Chapter 1

Shift policies in QR-like algorithms and feedback control of self-similar flows

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1.1 Description of the problem

FG **algorithms.** FG algorithms are generalizations of the well-known QR algorithm for calculating the eigenvalues of a matrix. Let $\mathcal{F}$ and $\mathcal{G}$ two closed subgroups of the general linear group $GL_n(\mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or

C). Assuming $\mathcal{F} \cap \mathcal{G} = \{I\}$, each matrix $A \in GL_n(\mathbb{F})$ has at most one factorization of the form $A = FG$, where $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Starting from a given matrix $B_0 \in GL_n(\mathbb{F})$, the FG *algorithm* produces a sequence of matrices B_m, $m = 1, 2, \ldots$, as follows: B_i is factored into a product $B_i = F_{i+1}G_{i+1}$ and this product is reversed to define $B_{i+1} := G_{i+1}F_{i+1}$. Thus

$$B_i = F_{i+1}G_{i+1} \Rightarrow B_{i+1} := G_{i+1}F_{i+1}. \tag{1.1}$$

What is typically expected from the FG algorithm is that the sequence of iterates $\{B_i\}_{i\geq 1}$ converges to a matrix whose diagonal entries are the eigenvalues of B_0. The sequence is indeed a sequence of similarity transformations, that is

$$B_{i+1} = F_{i+1}^{-1}B_iF_{i+1} = G_{i+1}B_iG_{i+1}^{-1}. \tag{1.2}$$

A key property of the algorithm is

$$B_m = F_{(m)}^{-1}B_0F_{(m)} = G_{(m)}B_0G_{(m)}^{-1} \tag{1.3}$$

$$B_0^m = F_{(m)}G_{(m)} \tag{1.4}$$

where the notation $F_{(m)}$ stands for the product $F_mF_{m-1}\ldots F_1$.

Shifted FG algorithms. In general, some iterate B_i of the algorithm may fail to have an FG factorization, in which case the algorithm breaks down. In addition, even if the algorithm converges, the convergence may be very slow. These two reasons motivate the introduction of *shift policies* in the standard FG algorithm. The simplest modification is of the form

$$B_i - \sigma_i I = F_{i+1}G_{i+1} \Rightarrow B_{i+1} := G_{i+1}F_{i+1} + \sigma_i I$$

which still ensures the self-similarity $B_{i+1} = F_{i+1}^{-1}B_iF_{i+1} = G_{i+1}B_iG_{i+1}^{-1}$. More general shift policies allow a polynomial

$$p_i(B_i) = \prod_j (B_i - \sigma_{i_j} I)$$

to replace B_i in the standard algorithm. This allows for several steps of the classic FG algorithm to be concatenated in one step of the generalized FG algorithm.

Shifted FG algorithms typically lead to improved convergence in the situations where the FG algorithm converges. In the situations where the FG algorithm does not converge, shift policies may be used to achieve a continuation of the algorithm, by avoiding certain singularities ([6]).

Open problem. The *analysis* of shifted algorithms and a systematic *design* of shift policies to improve convergence of the FG algorithms are essentially open problems in numerical analysis (See Section 4 for some existing results).

1.2 Reformulation of the problem

Self-similar flows. Self-similar flows are the solutions of differential equations that can be associated with FG algorithms because of the Lie group structure of the closed subgroups of $GL_n(\mathbb{F})$. Under (weak) extra conditions, self-similar flows continuously interpolate the solution of the FG algorithm, that is, the self-similar flow coincides at integer times with the iterates of the FG algorithm.

Let $\Lambda(\mathcal{F})$ and $\Lambda(\mathcal{G})$ be the Lie algebras associated with the Lie groups $\mathcal{F}$ and $\mathcal{G}$. Suppose that $\mathbb{F}^{n\times n} = \Lambda(\mathcal{F}) \oplus \Lambda(\mathcal{G})$. Then every matrix $M \in \mathbb{F}^{n\times n}$ can be expressed uniquely as a sum

$$M = \rho(M) + \nu(M)$$

where $\rho(M) \in \Lambda(\mathcal{F})$ and $\nu(M) \in \Lambda(\mathcal{G})$. Let $f(.)$ a function defined on $sp(B)$ where $sp(B)$ denotes the spectrum of B (this means $f(.)$ and its n first derivatives must be defined on an open set containing $sp(B)$). Then the solution of the differential equation

