Recursive all pass realizations subject to tangential constraints

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Abstract

Given d complex points λ_i and associated directions (of $\mathbb{C}^n$) z_i, we develop a recursive algorithm for obtaining a $\begin{bmatrix} I_2 & \\ & \Sigma \end{bmatrix}$-unitary realization $\{A, B, C, D\}$ of a Σ-unitary transfer matrix $U(\lambda)$ (and of its inverse) which satisfies $U(\lambda_i)z_i = 0, \quad i = 1, ..., d$. This algorithm is based on Σ-unitary transformations and has $O(nd^2)$ complexity.

Furthermore we introduce a modification to this algorithm that allows to work in real arithmetic in the case of self conjugate conditions.

1 Introduction

In this paper, we derive a recursive algorithm for constructing a $\begin{bmatrix} I_2 & \\ & \Sigma \end{bmatrix}$-unitary realization of a Σ-unitary transfer matrix satisfying tangential conditions.

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This work is motivated by [Fet 70] where this problem occurs in the factorization of transfer matrices. This problem also appears in [DymGoh 95] and [BGR 90] for constructing Schur functions satisfying tangential interpolation conditions. In these papers, such functions are constructed via Cauchy matrices.

An alternative algorithm to produce this construction is developed here. It is based on Σ-unitary transformations, avoids constructing and inverting a Cauchy matrix and has a complexity comparable to that obtained from the Cauchy approach. Finally we discuss a real arithmetic variant to obtain a $\begin{bmatrix} I_2 & \\ & \Sigma \end{bmatrix}$-orthogonal realization of a real Σ-orthogonal matrix subject to self conjugate tangential conditions.

The paper is also closely linked to [GVKDM 83] where the related problem is considered of factorizing a Σ-unitary transfer function into elementary factors.

The symbols $\mathbb{C}$, $\mathbb{D}$, $\partial\mathbb{D}$ and $\mathbb{R}$ denote respectively the complex plane, the open unit disk, the unit circle (alias the boundary of $\mathbb{D}$) and real numbers, as usual. Finally, $e_1, \ldots, e_n$ represent the standard basis in $\mathbb{C}^n$, i.e., e_i is the i-th column of the $n \times n$ identity matrix I_n.

We first analyse the simplest case $\Sigma = I_n$ in the sections 2-5. This case occurs in the construction of scattering matrices of lossless N-port systems [Bel 68] and in Youla's N-port synthesis procedure [YouTis 66]. The extension to $\Sigma = \begin{bmatrix} I_n & \\ & -I_m \end{bmatrix}$ is then considered in the sections 6-9.

2 Elementary all pass transfer matrices

In this section, we present a basic lemma which leads to a parametrization of all pass transfer matrices having d distinct complex zeros. This lemma extends easily to the case of self conjugate conditions, which are considered in the second part of this section. We first recall the following properties [YouTis 66], [GVKDM 83]:

Property 2.1 *A matrix $A(\lambda)$ satisfying $A(\lambda)A(1/\bar{\lambda})^* = I_n$ and having all its poles inside $\mathbb{D}$ is called an all pass transfer matrix. Such matrices always have a minimal realization $\{A, B, C, D\}$ such that*

$$S = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \quad S^*S = I,$$

(i.e. S is unitary). If $A(\lambda)$ has real coefficients, S can be chosen real as well (i.e. S is orthogonal).

Property 2.2 *Let D denote a $n \times (n-k)$ matrix satisfying $D^*D = I_{n-k}$. There there exists a unitary matrix U such that*

$$UD = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix}.$$

We first consider complex all pass transfer matrices of degree 1. The proof of the following lemma is strongly inspired by [Bel 68].

Lemma 2.3 *Let $A(\lambda)$ denote an $n \times n$ all pass transfer matrix of degree 1, with pole α. Then there exist a Householder matrix H and a unitary matrix U such that*

$$UA(\lambda)H = \begin{bmatrix} r(\lambda) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix},$$

where $r(\lambda)$ is an all pass transfer function of degree 1 admitting a unitary realization of the form

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} c & s \\ \hline s & -\bar{c} \end{array}\right], \quad s \in \mathbb{R}.$$

Proof. Let us consider a unitary realization of $A(\lambda)$:

$$S = \left[\begin{array}{c|c} \alpha & b^* \\ \hline c & D \end{array}\right], \quad S^*S = I_n$$

and let us choose a Householder transformation $H = H^*$ such that the last $n-1$ components of Hb are zero:

$$Hb = \gamma e_1, \quad |\gamma| = \|b\|_2.$$

Thus we have found a Householder matrix H such that $A(\lambda)H$ admits a unitary realization of the form

$$S\left[\begin{array}{c|c} 1 & \\ \hline & H \end{array}\right] = \left[\begin{array}{c|cccc} \alpha & \gamma & 0 & \cdots & 0 \\ \hline c & d_1 & & \hat{D} & \end{array}\right]$$

The matrix $\hat{D}$ satisfies $\hat{D}^*\hat{D} = I_{n-1}$ and hence there exists by property 2.2 a unitary matrix U which satisfies

$$U\hat{D} = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}.$$

Therefore, a unitary realization for $UA(\lambda)H$ is

$$\left[\begin{array}{c|c} 1 & \\ \hline & U \end{array}\right] S \left[\begin{array}{c|c} 1 & \\ \hline & H \end{array}\right] = \left[\begin{array}{c|cccc} \alpha & \gamma & 0 & \cdots & 0 \\ \hline \gamma e^{j\theta} & -\bar{\alpha}e^{j\theta} & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & 1 \end{array}\right].$$

Finally, the phase $e^{j\theta}$ can be incorporated in the unitary matrix U. This completes the proof since the unitarity of the realization allows us to interpret α and γ as complex cosine and sine. □

Remark 2.4 *Notice that if we start with a real all pass matrix $A(\lambda)$ then S, H and U will be real as well.*

We now extend this lemma to real all pass transfer matrices of degree 2 with complex conjugate poles.

Definition 2.5 *An $n \times n$ matrix A is called sign-symmetric if*

$$\Sigma A \Sigma = A^T,$$

where Σ denotes a diagonal matrix with ± 1 elements. For 2×2 matrices, we use $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ unless otherwise specified.

Lemma 2.6 *Let $B(\lambda)$ denote an $n \times n$ real all pass transfer matrix of degree 2, with complex conjugate poles. Then there exist two real Householder matrices H_1, H_2 and an orthogonal matrix U such that*

$$UB(\lambda)H_1H_2 = \begin{bmatrix} \hat{B}(\lambda) & 0 \\ 0 & I_{n-2} \end{bmatrix},$$

where $\hat{B}(\lambda)$ is a 2×2 real sign-symmetric all pass transfer matrix of degree 2 with complex conjugate poles admitting an orthogonal sign-symmetric realization of the form

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} \alpha+\delta & \beta & \gamma & 0 \\ -\beta & \alpha-\delta & \epsilon & \zeta \\ \hline -\gamma & \epsilon & \eta+\delta & \theta \\ 0 & -\zeta & -\theta & \eta-\delta \end{array}\right].$$

Proof. Let us consider an orthogonal realization of $B(\lambda)$ and an orthogonal state space transformation G_1

$$\left[\begin{array}{c|c} G_1 & \\ \hline & I_n \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} G_1^T & \\ \hline & I_n \end{array}\right] = \left[\begin{array}{c|c} G_1 A G_1^T & G_1 B \\ \hline C G_1^T & D \end{array}\right]$$

such that $G_1 A G_1^T$ is sign-symmetric. It is always possible to construct such a G_1. We denote respectively by b_1^T and b_2^T the first and second rows of $G_1 B$. Let us take a real Householder transformation $H_1 = H_1^T$ such that the last $n-1$ components of $H_1 b_1$ are zero. Thereafter, we choose a real Householder matrix $H_2 = H_2^T$ of the form

