On stability radii of generalized eigenvalue problems

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Abstract

In this paper, we extend a known characterization of the stability radius of a standard eigenvalue problem to the generalized eigenvalue problem as well as that of cyclic pencils occuring in periodic systems. We also extend some of these results to norms different than the 2-norm.

1 Introduction

The complex structured stability radius measures the ability of a matrix to preserve its stability under complex structured pertubations. Consider a partitioning of the complex plane $\mathbb{C}$ into two disjoint sets $\mathbb{C}_g$ and $\mathbb{C}_b$ such that $\mathbb{C}_g$ is open, i.e., $\mathbb{C} = \mathbb{C}_g \cup \mathbb{C}_b$. A matrix is called $\mathbb{C}_g$-stable if its spectrum $\Lambda(.)$ is contained in $\mathbb{C}_g$. The usual sets $\mathbb{C}_g$ are the open unit disk and the open left complex plane (stability in discret/continuous time) but it can also be chosen, for example, as a disk of radius R ($R < 1$) or the left halfspace $\{z : Re(z) < \alpha\}, \alpha < 0$ (strong stability in discrete/continuous time). Let us denote the singular values of a $p \times m$ matrix, ordered nonincreasingly, by $\sigma_k(.),\ k = 1, 2, ..., \min\{p, m\}$. The complex structured stability radius of a matrix triple $(A, B, C) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m} \times \mathbb{C}^{p\times n}$ (with A $\mathbb{C}_g$-stable), with respect to the p-norm, can be defined [1] as

$$r_{\mathbb{C}}(A, B, C) = \inf_{\Delta\in\mathbb{C}^{m\times p}} \{\|\Delta\|_p : \Lambda(A + B\Delta C) \not\subset \mathbb{C}_g\} \quad (1)$$

where we denote successively the matrix p-norm and the vector p-norm by

$$\|\Delta\|_p := \sup_{x\neq 0} \frac{\|\Delta x\|_p}{\|x\|_p} \quad (2)$$

and

$$\|x\|_p := (|x_1|^p + \cdots + |x_n|^p)^{1/p}\,. \quad (3)$$

We also define the vector and matrix ∞-norms as

$$\|x\|_\infty = \max_i |x_i|, \quad \|A\|_\infty = \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}|. \quad (4)$$

When $B = I$ and $C = I$, $r_{\mathbb{C}}(A, I, I)$ is usually abbreviated as $r_{\mathbb{C}}(A)$ and called the (unstructured) stability radius of A.

By continuity, we can easily rewrite (1) as

$$\begin{aligned} r_{\mathbb{C}}(A, B, C) &= \inf_{\Delta\in\mathbb{C}^{m\times p}} \{\|\Delta\|_p : \Lambda(A + B\Delta C) \cap \partial\mathbb{C}_g \neq \emptyset\} \\ &= \inf_{\lambda\in\partial\mathbb{C}_g} \left[\inf_{\Delta\in\mathbb{C}^{m\times p}} \{\|\Delta\|_p : \det(\lambda I - A - B\Delta C) = 0\} \right] \\ &= \inf_{\lambda\in\partial\mathbb{C}_g} \left[\inf_{\Delta\in\mathbb{C}^{m\times p}} \{\|\Delta\|_p : \det(I - \Delta C(\lambda I - A)^{-1}B) = 0\} \right]. \end{aligned} \quad (5)$$

Hence the key issue in the computation of the stability radius is to solve the following linear algebra problem: given $M \in \mathbb{C}^{p\times m}$, compute

$$\inf_{\Delta\in\mathbb{C}^{m\times p}} \{\|\Delta\|_p : \det(I - \Delta M) = 0\}.$$

This problem is solved by the following lemma:

Lemma 1

$$\inf_{\Delta\in\mathbb{C}^{m\times p}} \{\|\Delta\|_p : \det(I - \Delta M) = 0\} = \|M\|_p^{-1}$$