$$\dot{B} = [B, \rho(f(B))] = [\nu(f(B)), B],\ B(0) = B_0 \tag{1.5}$$

is self-similar, i.e. it satisfies

$$B(t) = F(t)^{-1} B_0 F(t) = G(t) B_0 G(t)^{-1}, \tag{1.6}$$

where F and G are solutions of

$$\dot{F} = F\rho(f(B(t)),\ F(0) = I$$

and

$$\dot{G} = G\nu(f(B(t)),\ G(0) = I.$$

The key property satisfied by these differential equations is

$$\exp(f(B_0)t) = F(t)G(t). \tag{1.7}$$

Property (1.6) is the continuous analogue of (1.3) while (1.7) is the continuous analogue of (1.4). The interpolating property of self-similar flows is obtained by choosing $f(\cdot) = \log(\cdot)$.

In shifted FG algorithms, the property (1.4) is replaced by a more general equation

$$p_m(B_0)\dots p_2(B_0)p_1(B_0) = F_{(m)}G_{(m)},$$

where the successive polynomials p_i are determined by the shift policy. The continuous analogue is obtained by switching at integer times between different functions $f_i(\cdot) = \log(p_i(\cdot))$. The shift policy is typically a feedback process since the shift is defined at iteration i as a function of the current "state" B_i. As a consequence, a shift policy defines a feedback control strategy for the nonlinear differential equation (1.5), the control variable being the function f. If the control f is switched at integer times based on some switching logic, a hybrid feedback control system is associated to the shifted FG algorithm. This reformulation leads to open questions of the following type:

- (Analysis) What are the limit sets and the convergence and robustness properties of the differential equation (1.5) ?

- (Synthesis) Design a switching controller which guarantees the absence of finite escape time (continuous analogue of the FG algorithm breaking down after a finite number of iterations) and ensures convergence to a desired equilibrium (stabilization problem).

1.3 Motivation and history of the problem

The existence of continuous analogues of matrix algorithms was noticed in early developments of numerical analysis (Rutishauser himself [5] derived a continuous analogue of the quotient-difference algorithm, a predecessor of the QR algorithm). The complete connection between (shifted) FG algorithms and their corresponding self-similar flows is more recent and is due to Watkins and Elsner [6].

The potential importance of such connections was brought to the attention of the control community by Brockett [2, 3] who established the connection between the "double-bracket" flow and several algorithmic problems, including the QR algorithm. This line of research has quickly developed over the last few years and is reported in a recent book by Helmke and Moore [4]. The double-bracket flow reveals the gradient nature of the self-similar flow associated with the QR algorithm (for symmetric matrices with distinct eigenvalues), thereby connecting FG algorithms to other optimization problems defined on Riemannian manifolds.

1.4 Available results and desired extensions

The QR like algorithms always start with a preliminary reduction to a matrix $B_1 = F_1^{-1}B_0F_1 = G_1B_0G_1^{-1}$ in so-called condensed form. For the QR algorithm his e.g. a Hessenberg form for general matrices and a tridiagonal form for symmetric or Hermitian matrices. These condensed forms are very useful since they allow to *estimate* a particular eigenvalue, which is then used as shift. In the case of real matrices one uses typically second order polynomial with two complex conjugate shifts. This procedure works very well for tridiagonal matrices where appropriate shift techniques have been proved to yield global and/or cubic convergence [7]. For Hessenberg matrices the results are already less complete and for the more general class of FG algorithms there are no solid results available : for some shifts, the algorithm may even break down because the factorization (1.1) may not exist.

So far, the connections between matrix algorithms and their continuous-time analogues were always made a posteriori and did not lead to results of direct use for numerical analysts. The situation may be different in the study of shift policies. The geometric study of the double bracket flow [1, 3] indicates that the current developments of modern mechanics may help understanding the dynamics of self-similar flows. In addition, the control of hybrid systems is a quickly expanding research area including the hybrid control of continuous-time systems. For these reasons, the authors have some hope that the study of shift policies from a control theoretic perspective will contribute to the field of numerical analysis.

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During the reviewing process of this manuscript, we were pointed out that Uwe Helmke (Würzburg University) had presented similar ideas at the MTNS conference of 1996 in St. Louis, but these were not reported in conference proceedings. He mentioned the issue of controllability of shifted QR algorithms, which we believe to be an important one.

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