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & \hat{H}_2 \end{array}\right]$$

such that the last $n-2$ components of $H_2 H_1 b_2$ are zero. Thus we have found two real Householder matrices so that $B(\lambda) H_1 H_2$ admits an orthogonal realization of the form

$$\left[\begin{array}{c|c} G_1 & \\ \hline & I_n \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} G_1^T & \\ \hline & H_1 H_2 \end{array}\right] = \left[\begin{array}{cc|ccccc} \alpha+\delta & \beta & \gamma & 0 & 0 & \cdots & 0 \\ -\beta & \alpha-\delta & \epsilon & \zeta & 0 & \cdots & 0 \\ \hline \multicolumn{2}{c|}{\hat{C}} & d_1 & d_2 & & \hat{D} & \end{array}\right].$$

The matrix $\hat{D}$ satisfies $\hat{D}^T \hat{D} = I_{n-2}$ and hence there exists by property 2.2 an orthogonal matrix U which satisfies

$$U\hat{D} = \left[\begin{array}{ccc} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{array}\right].$$

Therefore, an orthogonal realization for $UB(\lambda)H_1H_2$ is

$$\left[\begin{array}{c|c} G_1 & \\ \hline & U \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c|c} G_1^T & \\ \hline & H_1H_2 \end{array}\right] = \left[\begin{array}{cc|cccccc} \alpha+\delta & \beta & \gamma & 0 & 0 & \cdots & 0 \\ -\beta & \alpha-\delta & \epsilon & \zeta & 0 & \cdots & 0 \\ \hline & & & & 0 & \cdots & 0 \\ \multicolumn{2}{c|}{\tilde{C}} & \multicolumn{2}{c}{\tilde{D}} & & & \\ & & & & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & 0 & & & 1 \end{array}\right].$$

Then we can choose an orthogonal 2×2 matrix $\hat{U}$ (which can be integrated in the first orthogonal matrix) such that, by orthogonality of the realization,

$$\hat{U}\tilde{C} = \begin{bmatrix} -\gamma & \epsilon \\ 0 & -\zeta \end{bmatrix}.$$

Finally, the structure of $\hat{U}\tilde{D}$ results from the orthogonality of the realization. □

3 A minimal all pass parametrization

The lemmas developed in the previous section allow us to obtain a parametrization of all pass transfer matrices having d distinct complex zeros as well as real all pass transfer matrices with self conjugate conditions. These parametrizations are inspired from [Bel 68]. Let us first consider the complex case

Theorem 3.1 *An $n \times n$ all pass transfer matrix $A(\lambda)$ with d distinct complex zeros (of degree 1) λ_i can be decomposed as*

$$A(\lambda) = U_d A_d(\lambda) H_d A_{d-1}(\lambda) H_{d-1} \cdots A_1(\lambda) H_1, \qquad (1)$$

where

- *U_d is an $n \times n$ unitary matrix;*
- *H_i $(i = 1, \dots, d)$ are $n \times n$ Householder matrices;*

- *$A_i(\lambda)$ are $n \times n$ all pass transfer matrices (of degree 1) of the form*

$$A_i(\lambda) = \begin{bmatrix} r_i(\lambda) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \tag{2}$$

where $r_i(\lambda)$ are all pass transfer functions of degree 1 admitting unitary realizations of the form

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} c_i & s_i \\ \hline s_i & -\bar{c}_i \end{array}\right], \quad s_i \in \mathbb{R}. \tag{3}$$

Proof. It is well known that an $n \times n$ all pass transfer matrix of degree d (with d distinct complex zeros) can be decomposed as the product of d $n \times n$ all pass transfer matrices of degree 1. Such matrices can be written, from lemma 2.3, as

$$\hat{A}_i(\lambda) = \hat{U}_i \begin{bmatrix} r_i(\lambda) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \hat{H}_i, \quad i = 1, ..., d,$$

where $\hat{U}_i$ are $n \times n$ unitary matrices, H_i denote $n \times n$ Householder matrices and $r_i(\lambda)$ are characterized as earlier.

The theorem will be proved by induction. The case $d = 1$ is trivial. Let us assume the theorem proved for $d = 1, ..., k$ and let us prove it for $d = k+1$. It follows by induction that we can write

$$A(\lambda) = (\hat{U}_{k+1}\hat{A}_{k+1}(\lambda)\hat{H}_{k+1})(U_k A_k(\lambda) H_k \cdots A_1(\lambda) H_1).$$

Since $\hat{U}_{k+1}\hat{A}_{k+1}(\lambda)\hat{H}_{k+1}U_k$ is an all pass transfer matrix of degree 1, we finally obtain from lemma 2.3 that $\hat{U}_{k+1}\hat{A}_{k+1}(\lambda)\hat{H}_{k+1}U_k = U_{k+1}A_{k+1}(\lambda)H_{k+1}$ where U_{k+1} is an $n \times n$ unitary matrix and H_{k+1} represents an $n \times n$ Householder matrix. This completes the proof. □

It must be stressed that this parametrization is minimal. Indeed, it follows from [Bel 68, ch. 8] that the number of real parameters which characterize a $n \times n$ all pass transfer matrix of degree d is $2dn + n^2$. This value is equal to the number of real parameters of the above parametrization:

- $2n$ real parameters for each pole: $2n-2$ parameters for H_i and 2 parameters for $A_i(\lambda)$.
- n^2 real parameters for U_d.

Moreover, this parametrization allows us, as we will see in the next section, a great flexibility in the construction of such all pass transfer matrices.

This result extends very easily to real all pass transfer matrices $B(\lambda)$ with self conjugate poles. The proof of the following theorem is similar to the previous one.

Theorem 3.2 *An $n \times n$ real all pass transfer matrix $B(\lambda)$ with d_1 real poles and d_2 pairs of complex conjugate poles can be decomposed as*

$$B(\lambda) = U_{d_1+d_2} B_{d_1+d_2}(\lambda) H_{d_1+d_2,2} H_{d_1+d_2,1} \cdots B_{d_1+1}(\lambda) H_{d_1+1,2} H_{d_1+1,1} \\ B_{d_1}(\lambda) H_{d_1} \cdots B_1(\lambda) H_1, \quad (4)$$

where

- *$U_{d_1+d_2}$ is an $n \times n$ orthogonal matrix;*
- *$H_{i,1}$, $H_{i,2}$ $(i = d_1+1, ..., d_1+d_2)$ and H_j $(j = 1, ..., d_1)$ are $n \times n$ real Householder matrices;*
- *$B_i(\lambda)$ $(i = 1, ..., d_1)$ are $n \times n$ real all pass transfer matrices of degree 1 (corresponding to real poles $1/\lambda_i$) of the form (2) (with $c_i \in \mathbb{R}$);*
- *$B_i(\lambda)$ $(i = d_1+1, ..., d_1+d_2)$ are $n \times n$ real all pass transfer matrices of degree 2 (corresponding to conjugate poles) of the form*

$$B_i(\lambda) = \begin{bmatrix} \hat{B}_i(\lambda) & 0 \\ 0 & I_{n-2} \end{bmatrix}, \quad (5)$$

where $\hat{B}_i(\lambda)$ are 2×2 real sign-symmetric all pass transfer matrices of degree 2 with complex conjugate poles admitting orthogonal sign-symmetric realizations of the form

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} \alpha+\delta & \beta & \gamma & 0 \\ -\beta & \alpha-\delta & \epsilon & \zeta \\ \hline -\gamma & \epsilon & \eta+\delta & \theta \\ 0 & -\zeta & -\theta & \eta-\delta \end{array}\right]. \quad (6)$$

Once again, it is important to observe that this parametrization is minimal. Indeed, the number of real parameters of the above parametrization is:

- n real parameters for each real pole: $n-1$ parameters for H_i and another one for $B_i(\lambda)$;
- $2n$ real parameters for each pair of conjugate poles: $n-1$ parameters for $H_{i,1}$, $n-2$ parameters for $H_{i,2}$ and finally 3 parameters for $B_i(\lambda)$ which is completely determined by α, β and δ.
- $\frac{n(n-1)}{2}$ real parameters for $U_{d_1+d_2}$ (we can choose n signs as well).

Notice that the total number of real parameters corresponds exactly to that obtained from [Bel 68, ch. 8].