Proof: We first show that the p-norm of any matrix Δ satisfying $\det(I - \Delta M) = 0$ is greater or equal to $\|M\|_p^{-1}$. Indeed, there exists a p-unit vector v such that $\Delta M v = v$ and therefore

$$1 = \|v\|_p \leq \|\Delta\|_p \|M\|_p \|v\|_p = \|\Delta\|_p \|M\|_p.$$

Furthermore, such a bound can always be obtained by a rank 1 matrix $\Delta^* = w x^*$: let us consider a p-unit vector w such that $\|M\|_p = \|Mw\|_p =: \|y\|_p$. Then we define the vector x by

$$x_i = \begin{cases} \dfrac{|y_i|^p}{\overline{y_i}\, \|M\|_p^p} & \text{if } p \neq \infty \\ \delta_{ij} & \text{if } p = \infty \end{cases} \quad (6)$$

where j is the index of a component of x such that $|x_j| = \|x\|_\infty$.

The q-norm of x (with $\frac{1}{p} + \frac{1}{q} = 1$) is equal to $\frac{1}{\|M\|_p}$. Finally, we obtain

$$\|\Delta\|_p = \sup_{z\neq 0} \frac{\|w(x^* z)\|_p}{\|z\|_p} \leq \sup_{z\neq 0} \frac{\|w\|_p \|x\|_q \|z\|_p}{\|z\|_p} = \frac{1}{\|M\|_p}$$

and

$$(I - \Delta M)w = w - \Delta y = w - w\,(x^* y) = 0.$$

□

Combining lemma 1 with equation (5), it follows that we have to solve

$$\inf_{\lambda\in\partial\mathbb{C}_g}\{\|C(\lambda I - A)^{-1}B\|_p^{-1}\} = \left\{\sup_{\lambda\in\partial\mathbb{C}_g} \|C(\lambda I - A)^{-1}B\|_p^{-1}\right\}^{-1}$$

Since $C(\lambda I - A)^{-1}B$ is a rational matrix function in λ, this is a non-convex optimisation problem on $\partial\mathbb{C}_g$ in order to compute the stability radius via this characterization. This computation often leads to high difficulties and its complexity can even be non polynomial for certain norms (see [3]). In section 4, we point out some fast algorithms for solving this problem for the 2-norm.

2 Generalized eigenvalue problems

In section 1, we described the basic standard eigenvalue problem and the associated optimisation. In this section, we discuss generalized eigenvalue problems and present some new results for their resolution.

We are now interested in perturbations Δ_E and Δ_A on E and A which leads to $\mathbb{C}_g$-instability of the matrix $\lambda_E - A + \lambda\Delta_E - \Delta_A$. By similar developments as in (5), we obtain that $\Delta_\lambda := \lambda\Delta_E - \Delta_A$ has to verify

$$\det(I + \Delta_\lambda M_\lambda) = 0 \qquad (7)$$

with

$$M_\lambda = (\lambda E - A)^{-1}. \qquad (8)$$

We already know, from the proof of lemma 1, that

$$\|\Delta_\lambda\|_p \geq \|M_\lambda\|_p^{-1} \qquad (9)$$

and that this upper bound is attained by a rank 1 matrix Δ^*. But what can be said about Δ_E and Δ_A? Define the matrix

$$\Delta = [\Delta_E; \Delta_A]. \qquad (10)$$

The following theorem then gives similar results to lemma 1:

Theorem 1 $\inf_{\Delta\in\mathbb{C}^{n\times 2n}} \{\|\Delta\|_p : \det(I + \Delta M_\lambda) = 0\} =$

$$\frac{\|M_\lambda\|_p^{-1}}{\sqrt[p]{1+|\lambda|^p}} \;\; p\neq\infty; \qquad \frac{\|M_\lambda\|_\infty^{-1}}{\max(|\lambda|,1)} \;\; p=\infty. \qquad (11)$$

Proof: We prove this theorem for $p \neq \infty$. The case $p = \infty$ can be treated in a similar manner. We deduce from

$$\Delta_\lambda = [\Delta_E; \Delta_A] \begin{bmatrix} \lambda I \\ -I \end{bmatrix} \qquad (12)$$

that

$$\|[\Delta_E; \Delta_A]\|_p \geq \frac{\|\Delta_\lambda\|_2}{\sqrt[p]{1+|\lambda|^p}} \geq \frac{\|M_\lambda\|_p^{-1}}{\sqrt[p]{1+|\lambda|^p}}. \qquad (13)$$

