Model reduction of interconnected systems

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1 Introduction

Large scale linear systems are often composed of subsystems that interconnect to each other. Instead of reducing the entire system without taking into account its structure, it might make sense to reduce each subsystem (or a few of them) by taking into account its interconnection with the others subsystems in order to approximate the entire system. This is the purpose of this paper. Interconnected systems, also called aggregated systems have been studied in the eighties [1] in the model reduction framework, but received recently only little attention [2]. It turns out that many model reduction techniques such as weighted balanced truncation, controller reduction and second-order balanced truncation can be seen as particular interconnected model reduction techniques.

First, some words about the notation. Let us assume that we are given a large scale linear system $G(s)$. This system is composed of an interconnection of k sub-systems $T_i(s)$. Each subsystem is assumed to be a linear MIMO transfer function. Subsystem $T_j(s)$ has α_j inputs denoted by the vector a_j and β_j outputs denoted by the vector b_j :

$$b_i(s) = T_i(s)a_i(s). \tag{1}$$

Define $\alpha \doteq \sum_{i=1}^{k} \alpha_i$ and $\beta \doteq \sum_{j=1}^{k} \beta_j$. The inputs of each subsystem are either outputs of other subsystems or external input that do not depend on the other subsystems.

First, one can rewrite a transfer function from its subsystems via the use of an "interconnection matrix"

$$a_i(s) = u_i(s) + \sum_{j=1}^{k} K_{i,j} b_j(s). \tag{2}$$

Sometimes it is preferable to define the external output $u_i(s)$ as a linear combination of a global external output $u(s)$. This is written as $u_i(s) = H_i u(s)$, where $H_i \in \mathbb{C}^{\alpha_i \times m}$. Define

$$a(s) \doteq \begin{bmatrix} a_1(s)^T & \dots & a_k(s)^T \end{bmatrix}^T \quad , \quad b(s) \doteq \begin{bmatrix} b_1(s)^T & \dots & b_k(s)^T \end{bmatrix}^T ,$$

$$T(s) \doteq \begin{bmatrix} T_1(s) & & \\ & \ddots & \\ & & T_k(s) \end{bmatrix} \quad , \quad H \doteq \begin{bmatrix} H_1^T & \dots & H_k^T \end{bmatrix}^T .$$

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and finally the connectivity matrix K as follows

$$K \doteq \begin{bmatrix} K_{1,1} & \dots & K_{1,k} \\ \vdots & \ddots & \vdots \\ K_{k,1} & \dots & K_{k,k} \end{bmatrix}. \tag{3}$$

The Mc Millan degree of $T_i(s)$ is n_i and (A_i, B_i, C_i, D_i) is a minimal state space realization of $T_i(s)$. From these definitions, $T(s) = C(sI - A)^{-1}B + D$ with

$$C \doteq \begin{bmatrix} C_1 & & \\ & \ddots & \\ & & C_k \end{bmatrix} \quad , \quad A \doteq \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{bmatrix},$$
$$B \doteq \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{bmatrix} \quad , \quad D \doteq \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_k \end{bmatrix}.$$

The preceding equations can be rewritten as follows :

$$a(s) = Hu(s) + Kb(s) \quad , \quad b(s) = T(s)a(s). \tag{4}$$

The output of $G(s)$, denoted by $y(s)$ is a linear function of the outputs of the subsystems: $y(s) \doteq Fb(s)$, with $F \in \mathbb{C}^{p\times\beta}$. The input of $G(s)$ is the vector $u(s)$. From (4),

$$y(s) = F(I - T(s)K)^{-1}T(s)Hu(s). \tag{5}$$

In others words, $G(s) = F(I - T(s)K)^{-1}T(s)H$, with $F \in \mathbb{C}^{p\times\beta}$. Hence, a state space realization of $G(s)$ is given by (A_G, B_G, C_G, D_G) defined by (see for instance [3], pg 66)

$$\begin{aligned} C_G &\doteq F(I - DK)^{-1}C \quad , \quad A_G \doteq A + BK(I - DK)^{-1}C, \\ B_G &\doteq B(I - KD)^{-1}H \quad , \quad D_G \doteq FD(I - KD)^{-1}H. \end{aligned} \tag{6}$$

2 Balanced Truncation

Let us consider a transfer function

$$T(s) \doteq C(sI_n - A)^{-1}B,$$

which corresponds to the linear system

$$\mathcal{S}\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases} \quad , \quad u(t) \in \mathbb{R}^{\mathrm{m}}, x(t) \in \mathbb{R}^{\mathrm{n}}, y(t) \in \mathbb{R}^{\mathrm{p}} \tag{7}$$

If the matrix A is Hurwitz, the controllability and observability gramians, denoted respectively by P and Q are the unique solutions of the following equations

$$AP + PA^T + BB^T = 0 \quad , \quad A^TQ + QA + C^TC = 0.$$

These have the following energetic interpretation. If we apply an input $u(.) \in \mathcal{L}_2[-\infty, 0]$ to the system (7) for $t < 0$, the position of the state at time $t = 0$ (by assuming the zero initial condition $x(-\infty) = 0$) is equal to

$$x(0) = \int_{-\infty}^{0} e^{-At}Bu(t)dt \doteq \mathcal{C}_o u(t).$$

By assuming that a zero input is applied to the system for $t > 0$, then for all $t \geq 0$, the output $y(.) \in \mathcal{L}_2[0, +\infty]$ of the system (7) is equal to

$$y(t) = \mathcal{C}e^{\mathcal{A}t}x(0) \doteq \mathcal{O}_b x(0).$$

The so-called controllability operator $\mathcal{C}_o : \mathcal{L}_2[-\infty, 0] \mapsto \mathbb{R}^n$ (mapping past inputs $u(.)$ to the present state) and observability operator $\mathcal{O}_b : \mathbb{R}^n \mapsto \mathcal{L}_2[0, +\infty]$ (mapping the present state to future outputs $y(.)$) also have dual operators, respectively $\mathcal{C}_o^*$ and $\mathcal{O}_b^*$. It is easy to show that the controllability and observability gramians are related to those via the identities $\mathcal{P} = \mathcal{C}_o^*\mathcal{C}_o$ and $\mathcal{Q} = \mathcal{O}_b\mathcal{O}_b^*$ ([3]).