Remark 3.3 *The realization (6) is unique up to a state space transformation* Σ*.*

Remark 3.4 *When* $n=1$*, we do not need the Householder matrices any more and the 3 parameters* α*,* β *and* δ *are related by* $2\delta-\delta^2=1-\alpha^2-\beta^2$*. The number of real parameters is then equal to 1 for each real pole, 2 for each pair of conjugate poles and 1 sign can also be chosen for* $U_{d_1+d_2}$*. Therefore, the previous number of parameters is still correct for* $n=1$*.*

Moreover, the major advantage of such a parametrization is the great flexibility in constructing all pass transfer matrices. This flexibility will be employed in section 5 to develop a recursive algorithm for obtaining an all pass transfer matrix with self conjugate tangential conditions.

4 Tangential conditions and interpolation

In this section, we present a recursive algorithm using unitary transformations to obtain a unitary realization $\{A, B, C, D\}$ of an all pass transfer matrix which satisfies the following tangential conditions:

$$U(\lambda_i)z_i = 0, \quad i = 1, ..., d,$$

where the d distinct complex points λ_i and associated directions $z_i \in \mathbb{C}^n$ are given. This algorithm, based on the parametrization (1), leads also to a fast construction of a realization of $U(\lambda)$ as well as its inverse. The complexity of this algorithm is analyzed in the second part of this section.

The algorithm uses the following recursion: let us suppose that we know an all pass transfer matrix $A(\lambda) = A_k(\lambda)H_k \cdots A_1(\lambda)H_1$ which satisfies $A(\lambda_i)z_i = 0$ for the first k zeros ($A(\lambda) = I$ if $k = 0$) and where (like in theorem 3.1) $A_i(\lambda)$ are $n \times n$ all pass transfer matrices of the form (2) and H_i are $n \times n$ Householder matrices.

Then an all pass transfer matrix $\hat{A}(\lambda)$ satisfying $\hat{A}(\lambda_i)z_i = 0$ for $i = 1, ..., k+1$ can be obtained by the following procedure:

- evaluate $\hat{z}_{k+1} := A(\lambda_{k+1})z_{k+1}$;
- choose H_{k+1} such that the last $n-1$ components of $H_{k+1}\hat{z}_{k+1}$ are zero.

The new matrix $\hat{A}(\lambda) := A_{k+1}(\lambda)H_{k+1}A(\lambda)$ is then completely determined ($A_{k+1}(\lambda)$ is only characterized by λ_{k+1}) and verifies the asserted conditions.

It must be stressed that each all pass transfer matrix $A_i(\lambda)$ of degree 1 admits by lemma 2.3 a unitary realization of the form

$$\left[\begin{array}{c|cccc} c_i & s_i & 0 & \cdots & 0 \\ \hline s_i & -\bar{c}_i & & & \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \end{array}\right]. \tag{7}$$

Furthermore, if $\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ and $\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$ denote realizations of $R_1(\lambda)$ and $R_2(\lambda)$, then a realization of the product $R_1(\lambda)R_2(\lambda)$ is obtained by

$$\left[\begin{array}{cc|c} A_1 & & B_1 \\ & I & \\ \hline C_1 & & D_1 \end{array}\right] \left[\begin{array}{cc|c} I & & \\ & A_2 & B_2 \\ \hline & C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right].$$

Hence, the premultiplication by $A_i(\lambda)$ corresponds, in terms of realization, to the application of a Givens matrix to two rows of $\left[\begin{array}{cc|c} I & & \\ & A & B \\ \hline & C & D \end{array}\right]$, where $\{A, B, C, D\}$ is a realization of $H_iA_{i-1}(\lambda)\cdots A_1(\lambda)H_1$.

The algorithm resulting from this is the following one:

Start with the data

$$A = B = C = \emptyset; \quad D = I_n$$

and repeat the following steps for $i = 1, ..., d$:

Step 1 Determine the normalized vectors w_i associated with Householder matrices $H_i = I_n - 2w_i w_i^*$. This is done by evaluating the vector $\hat{z}_i$;

Step 2 (corresponding to the premultiplication by H_i)
Update C and D: $C \longleftarrow H_i C$; $D \longleftarrow H_i D$;

Step 3 (corresponding to the premultiplication by $A_i(\lambda)$)
Apply Givens rotation to $\left[\begin{array}{cc|c} I & & \\ & A & B \\ \hline & C & D \end{array}\right]$.

This algorithm requires essentially $4n^2d + 4nd^2$ (+ lower order terms) flops which subdivide into

- $2nd^2$ flops (+ l.o.t.) for step 1;
- $4n^2d + 2nd^2$ flops (+ l.o.t.) for steps 2 and 3.

This complexity is comparable to that obtained from the construction and inversion of a Cauchy matrix (cfr [BKO 95]).

It is important to note that the matrix A of the realization is upper triangular so that we have obtained a Schur-type realization. Furthermore, we can easily obtain a realization of $(A_d(\lambda)H_d \cdots A_1(\lambda)H_1)^{-1} = H_1 A_1(\lambda)^{-1} \cdots H_d A_d(\lambda)^{-1}$. Indeed, such a realization is given by similar steps as 2 and 3 but where the realization of $A_i(\lambda)^{-1}$ is now given by

$$\left[\begin{array}{c|cccc} 1/\bar{c}_i & s_i/\bar{c}_i & 0 & \cdots & 0 \\ \hline s_i/\bar{c}_i & -1/\bar{c}_i & & & \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \end{array}\right]. \quad (8)$$

Therefore, the construction of a realization of the inverse of the all pass transfer matrix only requires $4n^2d + 2nd^2$ extra flops.

Remark 4.1 *The number of real parameters of the solution provided by this algorithm, 2nd, is minimal since it is equal to the number of real parameters in the tangential constraints (if we assume a normalization $\|z_i\|_2 = 1$). Of course, this does not give all solutions of minimal degree. The degrees of freedom are precisely those of the leftover factor U_d which is not considered by the algorithm.*

5 Self conjugate tangential conditions

In this section, we present a recursive algorithm using orthogonal transformations to obtain an orthogonal realization $\{A, B, C, D\}$ of an all pass transfer matrix which satisfies the following self conjugate conditions (SCC)

$$(SCC)\left\{\begin{array}{lll} U(\lambda_i)z_i = 0 & \lambda_i \in \mathbb{R}, z_i \in \mathbb{R}^n, & i = 1, ...d_1; \\ \left\{\begin{array}{l} U(\lambda_i)z_i = 0 \\ U(\bar{\lambda}_i)\bar{z}_i = 0 \end{array}\right. & \lambda_i \in \mathbb{C} \setminus \mathbb{R}, z_i \in \mathbb{C}^n, & i = d_1 + 1, ..., d_1 + d_2, \end{array}\right.$$

where the $d_1 + 2d_2$ distinct poles and associated directions are given. This algorithm, based on the parametrization (4), leads to a fast construction of a realization of $U(\lambda)$ as well as its inverse. The complexity of this algorithm is analyzed in the second part of this section.

The algorithm uses the same recursion as in the previous section for the d_1 real poles λ_i but with real (instead of complex) Householder matrices and real orthogonal (instead of complex unitary) realizations of all pass transfer matrices of degree 1. We denote by $\hat{A}(\lambda)$ the real all pass transfer matrix obtained by imposing the d_1 real conditions in (SCC). Now we examine the conjugate poles. Let us assume that we know an all pass transfer matrix $B(\lambda) = B_{d_1+k}(\lambda)H_{d_1+k,2}H_{d_1+k,1} \cdots B_{d_1+1}(\lambda)H_{d_1+1,2}H_{d_1+1,1}\hat{A}(\lambda)$ (where $B(\lambda) = \hat{A}(\lambda)$ if $k = 0$) which satisfies the first $d_1 + k$ conditions (SCC) and where

- $H_{i,1}$ and $H_{i,2}$ are $n \times n$ real Householder matrices;
- $B_i(\lambda)$ denotes an $n \times n$ real all pass transfer matrix of degree 2 of the form
$$B_i(\lambda) = \begin{bmatrix} \hat{B}_i(\lambda) & 0 \\ 0 & I_{n-2} \end{bmatrix}$$
($\hat{B}_i(\lambda)$ satisfies the same conditions as in the theorem 3.2).

