Chapter 1

Projection of state space realizations

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1.0.1 Description of the problem

We consider two $m \times p$ strictly proper transfer functions

$$T(s) = C(sI_n - A)^{-1}B, \quad \hat{T}(s) = \hat{C}(sI_k - \hat{A})^{-1}\hat{B}, \tag{1.1}$$

of respective Mc Millan degrees n and $k < n$. We are interested in finding the necessary and sufficient conditions for the existence of projecting matrices $Z, V \in \mathbb{C}^{n \times k}$ such that

$$\hat{C} = CV, \quad \hat{A} = Z^T AV, \quad \hat{B} = Z^T B, \quad Z^T V = I_k \tag{1.2}$$

and in characterizing the set of all transfer functions $\hat{T}(s)$ that can be obtained via the projection equations (1.1,1.2). Only the image of the projecting matrices Z and V are important since choosing other bases satisfying the bi-orthogonality condition (1.2) amounts to a state-space transformation of the realization of $\hat{T}(s)$.

1.0.2 Motivation and history of the problem

Equation (1.2) arises naturally in the general framework of model reduction of large scale linear systems [1]. In this context we are given a transfer function $T(s)$ of Mc Millan degree n which we want to approximate by a transfer function

$\hat{T}(s)$ of smaller Mc Millan degree k, in order to solve a simpler analysis or design problem.

Classical model reduction techniques include *modal approximation* (where the dominant poles of the original transfer function are copied in the reduced order transfer function), *balanced truncation* and *optimal Hankel norm approximation* (related to the controllability and observability Grammians of the transfer function [10]). These methods either provide a global error bound between the original and reduced-order system and/or guarantee stability of the reduced order system. Unfortunately, their exact calculation involves $O(n^3)$ floating point operations even for systems with sparse model matrices $\{A, B, C\}$, which becomes untractable for a very large state dimension n.

A more recent approach involves *generalized Krylov spaces* ([3]) which are defined as the images of the *generalized Krylov matrices*

$$\left[(\sigma I_n - A)^{-1}B \cdots (\sigma I_n - A)^{-k}B\right] X, \quad X = \begin{bmatrix} x_0 & & \\ \vdots & \ddots & \\ x_{k-1} & \dots & x_0 \end{bmatrix} \tag{1.3}$$

and

$$\left[(\gamma I_n - A^T)^{-1}C^T \cdots (\gamma I_n - A^T)^{-\ell}C^T\right] Y, \quad Y = \begin{bmatrix} y_0 & & \\ \vdots & \ddots & \\ y_{\ell-1} & \dots & y_0 \end{bmatrix}. \tag{1.4}$$

These are related to the respective right and left *tangential interpolation* conditions

$$\left[T(s) - \hat{T}(s)\right] x(s) = O(s-\sigma)^k, \quad x(s) \doteq \sum_{i=0}^{k-1} x_i (s-\sigma)^i \tag{1.5}$$

and

$$\left[T(s) - \hat{T}(s)\right]^T y(s) = O(s-\gamma)^\ell, \quad y(s) \doteq \sum_{i=0}^{\ell-1} y_i (s-\gamma)^i. \tag{1.6}$$

In the most general form, one imposes such conditions in several points σ_i and γ_j as well as bi-tangential conditions (see [2],[5] for more details). The calculation of Krylov spaces and the solution of the corresponding tangential interpolation problem typically exploits the sparsity or the structure of the model matrices $\{A, B, C\}$ of the original system and are therefore efficient for large scale dynamical systems with such structure. Their drawbacks are that the resulting reduced order systems have no guaranteed error bound and that stability is not necessarily preserved.

The conjecture – and open problem – is that these methods are in fact quite universal (i.e. contain the classical methods as special cases) and can be formulated in terms of Sylvester equations and generalized eigenvalue problems. Tangential interpolation would then be a unifying procedure to construct reduced-order transfer functions in which only the interpolation points and tangential conditions need to be specified.

