer to formulas (15) and following for notation).

Step 0:

$$U_0\tilde{T}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\cdot\tilde{T}$$

$$=\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right] \text{ and } \rho_0=1$$

redefine

$$\tilde{T}=\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ \hline -2 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \quad \text{(zeros are shifted in).}$$

Step 1:

$$U_1\tilde{T}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}\cdot\tilde{T}$$

$$=\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right] \text{ and } \rho_1=2$$

redefine

$$\tilde{T}=\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{(zeros are shifted in).}$$

Step 2: $\tilde{T}$ has leading block of full rank: $\rho_2=3$. The rank function and the function $\mathcal{S}(i)$ are thus given as in Fig. 5. This means that the Smith form is $\Lambda(\lambda)=\text{diag}\{1,(\lambda-2),(\lambda-2)^2\}$ and A has a Jordan canonical form as (Jordan sizes 1 and 2):

$$A=M^{-1}\cdot\left[\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 2 & 1 \\ 0 & 0 & 2 \end{array}\right]\cdot M. \quad ■$$

We now show that the eigenstructure algorithm of Kublanovskaya [20] is similar to our rank search (except for some additional saving of work because of the special type of expansion $B+(\lambda-\alpha)I$).

This algorithm can be written as follows, keeping the same numbering as in Algorithm 4.1.

Algorithm 5.1

1) $j=0$; $\tilde{B}=B$,
2) compute a SVD of $\tilde{B}$ and its rank r_j: $\tilde{B}=U\Sigma V^*$,
3) compress rows of $\tilde{B}$ to full rank by a similarity transformation,

$$\tilde{B}\Leftarrow U^*\tilde{B}U=\Sigma V^*U=\left[\begin{array}{c|c}B_{1,1} & B_{1,2}\\ \hline \underbrace{0}_{r_j} & \underbrace{0}_{n_j}\end{array}\right]\begin{array}{l}\}\ r_j\\ \}\ n_j\end{array}$$

4) update $\tilde{B}\Leftarrow B_{1,1}$
5) if $n_j>0$ then $j=j+1$; go to 2
   else stop. ■

The n_j are the null ranks at each step and are equal to our ν_j. The r_j are not the rank indices (since the size of the block $\tilde{B}$ is decreasing) but are related to them:

$$\rho_j=r-n_j=\sum_{i=0}^{j} r_i. \tag{29}$$

In order to show the similarity between both algorithms we remark that multiplication with a constant invertible column transformation does *not* affect the rank search. This means that each block of $\tilde{T}$ may be right-multiplied by an invertible matrix. Let us put Algorithm 4.1 in the form (15)–(17) using the matrix $\tilde{T}=[\tilde{B}|I]$ and allow slight modifications in order to obtain Algorithm 5.1.

Steps 2 and 3: The SVD of $\tilde{B}$ gives r_j and U_j

$$U_j^*\tilde{T}=\left[\begin{array}{c|c}U_j^*\tilde{B} & U_j^*\end{array}\right]$$

and allowing a right transformation U_j

$$U_j^*\tilde{T}\left[\begin{array}{c|c}U_j & 0\\ \hline 0 & U_j\end{array}\right]=\left[\begin{array}{c|c}U_j^*\tilde{B}U_j & I\end{array}\right]=\left[\begin{array}{c|c|c|c}B_{1,1} & B_{1,2} & I_r & 0\\ \hline 0 & 0 & 0 & I_n\end{array}\right].$$

Steps 4 and 5: Redefine

$$\tilde{T}=\left[\begin{array}{c|c|c|c}B_{1,1} & B_{1,2} & I_r & 0\\ \hline 0 & I_n & 0 & 0\end{array}\right].$$

An additional row transformation can zero out $B_{1,2}$, and shows that we can work equally well on $\tilde{T}=[B_{1,1}|I_r]=[\tilde{B}|I_r]$ (this size reduction preserves null ranks, not rank indices).

Since $\rho_j=r-\nu_j=r-n_j$ the stop criterion is indeed $n_j=0$ or $\rho_j=r$, the normal rank of λI_r-A.