Then an all pass transfer matrix $\hat{B}(\lambda)$ satisfying the first d_1+k+1 conditions (SCC) can be obtained by the following procedure which essentially uses real arithmetic.

1. All complex n-vectors are decomposed into two real n-vectors: their real and complex parts. The evaluation of $B(\lambda_{d_1+k+1})z_{d_1+k+1}$ only requires premultiplications by $n \times n$ real Householder matrices and by $n \times n$ all pass matrices of degree 2 of the form (5) or of degree 1 of the form (2). The real Householder matrices can be applied separately to the real and complex parts of the vector in progress. On the other hand, the upper left blocks of the all pass matrices are first computed by complex arithmetic and then applied in real arithmetic to the first one/two row(s) of the obtained vector. We denote by $\hat{z}_{d_1+k+1}$ the normalized vector corresponding to $B(\lambda_{d_1+k+1})z_{d_1+k+1}$:

$$\hat{z}_{d_1+k+1} = u_{d_1+k+1} + j\ v_{d_1+k+1}.$$

2. We know by theorem 3.2 that there exist two real Householder matrices $H_{d_1+k+1,1}$ and $H_{d_1+k+1,2}$ and an $n \times n$ real all pass transfer matrix $B_{d_1+k+1}(\lambda)$ of the form (5) such that, for each phase factor $e^{j\phi}$, we have

$$B_{d_1+k+1}(\lambda_{d_1+k+1})H_{d_1+k+1,2}H_{d_1+k+1,1}\hat{z}_{d_1+k+1}e^{j\phi} = 0. \tag{9}$$

   Let us write the normalized vector of the nullspace of $B_{d_1+k+1}(\lambda_{d_1+k+1})$ as a function of the parameter δ:

$$B_{d_1+k+1}(\lambda_{d_1+k+1})\left(\begin{bmatrix} a(\delta) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + j\begin{bmatrix} b(\delta) \\ c(\delta) \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = 0$$

   where we can choose the phase such that the second component is purely imaginary. Indeed, once the parameter δ fixed, the other parameters can be deduced from the orthogonality of the realization and the location of the poles. We deduce from (9) that there exist two real Householder matrices and a 2×2 Givens matrix G corresponding to the phase factor $e^{j\phi}$ such that

$$\begin{bmatrix} a(\delta) & b(\delta) \\ 0 & c(\delta) \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} = H_{d_1+k+1,2} H_{d_1+k+1,1} \begin{bmatrix} u_{d_1+k+1} & v_{d_1+k+1} \end{bmatrix} G,$$

so that

$$\begin{bmatrix} a(\delta)^2 & a(\delta)b(\delta) \\ a(\delta)b(\delta) & b(\delta)^2 + c(\delta)^2 \end{bmatrix} = G^T \begin{bmatrix} \|u_{d_1+k+1}\|_2^2 & u_{d_1+k+1}^T v_{d_1+k+1} \\ u_{d_1+k+1}^T v_{d_1+k+1} & \|v_{d_1+k+1}\|_2^2 \end{bmatrix} G. \tag{10}$$

Finally, by inspection of the determinants of these two terms, we find that

$$a(\delta)^2 c(\delta)^2 = \|u_{d_1+k+1}\|_2^2 \|v_{d_1+k+1}\|_2^2 - \left(u_{d_1+k+1}^T v_{d_1+k+1}\right)^2. \tag{11}$$

Let us respectively denote the real and complex parts of the pole $1/\lambda_{d_1+k+1}$ by α_0 and β_0. We obtain from (11), the location of the poles and the orthogonality of the realization the following formulas for the parameters of the realization of $B_{d_1+k+1}(\lambda)$:

$$\Delta = \|u_{d_1+k+1}\|_2^2 \|v_{d_1+k+1}\|_2^2 - \left(u_{d_1+k+1}^T v_{d_1+k+1}\right)^2; \tag{12}$$

$$\alpha = \alpha_0; \tag{13}$$

$$r = 1 - \alpha_0^2 - \beta_0^2; \tag{14}$$

$$\delta = \frac{\beta_0 r \sqrt{1-4\Delta}}{2\sqrt{\beta_0^2 + r^2\Delta}}; \tag{15}$$

$$\beta = \sqrt{\beta_0^2 + \delta^2}; \tag{16}$$

$$\gamma = \sqrt{1 - (\alpha+\delta)^2 - \beta^2}; \tag{17}$$

$$\epsilon = \frac{2\beta\delta}{\gamma}; \tag{18}$$

$$\zeta = \sqrt{1 - (\alpha-\delta)^2 - \beta^2 - \epsilon^2}; \tag{19}$$

$$\eta = \alpha - \frac{\beta\epsilon}{\gamma}; \tag{20}$$

$$\theta = -\frac{\beta\zeta}{\gamma}. \tag{21}$$

It must be stressed that these parameters can always be determined since the Cholesky factorization of $I_2 - AA^T$ is guaranteed by choice of δ. All these parameters depend only on Δ, α_0 and β_0.

3. Once δ known, the complex factor $e^{j\phi}$ can easily be obtained from (10). Moreover, it can be integrated in the construction of the two real Householder matrices $H_{d_1+k+1,2}$ and $H_{d_1+k+1,1}$ which are applied to the real and complex parts of the vector in progress. Therefore we have

$$H_{d_1+k+1,2} H_{d_1+k+1,1} \begin{bmatrix} u_{d_1+k+1} & v_{d_1+k+1} \end{bmatrix} G = \begin{bmatrix} x & y \\ 0 & z \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} =: \begin{bmatrix} X & Y \end{bmatrix}.$$

This is merely the QR factorization of the matrix $\begin{bmatrix} u_{d_1+k+1} & v_{d_1+k+1} \end{bmatrix} G$, where xz is chosen (by $H_{d_1+k+1,2}$ and $H_{d_1+k+1,1}$) to have the same sign as $a(\delta)c(\delta)$.

Finally, the equation

$$\begin{bmatrix} a(\delta)^2 & a(\delta)b(\delta) \\ a(\delta)b(\delta) & b(\delta)^2 + c(\delta)^2 \end{bmatrix} = \begin{bmatrix} x^2 & xy \\ xy & y^2 + z^2 \end{bmatrix}$$

which results from (10) implies that $\begin{bmatrix} X & Y \end{bmatrix}$ belongs to the nullspace of $B_{d_1+k+1}(\lambda_{d_1+k+1})$.

The new real matrix $\hat{B}(\lambda) := B_{d_1+k+1}(\lambda) H_{d_1+k+1,2} H_{d_1+k+1,1} B(\lambda)$ is then completely determined. Moreover, it verifies $\hat{B}(\lambda_{d_1+k+1}) z_{d_1+k+1} = 0$ and automatically the conjugate condition too.

This algorithm requires essentially $4n^2 d + 4nd^2$ (+ lower order terms) **real** flops where $d = d_1 + 2d_2$ denotes the number of tangential conditions. Hence this second algorithm is faster (up to a factor 3) than the previous one which could also be applied to satisfy the self conjugate conditions.

Remark 5.1 *The same remark as remark 4.1 applies here as well. The number of real parameters of the solution constructed by this algorithm, $n(d_1 + 2d_2)$, is minimal since it is equal to the number of real parameters occuring* *in the tangential constraints. Here again the left factor $U_{d_1+d_2}$ represents the degrees of freedom of any minimal degree solution and is not constructed by the algorithm.*

Furthermore, we can easily obtain a realization of the inverse of the all pass transfer matrix. Indeed, such a realization is given by similar multiplications but where the realization of $B_i(\lambda)^{-1}$, $i = 1, ..., d_1$ is now given by

$$\left[\begin{array}{c|cccc} 1/c_i & s_i/c_i & 0 & \cdots & 0 \\ \hline s_i/c_i & -1/c_i & & & \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \end{array}\right]$$

and that of $B_j(\lambda)^{-1}$, $j = d_1 + 1, ..., d_1 + d_2$ by

$$\left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]$$

if

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

represents the realization of $B_j(\lambda)$.

Finally it must be observed that the realization of the inverse is still sign-symmetric and that its construction requires only $4n^2d + 2nd^2$ extra **real** flops.