1.0.3 Our conjecture

The error transfer function $E(s) \doteq T(s) - \hat{T}(s)$ is realized by the following pencil :

$$M - Ns \doteq \left[\begin{array}{cc|c} A & 0 & B \\ 0 & \hat{A} & \hat{B} \\ \hline C & -\hat{C} & 0 \end{array}\right] - s \left[\begin{array}{cc|c} I_n & & \\ & I_k & \\ \hline & & 0 \end{array}\right]. \tag{1.7}$$

The transmission zeros of the system matrix (i.e. the system zeros of its minimal part) can be chosen as interpolation points between $T(s)$ and $\hat{T}(s)$ since the normal rank of $E(s)$ drops below its normal rank. Therefore one can impose interpolation conditions of the type (1.5,1.6) for appropriate choices of $x(s)$ and $y(s)$ and generalized eigenvalues σ and γ of (1.7).

Our conjecture tries to give necessary and sufficient conditions for this in terms of the system zero matrix.

Conjecture 1.0.1 *A minimal state space realization of the strictly proper transfer function $\hat{T}(s)$ of Mc Millan degree k can be obtained by projection from a minimal state space realization of the strictly proper transfer function $T(s)$ of Mc Millan degree $n > k$ if and only if there exist two regular pencils, $M_r - sN_r$ and $M_l - sN_l$ such that the matrices $L, \hat{L}, R, \hat{R}, Q_l$ and Q_r of the following equations*

$$\begin{bmatrix} A - sI_n & 0 & B \\ 0 & \hat{A} - sI_k & \hat{B} \\ C & -\hat{C} & 0 \end{bmatrix} \begin{bmatrix} RN_r \\ \hat{R}N_r \\ Q_r \end{bmatrix} = \begin{bmatrix} R \\ \hat{R} \\ 0 \end{bmatrix} (M_r - sN_r), \tag{1.8}$$

$$\begin{bmatrix} A^T - sI_n & 0 & C^T \\ 0 & \hat{A}^T - sI_k & -\hat{C}^T \\ B^T & \hat{B}^T & 0 \end{bmatrix} \begin{bmatrix} LN_l \\ -\hat{L}N_l \\ Q_l \end{bmatrix} = \begin{bmatrix} L \\ -\hat{L} \\ 0 \end{bmatrix} (M_l - sN_l), \tag{1.9}$$

satisfy the following conditions :

1. $\begin{bmatrix} N_l^T L^T & -N_l^T \hat{L}^T & Q_l^T \end{bmatrix} (M - Ns) \begin{bmatrix} RN_r \\ \hat{R}N_r \\ Q_r \end{bmatrix} = 0,$

2. $\dim(Im(RN_r)) = \dim(Im(LN_l)) = k.$

Moreover, such matrices always exist provided $2k \leq 2n - m - p$.

The conditions given by our conjecture are at least sufficient. Indeed, from equations (1.9), (1.8) and the regularity assumption of $M_r - sN_r$ and $M_l - sN_l$, it follows that

$$CRN_r = \hat{C}\hat{R}N_r \quad , \quad N_l^T L^T B = N_l^T \hat{L}^T \hat{B}. \tag{1.10}$$

Then, from condition 1,

$$N_l^T L^T RN_r = N_l^T \hat{L}^T \hat{R}N_r \quad , \quad N_l^T L^T ARN_r = N_l^T \hat{L}^T \hat{A}\hat{R}N_r. \tag{1.11}$$

Finally, conditions 1 and 2 imply that the matrices $\hat{R}N_r$ and $\hat{L}N_l$ are right invertible. Defining $Z, V \in \mathbb{C}^{n\times k}$ by

$$Z = LN_l(\hat{L}N_l)^{-r}, \quad V = RN_r(\hat{R}N_r)^{-r}, \tag{1.12}$$

we can easily verify equations (1.1) and (1.2).