The use of similarity transformations in Algorithm 5.1 maintains the unit matrix as second block in $\tilde{T}$, whereby one can work only on $\tilde{B}$. Also, when proceeding to a next eigenvalue, an easy updating of $\tilde{B}$ is possible without restoring the original size r [20]. These modifications cannot be performed in the general case treated in this paper, because at each new step novel information might be shifted in, which is not the case here. Therefore, we may assert that Algorithm 4.1 is a straightforward generalization of Algorithm 5.1.

The connection with Kublanovskaya's algorithm allows us to make some considerations about numerical properties of the generalized eigenvalue problem. It is known that for the eigenvalue problem $\lambda I-A$ an expansion $(\alpha I-A)+(\lambda-\alpha)I$ is needed in order to retrieve the Jordan information of A [20]. Indeed, when such an expansion is not known, every attempt to retrieve a Jordan chain causes the multiple eigenvalue to split up in a cluster of not necessarily close eigenvalues [20]. Analogously in the case of the Smith–Macmillan information, we would end up with all first-order poles and zeros when no expansions are used as starting point.

When using Laurent expansion of the matrix $R(p)$, the problem can be converted to a rank-definition problem for which the singular value decomposition can be successfully used. Just as in the eigenvalue problem [20] we see how the SVD allows us, in a certain sense, to clear off the errors that would cause the poles or zeros to split up. Therefore, the knowledge of the eigenvalue and an expansion in that point are required.

VI. Applications

In this section we discuss briefly a number of applications of the theory previously treated. In each case we will refer to the relevant papers in the specific application. The numerical algorithm presented is to be introduced wherever a Toeplitz reduction is to be performed.

During revision of this paper other papers were published by Emre and Silverman [26] and by Van Dooren and Dewilde [2], covering part of the results of Section IV. However, the connection with the Smith–Macmillan information and the generalized eigenvalue problem are not shown. It is exactly this relation that generates the wide variety of applications summed up hereafter.

A) Coprime Factorization

When $R(p)$ is given in a partial fraction expansion (PFE)

$$R(p)=\sum_{i=1}^{k} R_{\alpha_i}(p)+R_0+R_\infty(p) \tag{30}$$

then we have the polar parts of the Laurent expansion in each pole explicitly given. The coprime factorization

$$R(p)=D^{-1}(p)\cdot N(p) \tag{31}$$

can be viewed [17] as the construction of a minimal operator $D(p)$ that displaces the finite poles of $R(p)$ at infinity in the product

$$D(p)\cdot R(p)=N(p). \tag{32}$$

The minimality of $D(p)$ ensures the coprimeness between $D(p)$ and $N(p)$ [17].

One proves [17], [2] that this can be done with the present structure algorithm when performed on each polar section $R_{\alpha_i}(p)$ consecutively. The structure of the poles α_i will indeed be reflected in $D(p)$ since the poles α_i must cancel in the product (32). The rank search of the Toeplitzes $T_{-1}(R_{\alpha_i}, \alpha_i)$ will give all necessary information to build up $D(p)$ and will also guarantee its minimality [17].

B) Realization

Using the PFE (30) the problem of realizing $D(p)$ boils down to determining its polar structure in each of its finite poles. In [2] we prove that the present ideas lead to a fast and stable algorithm for realizing each finite pole of $R(p)$. A total realization is then merely a block arrangement of these partial realizations. The superiority of this algorithm above other realization algorithms, both in fastness and stability, is also demonstrated in that paper.

C) Inversion of Systems

The problem here [11], [24] is to find an inverse (left or right) for the system

$$R(p) = C(pI - A)^{-1}B + D \tag{33}$$

where D is constant. If D is not invertible (left or right) transformations have to be performed on a growing Toeplitz matrix

$$T_j = \begin{bmatrix} D & CB & \cdots & CA^{j-1}B \\ & \ddots & \ddots & CB \\ & & \ddots & D \end{bmatrix} \tag{34}$$