Furthermore, such a lower bound is attained for

$$\Delta_E = \frac{\overline{\lambda}|\lambda|^{p-2}\Delta^*}{1+|\lambda|^p} \quad \text{and} \quad \Delta_A = \frac{-\Delta^*}{1+|\lambda|^p}. \qquad (14)$$

Indeed, we have

$$\begin{aligned}
\|[\Delta_E \Delta_A]\|_p &= \sup_{x_1,x_2\neq 0} \frac{\left\|\Delta^* \dfrac{\overline{\lambda}|\lambda|^{p-2}x_1 - x_2}{1+|\lambda|^p}\right\|_p}{\sqrt[p]{\|x_1\|_p^p + \|x_2\|_p^p}} \\
&= \frac{1}{1+|\lambda|^p} \sup_{x_1,x_2\neq 0} \frac{\left\|w\left(\overline{\lambda}|\lambda|^{p-2}x^* x_1 - x^* x_2\right)\right\|_p}{\sqrt[p]{\|x_1\|_p^p + \|x_2\|_p^p}} \\
&\leq \frac{1}{1+|\lambda|^p} \sup_{x_1,x_2\neq 0} \frac{\|w\|_p\|x\|_q\|\overline{\lambda}|\lambda|^{p-2}x_1 - x_2\|_p}{\sqrt[p]{\|x_1\|_p^p + \|x_2\|_p^p}} \\
&= \frac{1}{(1+|\lambda|^p)\|M_\lambda\|_p} \sup_{x_1,x_2\neq 0} \frac{\|\overline{\lambda}|\lambda|^{p-2}x_1 - x_2\|_p}{\sqrt[p]{\|x_1\|_p^p + \|x_2\|_p^p}} \\
&= \frac{1}{(1+|\lambda|^p)\|M_\lambda\|_p} \sqrt[q]{|\lambda|^p + 1} \\
&= \frac{1}{\sqrt[p]{1+|\lambda|^p}\|M_\lambda\|_p}
\end{aligned}$$

□

We easily deduce the following corollary from the rank of the solution:

Corollary 2

$$\inf_{\Delta\in\mathbb{C}^{n\times 2n}} \{\|\Delta\|_F : \det(I + \Delta M_\lambda) = 0\} = \frac{\|M_\lambda\|_2}{\sqrt{1+|\lambda|^2}}$$

Remark 3 *The bound (11) can e.g. be rewritten as*

$$\frac{\|(e^{j\omega}E - A)^{-1}\|_p^{-1}}{\sqrt[p]{2}} \quad : \textit{discrete time stability}$$

$$\frac{\|(j\omega E - A)^{-1}\|_p^{-1}}{\sqrt[p]{1+\omega^p}} \quad : \textit{continuous time stability}$$

$$\frac{\|(re^{j\omega}E - A)^{-1}\|_p^{-1}}{\sqrt[p]{1+r^p}} \quad : \textit{discrete time strong stability}$$

$$\frac{\|((\alpha + j\omega)E - A)^{-1}\|_p^{-1}}{\sqrt[p]{1+(\alpha^2+\omega^2)^{p/2}}} \quad : \textit{cont. time strong stability}$$

Remark 4 *For the discrete time stability, the optimal perturbations* Δ_E *and* Δ_A *are independant of the chosen norm:*

$$\Delta_E = \frac{\overline{\lambda}\Delta^*}{2} \quad \textit{and} \quad \Delta_A = \frac{-\Delta^*}{2}.$$