We now present the link with the Krylov techniques. Equations (1.8) and (1.9) give us the following Sylvester equations :

$$ARN_r - RM_r + BQ_r = 0 \quad , \quad A^T LN_l - LM_l + C^T Q_l = 0. \tag{1.13}$$

These Sylvester equations correspond to generalized left and right eigenspaces of the system zero matrix (1.7). More precisely, $Im(RN_r)$ and $Im(LN_l)$ can be expressed as *generalized Krylov spaces* of the form (1.3), (1.4). The choice of matrices M_l, N_l, M_r, N_r, Q_l and Q_r correspond to respectively left and right tangential interpolation conditions at the eigenvalues σ_i of $(M_r - sN_r)$ and γ_j of $(M_l - sN_l)$, that are satisfied between $T(s)$ and $\hat{T}(s)$ (see [5]). These eigenspaces correspond to disjoint parts of the spectrum of $M - Ns$ such that the product $N_l^T L^T RN_r = N_l^T \hat{L}^T \hat{R}N_r$ is invertible (see [5] for more details).

In other words, our conjecture is that any projected reduced-order transfer function can be obtained by imposing some interpolation conditions or some modal approximation conditions with respect to the original transfer function. Moreover, a solution always exists provided $2k \leq 2n - m - p$ (i.e. for all $\hat{T}(s)$ of sufficiently small degree k). If this turns out to be true, we could hope to find the interpolation conditions that yield e.g. the optimal Hankel norm or optimal H_∞ norm reduced order models using cheap interpolation techniques.

1.0.4 Available results

In the single input single output case, the following result was proved in [4] :

Theorem 1.0.1 *Let $T(s) = C(sI_n - A)^{-1}B$ and $\hat{T}(s) = \hat{C}(sI_k - \hat{A})^{-1}B$ be arbitrary strictly proper SISO transfer functions of Mc Millan degrees n and $k < n$, respectively. Then $\hat{T}(s)$ can be constructed via projection of $T(s)$ using equations (2).*

Moreover, by looking carefully at the proof, Conjecture 1.0.1 is satisfied for this particular case!

Independently, Halevi recently proved in [6] new results concerning the general framework of model order reduction via projection. The unknowns Z and V have $2nk$ parameters (or degrees of freedom), while (1.2) imposes $(2k+m+p)k$ constraints. He shows that the case $k = n - \frac{m+p}{2}$ corresponds to a finite number of solutions. Moreover, for the particular case $m = p$ and $k = n - m$, he shows that any pair of projecting matrices Z, V satisfying (1.2) can be seen as generalized eigenspaces of a certain matrix pencil. The matrix pencil used by Halevi can be linked to the system zero matrix of the error transfer function defined in equation (1.7).

Matrices Z and V satisfying (1.2) are also the k trailing rows of S^{-1}, respectively columns of S which transform the system $\{A,B,C\}$ to $\{S^{-1}AS, S^{-1}B, CS\}$:

$$\left[\begin{array}{c|c} S^{-1}AS - sI_n & S^{-1}B \\ \hline CS & 0 \end{array}\right] = \left[\begin{array}{cc|c} * & * & * \\ * & \hat{A} - sI_k & \hat{B} \\ \hline * & \hat{C} & 0 \end{array}\right]. \tag{1.14}$$

The existence of projecting matrices Z, V satisfying (1.1,1.2) is therefore related to the above sub-matrix problem. A square matrix $\hat{A}$ is said to be *embedded* in a square matrix A when there exists a change of coordinates S such that $\hat{A} - sI_k$ is a sub-matrix of $S^{-1}(A - sI_n)S$. Necessary and sufficient conditions for the embedding of such monic pencils are given in [9],[8].

As for monic pencils, we say that the pencil $\hat{M} - \hat{N}s$ is embedded in the pencil $M - Ns$ when there exist invertible matrices Le, Ri such that $\hat{M} - \hat{N}s$ is a sub-matrix of $Le(M - Ns)Ri$. Finding necessary and sufficient conditions for the embedding of such general pencils is still an open problem [7]. Nevertheless, one obtains from [9],[8],[7] *necessary* conditions on $(\hat{C}, \hat{A}, \hat{B})$ and (C, A, B) for $\begin{bmatrix} \hat{A} - sI_k & \hat{B} \\ \hat{C} & 0 \end{bmatrix}$ to be embedded in $\begin{bmatrix} A - sI_n & B \\ C & 0 \end{bmatrix}$. These obviously give *necessary* conditions for the existence of projecting matrices Z, V satisfying (1.1,1.2). We hope to be able to shed new light on the necessary and sufficient conditions for the embedding problem via the connections developed in this paper.

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