The next 4 sections are devoted to the more general case $\Sigma = \begin{bmatrix} I_n & \\ & -I_m \end{bmatrix}$. We present in section 6 a basic lemma which leads to a parametrization of Σ-unitary transfer matrices having d distinct complex zeros. This lemma also extends to the case of self conjugate conditions. Thereafter we obtain in section 7 a parametrization of Σ-unitary transfer matrices having d distinct complex zeros as well as real Σ-orthogonal transfer matrices with self conjugate conditions. Finally we develop in section 8 recursive algorithms for such Σ-unitary transfer matrices.

6 Elementary Σ-unitary transfer matrices

In this section, we present a basic lemma which leads to a parametrization of Σ-unitary transfer matrices having d distinct complex zeros. This lemma extends easily to the case of self conjugate conditions which are considered in the second part of this section. We first recall the following properties [GVKDM 83]:

Property 6.1 *A matrix $A(\lambda)$ satisfying $A(\lambda)\Sigma A(1/\bar{\lambda})^* = \Sigma$ and having all its poles inside $\mathbb{D}$ is called a Σ-unitary transfer matrix. Such matrices always have a minimal realization $\{A, B, C, D\}$ such that*

$$S = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \quad S^* \begin{bmatrix} I & \\ & \Sigma \end{bmatrix} S = \begin{bmatrix} I & \\ & \Sigma \end{bmatrix},$$

(i.e. S is $(I_2 \oplus \Sigma)$-unitary). If $A(\lambda)$ has real coefficients, S can be chosen real as well (i.e. S is $(I_2 \oplus \Sigma)$-orthogonal).

Property 6.2 *Let D denote an $(n+m) \times (n+m-k)$ matrix satisfying $D^*\Sigma D = \Sigma'$. There there exists a Σ-unitary matrix U such that*

$$UD = \begin{bmatrix} 0 \\ \Sigma' \end{bmatrix}.$$

We first consider complex Σ-unitary transfer matrices of degree 1. The proof of the following lemma is strongly inspired from [Bel 68].

Lemma 6.3 *Let $A(\lambda)$ denote an $(n+m)\times(n+m)$ Σ-unitary transfer matrix of degree 1, with pole α. Then there exist a skew-Householder matrix H and a Σ-unitary matrix U such that*

$$UA(\lambda)H = \begin{bmatrix} r(\lambda) & & \\ & I_{n-1} & \\ & & -I_m \end{bmatrix},$$

where $r(\lambda)$ is an all pass transfer function of degree 1 admitting a unitary realization of the form

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} c & s \\ \hline s & -\bar{c} \end{array}\right], \quad s \in \mathbb{R}.$$

Proof. Let us consider a $\begin{bmatrix} 1 & \\ & \Sigma \end{bmatrix}$-unitary realization of $A(\lambda)$:

$$\left[\begin{array}{c|c} \alpha & b^* \\ \hline c & D \end{array}\right], \quad S^*\Sigma S = \Sigma.$$

We deduce from the $\begin{bmatrix} 1 & \\ & \Sigma \end{bmatrix}$-unitary of the realization that

$$b^*\Sigma b = 1 - \alpha\bar{\alpha} > 0.$$

Hence we can choose a skew-Householder transformation $H = H^*$ such that the last $n + m - 1$ components of Hb are zero:

$$Hb = \gamma e_1, \quad |\gamma| = b^*\Sigma b.$$

Thus we have found a skew-Householder matrix H such that $A(\lambda)H$ admits a $\begin{bmatrix} 1 & \\ & \Sigma \end{bmatrix}$-unitary realization of the form

$$S\left[\begin{array}{c|c} 1 & \\ \hline & H \end{array}\right] = \left[\begin{array}{c|cccc} \alpha & \gamma & 0 & \cdots & 0 \\ \hline c & d_1 & & \hat{D} & \end{array}\right].$$

The matrix $\hat{D}$ satisfies $\hat{D}^*\Sigma\hat{D} = \begin{bmatrix} I_{n-1} & \\ & -I_m \end{bmatrix}$ and hence there exists by property 6.2 a Σ-unitary matrix U such that

$$U\hat{D} = \begin{bmatrix} 0 & 0 \\ I_{n-1} & 0 \\ 0 & -I_m \end{bmatrix}.$$

Therefore, a unitary realization for $UA(\lambda)H$ is

$$\left[\begin{array}{c|c} 1 & \\ \hline & U \end{array}\right] S \left[\begin{array}{c|c} 1 & \\ \hline & H \end{array}\right] = \left[\begin{array}{c|ccc} \alpha & \gamma & 0 & 0 \\ \hline \gamma e^{j\theta} & -\bar{\alpha}e^{j\theta} & 0 & 0 \\ 0 & 0 & I_{n-1} & 0 \\ 0 & 0 & 0 & -I_m \end{array}\right].$$

Finally, the phase $e^{j\theta}$ can be incorporated in the Σ-unitary matrix U. This completes the proof since the $\begin{bmatrix} 1 & \\ & \Sigma \end{bmatrix}$-unitarity of the realization allows us to interpret α and γ as complex sine and cosine. □

Remark 6.4 *Notice that if we start with a real Σ-orthogonal matrix $A(\lambda)$ then S, H and U will be real as well.*

We now extend this lemma to real Σ-orthogonal transfer matrices of degree 2 with complex conjugate poles.

Lemma 6.5 *Let $B(\lambda)$ denote an $(n+m)\times(n+m)$ real Σ-orthogonal transfer matrix of degree 2, with complex conjugate poles. Then there exist two real skew-Householder matrices H_1, H_2 and a Σ-orthogonal matrix U such that $UB(\lambda)H_1H_2$ has one of the following forms*

$UB(\lambda)H_1H_2$		*realization of* $\hat{B}(\lambda) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$	*symmetry of realization*	*signature of* $I_2 - AA^T$
$\begin{bmatrix} I_{n-2} & & \\ & \hat{B}(\lambda) & \\ & & -I_m \end{bmatrix}$	(22)	$\left[\begin{array}{cc\|cc} \alpha+\delta & \beta & 0 & \gamma \\ -\beta & \alpha-\delta & \zeta & \epsilon \\ \hline 0 & \zeta & \eta+\delta & \theta \\ \gamma & -\epsilon & -\theta & \eta-\delta \end{array}\right]$	I_4-*orthogonal*	*(+,+)*
$\begin{bmatrix} I_{n-1} & & \\ & \hat{B}(\lambda) & \\ & & -I_{m-1} \end{bmatrix}$	(23)	$\left[\begin{array}{cc\|cc} \alpha+\delta & \beta & \gamma & \nu \\ -\beta & \alpha-\delta & \epsilon & \zeta \\ \hline -\gamma & \epsilon & \eta+\delta & \theta \\ \nu & -\zeta & -\theta & \eta-\delta \end{array}\right]$	$I_3 \oplus (-I_1)$-*orthogonal* $\nu\gamma = 0$	*(+,-)*
$\begin{bmatrix} I_{n-2} & & \\ & \hat{B}(\lambda) & \\ & & -I_{m-1} \end{bmatrix}$	(24)	$\left[\begin{array}{cc\|ccc} \alpha+\delta & \beta & 0 & \gamma & 0 \\ -\beta & \alpha-\delta & \nu & \epsilon & \nu \\ \hline 0 & \nu & \eta & \theta & \kappa \\ -\gamma & \epsilon & \theta & \tau & \theta \\ 0 & -\nu & -\kappa & -\theta & \psi \end{array}\right]$	$I_4 \oplus (-I_1)$-*orthogonal* $\nu \in \{0,1\}$	*(+,0)*
$\begin{bmatrix} I_{n-2} & & \\ & \hat{B}(\lambda) & \\ & & -I_{m-1} \end{bmatrix}$	(25)	$\left[\begin{array}{cc\|ccc} \delta & \beta & 0 & 1 & 1 \\ -\beta & -\delta & \zeta & \gamma & \epsilon \\ \hline 0 & \zeta & \eta & \theta & \kappa \\ -1 & \gamma & \theta & \tau & \xi \\ 1 & -\epsilon & -\kappa & -\xi & \psi \end{array}\right]$	$I_4 \oplus (-I_1)$-*orthogonal*	*(+,-)*

Proof. Let us consider a $\begin{bmatrix} I_2 & \\ & \Sigma \end{bmatrix}$-orthogonal realization of $B(\lambda)$ and an orthogonal state space transformation G_1

$$\left[\begin{array}{c|c} G_1 & \\ \hline & I_{n+m} \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} G_1^T & \\ \hline & I_{n+m} \end{array}\right] = \left[\begin{array}{c|c} G_1AG_1^T & G_1B \\ \hline CG_1^T & D \end{array}\right]$$

such that $G_1AG_1^T$ is sign-symmetric. There are essentially two solutions for such a state space transformation which are permutations of each other: G_1 and $G_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. In addition a diagonal similarity scaling Σ can be applied as well. This freedom is sometimes exploited below. We denote respectively by $\hat{B}$ and $\hat{C}$ the new matrices $G_1 B$ and $C G_1^T$.