until a modified D is found that can be inverted. It is easy to show that $D + CBp^{-1} + CABp^{-2} + \cdots$ is an expansion of $R(p)$ at infinity and that the Toeplitz search of Silverman [11] is nothing but our rank search at the point $p = \infty$. This will thus deliver the zero structure of $R(p)$ at $p = \infty$. As soon as D has full rank (corresponding to a rank index which equals the normal rank), the zero structure is completely determined, and the system can be inverted. This alternative way of looking at the problem gives an easy understanding of the inversion problem. Moreover, when one has to cope with a polynomial D in (33), the extension is trivial since the expansion of $R(p)$ is

$$R(p) = D_l p^l + \cdots D_1 p + D_0 + CBp^{-1} + CABp^{-2} + \cdots \tag{35}$$

and the Toeplitz matrix becomes

$$T_j = \begin{bmatrix} D_l & D_0 & \cdots & CA^{j-1}B \\ & \ddots & \ddots & D_0 \\ & & \ddots & D_l \end{bmatrix}. \tag{36}$$

D) Linearization of Polynomial Matrices

When using present ideas on the polynomial matrix

$$D(p) = D_0 + D_1 p + \cdots D_l p^l \tag{37}$$

we can obtain a minimal linearization $pS - T$ which has the same zero structure as $D(p)$. In order to insure minimality, a rank search of

$$T_{-1} = \begin{bmatrix} D_l & & D_1 \\ & \ddots & \vdots \\ & & D_l \end{bmatrix} \tag{38}$$

is required [27]. Linearizations of polynomial matrices have the advantage of bringing the problem in a suitable form for computation of its zero structure; numerically stable algorithms have been developed [28], [29] for computation of generalized eigenvalue problems of the type $\lambda S - T$.

E) Eigenstructure of Polynomial and Rational Matrices

When using jointly the realization algorithm *B*) and the linearization algorithm *D*) on an arbitrary rational matrix we obtain a linearization of $R(p)$ in the sense that it has the same zero structure as $R(p)$ [27]. When the $m \times n$ matrix $R(p)$ has normal rank r smaller than m and/or n this linearization also yields interesting results about the role of a left and/or right dual basis for the zero structure of the matrix [27], as well as a unifying theory for inversion of rational matrices.

F) Factorization of Rational Matrices

When factoring $R(p)$ as $R_1(p) \cdot R_2(p)$ (eventually spectral factorization) a choice of poles and zeros must be done. Also the spaces which accompany those points [14], [16], [18] have to be determined. A handsome way of doing this is by determining the realization of $R(p)$ with the present algorithm [2], and by computing the zeros with the linearization of $R(p)$. A factorization of $R(p)$ can then be computed (if it exists) by updating its realization so that it splits into the tandem system $R_1(p) \cdot R_2(p)$ [30].

VII. Conclusions

The numerically stable algorithm presented in this paper is derived in a particularly simple way from the Smith–Macmillan theory. The major concern of the paper is to provide a good insight in the algebraic structure of this generalized eigenvalue problem. The connection with the Kublanovskaya algorithm, e.g., shows how fragile the Smith–Macmillan information is and how heavily it relies on the knowledge of poles and zeros and expansions in these points. In some applications, where only partial results are requested, such information is indeed available (see applications *C*) and *D*)). In other cases one can solve the problem without the knowledge of poles and zeros but with the implications thereof (absence of multiple poles/zeros, eventual unstability, etc.). The algorithm has been implemented on computer and has been used in a

number of applications. Another interesting aspect of the theory is the linkage it provides between a number of seemingly disjoint topics: 1) the Kublanovskaya algorithm in numerical analysis as described in [20], 2) the realization theory of a transfer function in ABCD form, 3) the inversion theory as used in coding theory and the so-called structure algorithm, 4) factorization theory spectral or general, and 5) network realization theory through coprime factorization [19]. Some features of the theory may appear almost trivial, but seen against the proliferation of difficult arguments found in the literature on this very topic, it seems useful to stress the basic simplicity obtained.

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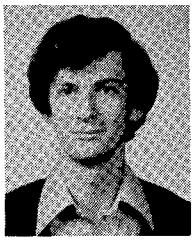

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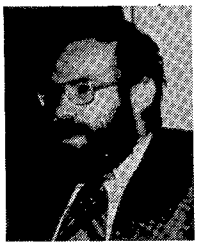

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