The perturbations Δ_E and Δ_A have sometimes a particular structure. One is e.g. interested in $\mathbb{C}_g$-stability of the matrix

$$\lambda\left(E + F\Delta_E G\right) - \left(A + B\Delta_A C\right). \tag{15}$$

By similar developments to those in (5), we obtain that

$$\det\left[(\lambda E - A) + \lambda F \Delta E G - B\Delta_A C\right] = 0$$
$$\Updownarrow$$
$$\det\left[I + (\lambda E - A)^{-1}\lambda F\Delta_E G - (\lambda E - A)^{-1}B\Delta_A C\right] = 0$$
$$\Updownarrow$$
$$\det\left[I + (\lambda E - A)^{-1}\left[\lambda F, -B\right]\begin{bmatrix}\Delta_E & \\ & \Delta_A\end{bmatrix}\begin{bmatrix}G \\ C\end{bmatrix}\right] = 0$$
$$\Updownarrow$$
$$\det\left[I + \begin{bmatrix}\Delta_E & \\ & \Delta_A\end{bmatrix}\begin{bmatrix}G \\ C\end{bmatrix}(\lambda E - A)^{-1}\left[\lambda F, -B\right]\right] = 0.$$

Once again, the key issue in the computation of the stability radius is to solve a minimization problem: given $M_\lambda \in \mathbb{C}^{p\times n}$, compute

$$\inf_{\Delta\in\mathcal{D}}\{\|\Delta\|_p : \det(I - \Delta M_\lambda) = 0\} \tag{16}$$

with

$$\mathcal{D} = \{\Delta : \Delta = \begin{bmatrix}\Delta_E & \\ & \Delta_A\end{bmatrix}\}$$

This minimization is solved in [2] for the 2-norm by the following lemma:

Lemma 5

$\inf_{\Delta\in\mathcal{D}}\{\|\Delta\|_2 : \det(I - \Delta M_\lambda) = 0\}$

$$= \left(\inf_{\gamma>0}\left\|\begin{bmatrix}M_{11} & \gamma M_{12} \\ \gamma^{-1}M_{12} & M_{22}\end{bmatrix}\right\|_2\right)^{-1}$$

where

$$M = \begin{bmatrix}M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix} \tag{17}$$

is partitioned conformably to Δ.

Remark 6 *This infimum is reached for a* **real non zero** *parameter* γ *if* M_{12} *and* M_{21} *are not zero.*

Remark 7 *This lemma can not be extended to arbitrary p-norm as shown by the following* 2×2 *counter-example:*

$$M = \begin{bmatrix}1 & 1 \\ -1 & 1\end{bmatrix}; \quad \min_{\gamma>0}\left\|\begin{bmatrix}1 & \gamma \\ \dfrac{-1}{\gamma} & 1\end{bmatrix}\right\|_1 = \|M\|_1 = 2$$

and

$$\inf_{\Delta\in\mathcal{D}}\{\|\Delta\|_1 : \det(I - \Delta M_\lambda) = 0\} = \frac{\sqrt{2}}{2}.$$

3 Discrete-time periodic systems

We now consider the periodically time-varying system

$$E_k x_{k+1} = A_k x_k, \quad k \in \mathbb{Z}_0 \tag{18}$$

where $\mathbb{Z}$ is the set of integers. Let us denote by K the smallest positive integer for which $E_k = E_{k+K}$, $A_k = A_{k+K}$, $\forall k \in \mathbb{Z}$ and by Z the block left shift matrix of size $Kn \times Kn$:

$$Z = \begin{bmatrix} & & & I_n \\ I_n & & & \\ & \ddots & & \\ & & I_n & \end{bmatrix}.$$

We use the standard notation I_n for the identity matrix of size n and $\mathrm{Diag}((A_i)_{i=1,\ldots,K})$ to represent the block diagonal matrix whose elements are the matrices A_i. The underlying eigenvalue problem now involves the matrix

$$M^{-1} = \lambda\mathcal{E} - \mathcal{A} = \lambda\ Diag((E_i)_{i=1,\ldots,K}) - Diag((A_i)_{i=1,\ldots,K})\ Z.$$