Let us consider the eigenvalues of $\hat{B}\Sigma\hat{B}^T = I_2 - AA^T$. These eigenvalues are equal to $1 - \sigma_i(A)^2$, $i = 1, 2$, where $\sigma_i(A)$ denote the singular values of A. Moreover, at least one of these singular values is strictly less than 1 because A has stable eigenvalues and one always has $\sigma_{\min}(A) \leq |\lambda_i(A)| \leq \sigma_{\max}(A)$. Therefore, at least one eigenvalue of $\hat{B}\Sigma\hat{B}^T$ is strictly positive.

We denote respectively by b_1^T and b_2^T the first and second rows of $\hat{B}$. We consider three cases:

- $\boxed{b_1^T\Sigma b_1 > 0 \text{ or } b_2^T\Sigma b_2 > 0}$

  We may suppose without loss of generality that $b_1^T\Sigma b_1 > 0$ (otherwise we permute b_1 and b_2). So we can find a real skew-Householder transformation $H_1 = H_1^T$ such that only the n-th component of $H_1 b_1$ is not equal to zero. Let us denote by $\tilde{b}_2$ the vector formed by the first $n-1$ and the last m components of $H_1 b_2$. The choice of the second real skew-Householder matrix

  $$H_2 = \begin{bmatrix} H_{11} & & H_{12} \\ & 1 & \\ H_{21} & & H_{22} \end{bmatrix}$$

  depends on the sign of $\tilde{b}_2^T\Sigma\tilde{b}_2$.

  1. $\tilde{b}_2^T\Sigma\tilde{b}_2 > 0$

     We then choose a real skew-Householder matrix $H_2 = H_2^T$ such that the first $n-2$ and the last m components of $H_2 H_1 b_2$ are zero. Thus we have found two real skew-Householder matrices so that $B(\lambda)H_1H_2$ admits a $\begin{bmatrix} I_2 & \\ & \Sigma \end{bmatrix}$-orthogonal realization of the form

$$\left[\begin{array}{c|c} G_1 & \\ \hline & I_{n+m} \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} G_1^T & \\ \hline & H_1H_2 \end{array}\right] = \left[\begin{array}{cc|ccccccccc} \alpha+\delta & \beta & 0 & \cdots & 0 & 0 & \gamma & 0 & \cdots & 0 \\ -\beta & \alpha-\delta & 0 & \cdots & 0 & \zeta & \epsilon & 0 & \cdots & 0 \\ \hline \multicolumn{2}{c|}{\hat{C}} & \multicolumn{3}{c}{\hat{D}_1} & d_1 & d_2 & \multicolumn{3}{c}{\hat{D}_2} \end{array}\right].$$

     The matrix $\hat{D} := \begin{bmatrix} \hat{D}_1 & \hat{D}_2 \end{bmatrix}$ satisfies $\hat{D}^T\Sigma\hat{D} = \begin{bmatrix} I_{n-2} & \\ & -I_m \end{bmatrix}$ and hence there exists by property 6.2 a Σ-orthogonal matrix U such that

     $$U\hat{D} = \begin{bmatrix} 0 & 0 \\ I_{n-2} & \\ & -I_m \end{bmatrix}.$$

Therefore, a $\begin{bmatrix} I_2 & \\ & \Sigma \end{bmatrix}$-orthogonal realization for $UB(\lambda)H_1H_2$ is

$$\left[\begin{array}{c|c} G_1 & \\ \hline & U \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c|c} G_1^T & \\ \hline & H_1H_2 \end{array}\right] = \left[\begin{array}{cc|ccc|cc|ccc} \alpha+\delta & \beta & 0 & \cdots & 0 & 0 & \gamma & 0 & \cdots & 0 \\ -\beta & \alpha-\delta & 0 & \cdots & 0 & \zeta & \epsilon & 0 & \cdots & 0 \\ \hline 0 & 0 & 1 & & & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & & & 1 & 0 & 0 & 0 & \cdots & 0 \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\tilde{C}$}} & 0 & \cdots & 0 & \multicolumn{2}{c|}{\multirow{2}{*}{$\tilde{D}$}} & 0 & \cdots & 0 \\ & & 0 & \cdots & 0 & & & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & & & -1 \end{array}\right].$$

Then we can choose an orthogonal 2×2 matrix $\hat{U}$ (which can be integrated in the first Σ-orthogonal matrix) such that, by $\begin{bmatrix} I_2 & \\ & \Sigma \end{bmatrix}$-orthogonality of the realization,

$$\hat{U}\tilde{C} = \begin{bmatrix} 0 & \zeta \\ \gamma & -\epsilon \end{bmatrix}.$$

Finally, the structure of $\hat{U}\tilde{D}$ results from the $\begin{bmatrix} I_2 & \\ & \Sigma \end{bmatrix}$-orthogonality of the realization.

2. $\tilde{b}_2^T\Sigma\tilde{b}_2 < 0$
   In this case, we choose a real skew-Householder matrix $H_2 = H_2^T$ such that the first $n-1$ and the last $m-1$ components of $H_2H_1b_2$ are zero. Similar arguments to the previous case then lead to the second parametrization with $\nu = 0$.
3. $\tilde{b}_2^T\Sigma\tilde{b}_2 = 0$
   We then choose the second skew-Householder matrix H_2 such that $H_2H_1b_2 = \begin{bmatrix} 0 & \cdots & 0 & \nu & \epsilon & \nu & 0 & \cdots & 0 \end{bmatrix}^T$, $\nu \in \{0,1\}$. Similar arguments to the first two cases finally lead to the third parametrization.

- $\boxed{b_i^T\Sigma b_i < 0 \text{ and } b_j^T\Sigma b_j \leq 0,\ i,j \in \{1,2\},\ i \neq j}$

We may suppose without loss of generality that $b_1^T \Sigma b_1 < 0$ (otherwise we permute b_1 and b_2). So we can find a real skew-Householder transformation $H_1 = H_1^T$ such that only the $(n+1)$-th component of $H_1 b_1$ is not equal to zero. Let us denote by $\tilde{b}_2$ the vector formed by the first n and the last $m-1$ components of $H_1 b_2$. The trace of $\hat{B}\Sigma\hat{B}^T$ is strictly negative and hence the signature of this matrix is (+,-) and its determinant is strictly negative. Therefore, we deduce that $\tilde{b}_2^T \Sigma \tilde{b}_2 = det(\hat{B}\Sigma\hat{B}^T)/b_1^T \Sigma b_1$ is strictly positive. We then choose the real skew-Householder matrix $H_2 = H_2^T$ of the form

$$\begin{bmatrix} H_{11} & & H_{12} \\ & 1 & \\ H_{21} & & H_{22} \end{bmatrix}$$

such that the first $n-1$ and the last $m-1$ components of $H_2 H_1 b_2$ are zero. Similar arguments to the previous cases then lead to the second parametrization with $\gamma = 0$.

- $\boxed{b_1^T \Sigma b_1 = 0 \text{ and } b_2^T \Sigma b_2 = 0}$

  Since at least one eigenvalue of $\hat{B}\Sigma\hat{B}^T$ is strictly positive, this implies that $b_1^T \Sigma b_2 \neq 0$. We may suppose without loss of generality that $b_1^T \Sigma b_2 < 0$ (otherwise we apply the similarity Σ). This does not affect the sign-symmetry of A. The first real skew-Householder matrix H_1 is chosen such that $H_1 b_1 = e_n + e_{n+1}$. Let us denote by $\tilde{b}_2$ the vector formed by the first $n-1$ and the last $m-1$ components of $H_1 b_2$. We deduce from the structure of $H_1 b_1$ and from the inequality $b_1^T \Sigma b_2 < 0$ that $\tilde{b}_2^T \Sigma \tilde{b}_2$ is strictly positive. We then choose the real skew-Householder matrix $H_2 = H_2^T$ of the form

$$\begin{bmatrix} H_{11} & & H_{12} \\ & I_2 & \\ H_{21} & & H_{22} \end{bmatrix}$$

  such that the first $n-2$ and the last $m-1$ components of $H_2 H_1 b_2$ are zero. Similar arguments to the previous cases then lead to the fourth parametrization. □

Remark 6.6 *It must be stressed that this parametrization is minimal in the generic cases (22) and* (23). *Indeed, it follows from [Bel 68, ch. 8] that the* *number of real parameters which characterize a $(n+m)\times(n+m)$ Σ-unitary transfer matrix of degree 2 is $2(n+m)+(n+m)^2$. This value is equal to the number of real parameters of the above parametrization:*

- $(n+m-1)$ *real parameters for the first skew-Householder matrix H_1;*
- $(n+m-2)$ *real parameters for H_2;*
- 3 *real parameters α, β, δ for $\hat{B}(\lambda)$ which is completely determined by them;*
- $(n+m)^2$ *real parameters for U_d.*

The number of parameters in the other two cases (24) and (25) differs from this value. This difference is due to the relations existing between the two eigenvalues of $I_2 - AA^T$: in the first case, one of them is equal to zero; in the second case, their sum is zero.

7 A minimal Σ-unitary parametrization

The lemmas developed in the previous section allow us to obtain a parametrization of Σ-unitary transfer matrices having d distinct complex zeros as well as real Σ-orthogonal transfer matrices with self conjugate conditions. These parametrizations are inspired from [Bel 68]. Let us first consider the complex case

Theorem 7.1 *A $(n+m)\times(n+m)$ Σ-unitary transfer matrix $A(\lambda)$ with d distinct complex zeros (of degree 1) λ_i can be decomposed as*

$$A(\lambda) = U_d A_d(\lambda) H_d A_{d-1}(\lambda) H_{d-1} \cdots A_1(\lambda) H_1, \tag{26}$$

where

- U_d *is an $(n+m)\times(n+m)$ Σ-unitary matrix;*
- H_i $(i=1,\ldots,d)$ *are $(n+m)\times(n+m)$ skew-Householder matrices;*
- $A_i(\lambda)$ *are $(n+m)\times(n+m)$ Σ-unitary transfer matrices (of degree 1) of the form*

$$A_i(\lambda) = \begin{bmatrix} r_i(\lambda) & & \\ & I_{n-1} & \\ & & -I_m \end{bmatrix}, \tag{27}$$

where $r_i(\lambda)$ is an all pass transfer function of degree 1 admitting a unitary realization of the form

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} c_i & s_i \\ \hline s_i & -\bar{c}_i \end{array}\right], \quad s_i \in \mathbb{R}. \tag{28}$$

Proof. This proof is similar to that of theorem 3.1 and is left to the reader. □

It must be stressed that this parametrization is minimal. Indeed, it follows from [Bel 68, ch. 8] that the number of real parameters which characterize a $(n+m)\times(n+m)$ Σ-unitary transfer matrix of degree d is $2d(n+m)+(n+m)^2$. This value is equal to the number of real parameters of the above parametrization:

- $2n$ real parameters for each pole: 2n-2 parameters for H_i and 2 parameters for $A_i(\lambda)$.
- n^2 real parameters for U_d.

Moreover, this parametrization allows us, as we will see in the next section, a great flexibility in the construction of such Σ-unitary transfer matrices.

This result extends easily to real Σ-orthogonal transfer matrices $B(\lambda)$ with self conjugate poles. The proof of the following theorem is similar to the previous one.

Theorem 7.2 *An $(n+m)\times(n+m)$ real Σ-orthogonal transfer matrix $B(\lambda)$ with d_1 real poles and d_2 pairs of complex conjugate poles can be decomposed as*

$$B(\lambda) = U_{d_1+d_2}B_{d_1+d_2}(\lambda)H_{d_1+d_2,2}H_{d_1+d_2,1}\cdots B_{d_1+1}(\lambda)H_{d_1+1,2}H_{d_1+1,1} \\ B_{d_1}(\lambda)H_{d_1}\cdots B_1(\lambda)H_1, \tag{29}$$

where

- *$U_{d_1+d_2}$ is an $(n+m)\times(n+m)$ Σ-orthogonal matrix;*
- *$H_{i,1}$, $H_{i,2}$ $(i = d_1+1,\ldots,d_1+d_2)$ and H_j $(j=1,\ldots,d_1)$ are $(n+m)\times(n+m)$ real skew-Householder matrices;*

- $B_i(\lambda)$ $(i = 1, ..., d_1)$ *are* $(n+m) \times (n+m)$ *real* Σ*-orthogonal transfer matrices of degree 1 (corresponding to real poles* $1/\lambda_i$*) of the form (27) (with* $c_i \in \mathbb{R}$*);*

- $B_i(\lambda)$ $(i = d_1+1, ..., d_1+d_2)$ *are* $(n+m) \times (n+m)$ *real* Σ*-orthogonal transfer matrices of degree 2 (corresponding to conjugate poles) of the form (22), (23), (24) or (25).*

Remark 7.3 *The case* $n = 1$ *can be treated in a manner similar to remark 3.4.*

Moreover, the major advantage of such a parametrization is the great flexibility in constructing Σ-orthogonal transfer matrices. This flexibility will be employed in the next section to develop a recursive algorithm for obtaining a Σ-orthogonal transfer matrix with self conjugate tangential conditions.

8 Σ-unitary interpolation

In this section, we present a recursive algorithm using Σ-unitary transformations to obtain a $\begin{bmatrix} I_2 & \\ & \Sigma \end{bmatrix}$-unitary realization $\{A, B, C, D\}$ of a Σ-unitary transfer matrix which satisfies the following tangential conditions:

$$U(\lambda_i)z_i = 0, \quad i = 1, ..., d,$$

where the d distinct complex points λ_i and associated directions $z_i \in \mathbb{C}^n$ are given. This algorithm, based on the parametrization (26), leads also to a fast construction of a realization of $U(\lambda)$ as well as its inverse. Furthermore we introduce in the second part of this section a modification to this algorithm that allows to work in real arithmetic in the case of self conjugate conditions.

Similar developments to those in section 4 lead in the complex case to the following algorithm:

Start with the data

$$A = B = C = \emptyset; \quad D = I_n$$

and repeat the following steps for $i = 1, ..., d$:

Step 1 Determine the Σ-normalized vectors w_i associated with skew-Householder matrices $H_i = \Sigma - \frac{2w_i w_i^*}{w_i^* \Sigma w_i}$. This is done by evaluating the vectors $\hat{z}_i$ which satisfy $\hat{z}_i^* \Sigma \hat{z}_i > 0$ (cfr appendix 1);

Step 2 (corresponding to the premultiplication by H_i)
Update C and D: $C \longleftarrow H_i C$; $D \longleftarrow H_i D$;

Step 3 (corresponding to the premultiplication by $A_i(\lambda)$)
Apply Givens rotation and Σ to $\left[\begin{array}{cc|c} I & & \\ & A & B \\ \hline & C & D \end{array}\right]$.

This algorithm requires essentially $4n^2d + 4nd^2$ (+ lower order terms) flops which subdivide into

- $2nd^2$ flops (+ l.o.t.) for step 1;
- $4n^2d + 2nd^2$ flops (+ l.o.t.) for steps 2 and 3.

This complexity is comparable to that obtained from the construction and inversion of a Cauchy matrix (cfr [BKO 95]).