We are interested in perturbations $\Delta\mathcal{E}$ and $\Delta\mathcal{A}$ having the same structure as $\mathcal{E}$ and $\mathcal{A}$. By a similar reasoning as in (5), we obtain that $\lambda\Delta\mathcal{E} - \Delta\mathcal{A}$ has to verify

$$\det(I + (\lambda\Delta\mathcal{E} - \Delta\mathcal{A})M) = 0.$$

The following theorem is the periodic version of lemma 5 and solves the minimization problem

$$\inf_{\Delta\in\mathcal{D}}\{\|\Delta\|_2 : \det(I + (\lambda\Delta\mathcal{E} - \Delta\mathcal{A})M) = 0\}$$

with

$$\mathcal{D} = \{\Delta : \Delta = Diag(E_1, ..., E_K, A_1, ..., A_K)\}.$$

Theorem 8

$\inf_{\Delta\in\mathcal{D}}\{\|\Delta\|_2 : \det(I + (\lambda\Delta\mathcal{E} - \Delta\mathcal{A})M) = 0\}$

$$= \left(\min_{e_i,a_i}\|D_1(e_i,a_i)MD_2(e_i,a_i)\|_2\right)^{-1}$$

$$= \max_{e_i,a_i}\left\{\sigma_{min}\left(D_2(e_i,a_i)^{-1}M^{-1}D_1(e_i,a_i)^{-1}\right)\right\}$$

where

$$D_1(e_i,a_i) = Diag\left(\left(\sqrt{e_i^2 + a^2_{1+(i\ mod\ K)}}I_n\right)_{i=1,\ldots,K}\right)$$
$$=: Diag\left((\alpha_i I_n)_{i=1,\ldots,K}\right) \tag{19}$$

$$D_2(e_i,a_i) = Diag\left(\left(\sqrt{e_i^{-2} + a_i^{-2}}I_n\right)_{i=1,\ldots,K}\right);$$
$$=: Diag\left((\beta_i I_n)_{i=1,\ldots,K}\right) \tag{20}$$

$$e_1 = 1. \tag{21}$$

Proof: To reduce the conservatism caused by the structure of Δ, we resort to the widely used technique of scaling. It turns out that this scaling completely eliminates the conservatism. Let us choose a block diagonal matrix D of the form

$$D = Diag(e_1 I_n, ..., e_K I_n, a_1 I_n, ..., a_K I_n), \qquad e_i, a_i \in \mathbb{R}^+ \cup \{\infty\}$$

commuting with Δ. We then have that

$$\det(I + (\lambda\Delta\mathcal{E} - \Delta\mathcal{A})M) = 0$$
$$\Updownarrow$$
$$\det\left(I + \left[\begin{array}{cc} \lambda I_{Kn} & -I_{Kn} \end{array}\right] \Delta \left[\begin{array}{c} I_{Kn} \\ I_{Kn} Z \end{array}\right] M\right) = 0$$
$$\Updownarrow$$
$$\det\left(I + \Delta D \left[\begin{array}{c} I_{Kn} \\ I_{Kn} Z \end{array}\right] M \left[\begin{array}{cc} \lambda I_{Kn} & -I_{Kn} \end{array}\right] D^{-1}\right) = 0.$$

We deduce, from lemma 1, that

$$\begin{aligned} \|\Delta\|_2 &\geq \left\| D \left[\begin{array}{c} I_{Kn} \\ I_{Kn} Z \end{array}\right] M \left[\begin{array}{cc} \lambda I_{Kn} & -I_{Kn} \end{array}\right] D^{-1} \right\|_2^{-1} \\ &= \left\| \left[\begin{array}{c} Diag\left((e_i I_n)_{i=1,...,K}\right) \\ Diag\left((a_i I_n)_{i=1,...,K}\right) Z \end{array}\right] M \left[\begin{array}{c} Diag\left((\lambda e_i^{-1} I_n)_{i=1,...,K}\right) \\ Diag\left((-a_i^{-1} I_n)_{i=1,...,K}\right) \end{array}\right]^T \right\|_2^{-1} \\ &= \|U D_1(e_i, a_i) M D_2(e_i, a_i) V^*\|_2^{-1} \end{aligned}$$

for some isometries U and V. We conclude that

$$\begin{aligned} \|\Delta\|_2 &\geq \sup_{e_i, a_i} \|D_1(e_i, a_i) M D_2(e_i, a_i)\|_2^{-1} \\ &= \sup_{e_i, a_i} \left\{\sigma_{min}\left(D_2(e_i, a_i)^{-1} M^{-1} D_1(e_i, a_i)^{-1}\right)\right\} \end{aligned}$$

We now show that the supremum

$$\sup_{e_i, a_i} \left\{\sigma_{min}\left(D_2(e_i, a_i)^{-1} M^{-1} D_1(e_i, a_i)^{-1}\right)\right\}$$

is attained for **real non zero** parameters $\widehat{e_i}$ and $\widehat{a_i}$.