It is important to note that the matrix A of the realization is upper triangular so that we have obtained a Schur-type realization. Furthermore, we can easily obtain a realization of

$$(A_d(\lambda)H_d \cdots A_1(\lambda)H_1)^{-1} = \Sigma H_1 \Sigma A_1(\lambda)^{-1} \cdots \Sigma H_d \Sigma A_d(\lambda)^{-1}.$$

Indeed, such a realization is given by similar steps as 2 and 3 but where the realization of $A_i(\lambda)^{-1}$ is now given by (8). Therefore, the construction of a realization of the inverse of the Σ-unitary transfer matrix only requires $4n^2d + 2nd^2$ extra flops.

Remark 8.1 *A similar remark as 4.1 applies here as well.*

We now examine the self conjugate conditions (SCC). Similar developments to those in section 5 lead to the following procedure:

1. The algorithm uses the same recursion as in the complex case for the d_1 real poles λ_i but with real (instead of complex) skew-Householder matrices and real $\begin{bmatrix} I_2 & \\ & \Sigma \end{bmatrix}$-orthogonal (instead of complex $\begin{bmatrix} I_2 & \\ & \Sigma \end{bmatrix}$-unitary) realizations of Σ-orthogonal transfer matrices of degree 1.

2. Now we examine the conjugate poles. A similar recursion to that of section 5 applies here as well but where now

   - $H_{i,1}$ and $H_{i,2}$ are now $(n+m) \times (n+m)$ real skew-Householder matrices;
   - $B_i(\lambda)$ denote $(n+m)\times(n+m)$ real Σ-orthogonal transfer matrices of degree 2 (corresponding to conjugate poles) of the form (22), (23), (24) or (25).

   This recursion has the following characteristics:

   - All complex n-vectors are decomposed into two real n-vectors: their real and complex parts. The evaluation of the vector in progress only requires premultiplications by $(n+m) \times (n+m)$ real skew-Householder matrices and by $(n+m) \times (n+m)$ Σ-orthogonal matrices of degree 1 of the form (27) or of degree 2 of the form (22), (23), (24) or (25). The real Householder matrices can be applied separately to the real and complex parts of the vector in progress. On the other hand, the blocks $\hat{B}_i(\lambda)$ of the Σ-orthogonal matrices are first computed by complex arithmetic and then applied in real arithmetic to the correponding rows of the obtained vector. We denote by $\hat{z}_{d_1+k+1}$ the Σ-normalized vector corresponding to $B(\lambda_{d_1+k+1})z_{d_1+k+1}$:

   $$\hat{z}_{d_1+k+1} = u_{d_1+k+1} + j\ v_{d_1+k+1}.$$

   This normalization is possible since any vector v in the nullspace of a Σ-orthogonal matrix satisfies $v^*\Sigma v > 0$ (cfr appendix 1).

   - We know by theorem 7.2 that there exist two real skew-Householder matrices $H_{d_1+k+1,1}$ and $H_{d_1+k+1,2}$ and an $(n+m) \times (n+m)$ real Σ-orthogonal transfer matrix $B_{d_1+k+1}(\lambda)$ such that, for each phase factor $e^{j\phi}$, we have

   $$B_{d_1+k+1}(\lambda_{d_1+k+1})H_{d_1+k+1,2}H_{d_1+k+1,1}\hat{z}_{d_1+k+1}e^{j\phi} = 0. \qquad (30)$$

A similar argumentation as in the section 5 then leads to the following procedure:

- first evaluate

$$\Delta = \left(u_{d_1+k+1}^T \Sigma u_{d_1+k+1}\right)\left(v_{d_1+k+1}^T \Sigma v_{d_1+k+1}\right) - \left(u_{d_1+k+1}^T \Sigma v_{d_1+k+1}\right)^2;$$

- then choose the Σ-orthogonal realization (22) (respectively (23), (24)) if $\Delta > 0$ (respectively $< 0, = 0$);
- in all cases, the parameter δ is equal to

$$\frac{\beta_0 r\sqrt{1-4\Delta}}{2\sqrt{\beta_0^2 + r^2\Delta}}$$

where r is defined by (14). The other parameters of the realization are then obtained from the location of the poles and the $\begin{bmatrix} I_2 & \\ & \Sigma \end{bmatrix}$-orthogonality of the realization.

- Once δ known, the complex factor $e^{j\phi}$ can easily be obtained by an equation similar to (10). Moreover, it can be integrated in the calculus of the two real skew-Householder matrices $H_{d_1+k+1,2}$ and $H_{d_1+k+1,1}$ which are applied to the real and complex parts of the vector in progress.

The new real matrix $\hat{B}(\lambda) := B_{d_1+k+1}(\lambda) H_{d_1+k+1,2} H_{d_1+k+1,1} B(\lambda)$ is then completely determined. Moreover, it verifies $\hat{B}_{d_1+k+1}(\lambda_{d_1+k+1}) z_{d_1+k+1} = 0$ and automatically the conjugate conditions too.

This algorithm requires essentially $4n^2d + 4nd^2$ (+ lower order terms) **real** flops where $d = d_1 + 2d_2$ denotes the number of tangential conditions. Hence this second algorithm is faster (up to a factor 3) than the previous one which could also be applied to satisfy the self conjugate conditions.

Furthermore, in analogy to section 5, we can easily obtain a realization of the inverse of the Σ-unitary transfer matrix.

Finally it must be observed that the realization of the inverse is still sign-symmetric and that its construction requires only $4n^2d + 2nd^2$ extra **real** flops.

9 Concluding remarks

In this paper we have presented a parametrization of all pass (respectively Σ-unitary) transfer matrices which leads to a recursive algorithm for constructing a unitary (respectively $\begin{bmatrix} I_2 & \\ & \Sigma \end{bmatrix}$-unitary) realization of an all pass (respectively Σ-unitary) transfer matrix subject to tangential interpolation constraints. This algorithm was developed in the discrete case but works as well in the continuous one: the unitary realization (7) has then to be replaced by

$$\left[\begin{array}{c|cccc} a & \sqrt{-2a} & 0 & \cdots & 0 \\ \hline -\sqrt{-2a} & 1 & & & \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \end{array}\right]$$

while the real realization (6) becomes

$$\left[\begin{array}{cc|cc} \alpha+\delta & \beta & \sqrt{-2(\alpha+\delta)} & 0 \\ -\beta & \alpha-\delta & 0 & \sqrt{-2(\alpha-\delta)} \\ \hline -\sqrt{-2(\alpha+\delta)} & 0 & 1 & 0 \\ 0 & -\sqrt{-2(\alpha-\delta)} & 0 & 1 \end{array}\right].$$

The rest of the algorithm remains the same and so does its complexity.

Σ-orthogonal transfer matrices can also be treated in the continuous case. We then have to consider the following equations

$$A + A^T = B\Sigma B^T, \quad trace(B\Sigma B^T) = -4\alpha > 0,$$

and the different cases of lemma 6.5 are retrieved here as well. We do not develop this here since it is completely analogous.

A Appendix 1

We prove here that any vector v_0 in the nullspace of a Σ-orthogonal transfer matrix $H(\lambda_0)$ verifies $v_0^*\Sigma v_0 > 0$. Let us consider a $\begin{bmatrix} I_2 & \\ & \Sigma \end{bmatrix}$-orthogonal realization of the matrix

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

For every pair (λ_0, v_0) such that $v_0 \neq 0$, $H(\lambda_0)v_0 = 0$, there exists a vector $z_0 \neq 0$ such that

$$\left[\begin{array}{c|c} A - \lambda_0 I_2 & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c} z_0 \\ \hline v_0 \end{array}\right] = 0$$

i.e.

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c} z_0 \\ \hline v_0 \end{array}\right] = \lambda_0 \left[\begin{array}{c} z_0 \\ \hline 0 \end{array}\right].$$

Finally, we deduce from the $\begin{bmatrix} I_2 & \\ & \Sigma \end{bmatrix}$-orthogonality of the realization that

$$z_0^* z_0 - v_0^* \Sigma v_0 = |\lambda_0|^2 z_0^* z_0$$

and hence from the stability of the transfer matrix

$$v_0^* \Sigma v_0 = (1 - |\lambda_0|^2) z_0^* z_0 > 0.$$

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