1. Let us first show that no e_i, at the optimum, can grow unboundedly. The structure of $M^{-1} = \lambda\mathcal{E} - \mathcal{A}$ implies the following relation to avoid a zero singular value
$$(e_i \to \infty) \Longrightarrow (e_{1+(i \bmod K)} \to \infty). \qquad (22)$$
Indeed, let us suppose that e_i tends to infinity and $e_{1+(i \bmod K)}$ does not grow unboundedly. The $(i+1, i)$ scaled block of M^{-1} is then equal to zero and therefore none of the (j, j) $(j = 1, ..., K)$ scaled blocks can be equal to zero. We then deduce recursively (from the non zero (j, j) $(j = i, ..., 1)$ scaled blocks) that $e_{i-1}, ..., e_1$ grow to infinity. This is in contradiction with $e_1 = 1$. Hence we have proved that $(e_i \to \infty)$ implies $(e_1 \to \infty)$, which once again contradicts $e_1 = 1$.
2. Therefore, no a_i can grow unboundedly at the optimum since
$$(a_i \to \infty) \Longrightarrow (e_i \to \infty) \text{ or } (e_{\ulcorner i-1 \urcorner} \to \infty) \qquad (23)$$
where we define $\ulcorner a \urcorner$ by
$$\ulcorner a \urcorner = (a - 1) \bmod K + 1.$$
3. By a similar reasoning, we can prove that neither e_i nor a_i can tend to 0 at the optimum.

Let us finally construct a matrix Δ such that

$$\|\Delta\|_2 = \sigma_{min}\left(D_2(\widehat{e_i}, \widehat{a_i})^{-1} M^{-1} D_1(\widehat{e_i}, \widehat{a_i})^{-1}\right) =: \underline{\sigma}.$$

We can choose a pair of left and right singular vectors $u = \left[\begin{array}{c} u_1 \\ \vdots \\ u_K \end{array}\right]$ and $v = \left[\begin{array}{c} v_1 \\ \vdots \\ v_K \end{array}\right]$ corresponding to $\underline{\sigma}$ such that [2], by optimality of the parameters e_i and a_i, that, u and v satisfy

$$u^* \frac{\partial\left(D_2(e_i, a_i)^{-1} M^{-1} D_1(e_i, a_i)^{-1}\right)}{\partial e_i}(\widehat{e_i}, \widehat{a_i}) v = 0 \qquad (24)$$

and

$$u^* \frac{\partial\left(D_2(e_i, a_i)^{-1} M^{-1} D_1(e_i, a_i)^{-1}\right)}{\partial a_i}(\widehat{e_i}, \widehat{a_i}) v = 0. \qquad (25)$$

Because of the special structure of D_1, D_2 and M^{-1}, these equations can be rewritten as

$$\frac{\|u_i\|_2}{\|v_i\|_2} = \frac{\widehat{e_i}^2 \beta_i}{\alpha_i} \qquad (26)$$

and

$$\frac{\|u_i\|_2}{\|v_{\ulcorner i-1 \urcorner}\|_2} = \frac{\widehat{a_i}^2 \beta_i}{\alpha_{\ulcorner i-1 \urcorner}}. \qquad (27)$$

We define ΔE_i and ΔA_i by

$$\Delta E_i = -\frac{\overline{\lambda}\, \underline{\sigma}\, \alpha_i}{\beta_i\, \widehat{e_i}^2\, \|v_i\|_2^2} u_i v_i^*; \qquad (28)$$

$$\Delta A_i = \frac{\underline{\sigma}\, \alpha_{\ulcorner i-1 \urcorner}}{\beta_i\, \widehat{a_i}^2\, \|v_{\ulcorner i-1 \urcorner}\|_2^2} u_i v_{\ulcorner i-1 \urcorner}^*. \qquad (29)$$

We deduce from (26) and (27) that

$$\|\Delta E_i\|_2 = \|\Delta A_i\|_2 = \underline{\sigma}$$

and therefore

$$\|\Delta\|_2 = \max\left(\|\Delta E_i\|_2, \|\Delta A_i\|_2\right) = \underline{\sigma}.$$

Furthermore, the matrix $I + (\lambda\Delta\mathcal{E} - \Delta\mathcal{A})M$ is singular since

$$\begin{aligned}
&\left(I + \Delta D \begin{bmatrix} I_{Kn} \\ I_{Kn}Z \end{bmatrix} M \begin{bmatrix} \lambda I_{Kn} & -I_{Kn} \end{bmatrix} D^{-1}\right) Vu \\
&\quad = (I + \Delta U D_1(\widehat{e_i}, \widehat{a_i}) M D_2(\widehat{e_i}, \widehat{a_i}) V^*)\, Vu \\
&\quad = Vu + \Delta U D_1(\widehat{e_i}, \widehat{a_i}) D_2(\widehat{e_i}, \widehat{a_i}) u \\
&\quad = Vu + \frac{1}{\underline{\sigma}} \Delta U v.
\end{aligned}$$

Moreover, we have

$$V = \begin{bmatrix} Diag\left((\overline{\lambda}\widehat{e_i}^{-1} I_n)_{i=1,\ldots,K}\right) \\ Diag\left((-\widehat{a_i}^{-1} I_n)_{i=1,\ldots,K}\right) \end{bmatrix} D_2(\widehat{e_i}, \widehat{a_i})^{-1} \quad (30)$$

and

$$U = \begin{bmatrix} Diag\left((\widehat{e_i} I_n)_{i=1,\ldots,K}\right) \\ Diag\left((\widehat{a_i} I_n)_{i=1,\ldots,K}\right) Z \end{bmatrix} D_1(\widehat{e_i}, \widehat{a_i})^{-1}. \quad (31)$$

Therefore, $Vu + \frac{1}{\underline{\sigma}}\Delta U v$ can be rewritten as

$$\begin{bmatrix} \overline{\lambda}\widehat{e_1}^{-1}\beta_1^{-1}u_1 \\ \vdots \\ \overline{\lambda}\widehat{e_K}^{-1}\beta_K^{-1}u_K \\ -\widehat{a_1}^{-1}\beta_1^{-1}u_1 \\ -\widehat{a_2}^{-1}\beta_2^{-1}u_2 \\ \vdots \\ -\widehat{a_K}^{-1}\beta_K^{-1}u_K \end{bmatrix} + \frac{\Delta}{\underline{\sigma}} \begin{bmatrix} \widehat{e_1}\alpha_1^{-1}v_1 \\ \vdots \\ \widehat{e_K}\alpha_K^{-1}v_K \\ \widehat{a_1}\alpha_K^{-1}v_K \\ \widehat{a_2}\alpha_1^{-1}v_1 \\ \vdots \\ \widehat{a_K}\alpha_{K-1}^{-1}v_{K-1} \end{bmatrix}$$

which is equal to the zero vector by (26), (27), (28) and (29). This equality completes the proof. □

Remark 9 *Similar results can be obtained for time-invariant matrices A and E with periodic perturbations. The corresponding version of theorem 8 is the following:*

Theorem 10
$\inf_{\Delta\in\mathcal{D}}\left\{\|\Delta\|_2 : \det(I + (\lambda\Delta\mathcal{E} - \Delta\mathcal{A})\,(\lambda\mathcal{E} - \mathcal{A})^{-1}) = 0\right\}$

$$= \frac{\left\|(\lambda\mathcal{E} - \mathcal{A})^{-1}\right\|_2^{-1}}{2} = \frac{\sigma_{\min}(\lambda\mathcal{E} - \mathcal{A})}{2}.$$

We observe that no diagonal scalings are needed any more. This remark is particularly relevant for recurrences of the type $Ex_{k+1} = Ax_k$ which under rounding errors are known to satisfy **exactly**

$$(E + \Delta E_{k+1})\overline{x_{k+1}} = (A + \Delta A_k)\overline{x_k}$$

where $\overline{x}$ denotes the **rounded** *version of the vector x.*

Remark 11 *Similar results can also be obtained for the 2-norm of $[\Delta\mathcal{E},\ \Delta\mathcal{A}]$ and time-variant matrices E_i and A_i*

Theorem 12
$\inf_{\Delta\mathcal{E},\Delta\mathcal{A}}\left\{\|[\Delta\mathcal{E},\ \Delta\mathcal{A}]\|_2 : \det(I + (\lambda\Delta\mathcal{E} - \Delta\mathcal{A})\,(\lambda\mathcal{E} - \mathcal{A})^{-1}) = 0\right\}$

$$\begin{aligned}
&= \left(\min_{d_i}\left\|D_1(d_i)\,(\lambda\mathcal{E} - \mathcal{A})^{-1} D_2(d_i)\right\|_2\right)^{-1} \\
&= \max_{d_i}\left\{\sigma_{min}\left(D_2(d_i)^{-1}\,(\lambda\mathcal{E} - \mathcal{A})\,D_1(d_i)^{-1}\right)\right\}
\end{aligned}$$

where

$$D_1(d_i) = Diag\left(\left(\sqrt{d_i^2 + d_{i \ mod \ K+1}^2} I_n\right)_{i=1,\ldots,K}\right) \quad (32)$$

$$D_2(d_i) =: Diag\left(\left(d_i^{-1} I_n\right)_{i=1,\ldots,K}\right); \quad (33)$$

$$d_1 = 1 \quad (34)$$

or time-invariant matrices E and A with periodic perturbations:

Theorem 13
$\inf_{\Delta\mathcal{E},\Delta\mathcal{A}}\left\{\|[\Delta\mathcal{E},\ \Delta\mathcal{A}]\|_2 : \det(I + (\lambda\Delta\mathcal{E} - \Delta\mathcal{A})\,(\lambda\mathcal{E} - \mathcal{A})^{-1}) = 0\right\}$

$$= \frac{\left\|(\lambda\mathcal{E} - \mathcal{A})^{-1}\right\|_2^{-1}}{\sqrt{2}} = \frac{\sigma_{\min}(\lambda\mathcal{E} - \mathcal{A})}{\sqrt{2}}.$$

4 Numerical algorithms

In earlier sections, we derived analytic expressions for the stability radii of several generalizations of the classical eigenvalue problem. These expressions were in terms of norms of a matrix function $H(\lambda)$ where λ varied over $\partial\mathbb{C}_g$, the boundary between the so called stable and unstable regions of the complex plane. This is in fact a **non convex** optimization problem but for the specific case of the 2-norm, there are quite performant algorithms that find the **global optimum** for such a problem. We illustrate the general procedure for the discrete time strong stability:

$$\inf_{\omega} \frac{\|(re^{j\omega}E - A)^{-1}\|_2^{-1}}{\sqrt{1+r^2}} = \frac{r}{\sqrt{1+r^2}}\left(\sup_{\omega}\left\|(e^{j\omega}E - \frac{A}{r})^{-1}\right\|_2\right)^{-1}.$$

The function

$$\left\|(e^{j\omega}E - \frac{A}{r})^{-1}\right\|_2 = \sigma_{\max}\left(e^{j\omega}E - \frac{A}{r}\right)^{-1}$$

is a non convex function of ω but it is easy to test if a particular value of ξ is below the optimal value

$$\xi_{\max} = \sup_{\omega} \sigma_{\max}\left(e^{j\omega}E - \frac{A}{r}\right)^{-1}. \quad (35)$$

It turns out [4] that $(e^{j\theta}E - \frac{A}{r})^{-1}$ has singular value ξ iff $e^{j\theta}$ is an eigenvalue of

$$\lambda\begin{bmatrix} E & \frac{1}{\xi}I \\ 0 & \frac{A^*}{r} \end{bmatrix} - \begin{bmatrix} \frac{A}{r} & 0 \\ \frac{1}{\xi}I & E^* \end{bmatrix}.$$

This correspondance between the generalized singular values of a transfer matrix along the unit circle and the generalized unit norm eigenvalues of a related Hamiltonian matrix yields a simple bisection algorithm [4] to compute the above maximization. This algorithm, along with additional information, then leads to a quadratically convergent algorithm.

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