Another physical interpretation of the gramians is the following. The controllability matrix arises from the following optimization problem. Let

$$J(v(t), a, b) \doteq \int_a^b v(t)^T v(t) dt$$

be the *energy* of the vector function $v(t)$ in the interval $[a, b]$. Then (see [4])

$$\min_{\mathcal{C}_0 u(t) = x_0} J(u(t), -\infty, 0) = x_0^T P^{-1} x_0, \tag{8}$$

and, symmetrically, we have the dual property

$$\min_{\mathcal{O}_b^* y(t) = x_0} J(y(t), -\infty, 0) = x_0^T Q^{-1} x_0. \tag{9}$$

Two essential algebraic properties of gramians $\mathcal{P}$ and $\mathcal{Q}$ are as follows. First, under a coordinate transformation $x(t) = S\bar{x}(t)$, the new gramians $\bar{\mathcal{P}}$ and $\bar{\mathcal{Q}}$ corresponding to the state-space realization $(\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{\mathcal{C}}) = (S^{-1}\mathcal{A}S, S^{-1}\mathcal{B}, \mathcal{C}S)$ undergo the following (so-called *contragradient*) transformation :

$$\bar{\mathcal{P}} = S^{-1}\mathcal{P}S^{-T} \quad \bar{\mathcal{Q}} = S^T \mathcal{Q} S. \tag{10}$$

This implies that there exists a state-space realization $(A_{bal}, B_{bal}, C_{bal})$ of $T(s)$ such that the corresponding gramians are equal and diagonal $\bar{\mathcal{P}} = \bar{\mathcal{Q}} = \Sigma$ [3]. Secondly, because these gramians appear in the solutions of the optimization problems (8) and (9), they tell something about the energy that goes through the system, and more specifically, about the distribution of this energy among the state variables. The idea is to perform a state space transformation that gives equal and diagonal gramians and to keep only the more controllable and observable states. We show how to apply the preceding idea to a set of interconnected systems.

3 Interconnected Systems Balanced Truncation

This section is inspired from [5] and [6]. Let us consider the controllability and observability gramians of $G(s)$:

$$A_G P_G + P_G A_G^T + B_G B_G^T = 0 \quad , \quad A_G^T Q_G + Q_G A_G + C_G^T C_G = 0.$$

Let us decompose

$$P_G = \begin{bmatrix} P_{1,1} & \dots & P_{1,k} \\ \vdots & \ddots & \vdots \\ P_{k,1} & \dots & P_{k,k} \end{bmatrix} \quad , \quad Q_G = \begin{bmatrix} Q_{1,1} & \dots & Q_{1,k} \\ \vdots & \ddots & \vdots \\ Q_{k,1} & \dots & Q_{k,k} \end{bmatrix},$$

where $P_{i,j} \in \mathbb{C}^{n_i \times n_j}$. If we perform a state space transformation Φ_i to the state $\bar{x}_i(t) = \Phi_i x_i(t)$ of each interconnected transfer function $T_i(s)$, or to the linear system

$$\mathcal{S}\left\{\begin{array}{rcl} \dot{x}_i(t) & = & A_i x_i(t) + B_i u_i(t) \\ y_i(t) & = & C_i x_i(t) + D_i u_i(t) \end{array}\right. , \quad u(t) \in \mathbb{R}^{\mathrm{m}}, x(t) \in \mathbb{R}^{\mathrm{n_i}}, y(t) \in \mathbb{R}^{\mathrm{p}}, \qquad (11)$$

we actually perform a state space transformation

$$\Phi \doteq \begin{bmatrix} \Phi_1 & & \\ & \ddots & \\ & & \Phi_k \end{bmatrix}$$

to the realization $(\bar{A}, \bar{B}, \bar{C}, \bar{D}) = (\Phi A \Phi^{-1}, \Phi B, C\Phi^{-1}, D)$ of $T(s)$. This, in turn implies that $(\bar{A}_G, \bar{B}_G, \bar{C}_G, \bar{D}_G) = (\Phi A_G \Phi^{-1}, \Phi B_G, C_G \Phi^{-1}, D_G)$. From this,

$$(\bar{P}_G, \bar{Q}_G) = (\Phi P_G \Phi^T, \Phi^{-T} Q_G \Phi^{-1}),$$

i.e. they also perform a contragradient transformation. This implies that $(\bar{P}_{i,i}, \bar{Q}_{ii}) = (\Phi_i P_{i,i} \Phi_i^T, \Phi_i^{-T} Q_{i,i} \Phi_i^{-1})$, which is a contra-gradient transformation that only depends on the state space transformation on x_i, i.e. on the state space associated to $T_i(s)$.

It can be shown that the minimal past input energy necessary to reach $x_i(0) = x_i^*$ over all initial input condition $x_j(0), j \neq i$, is $x_i^* P_{i,i}^{-1} x_i$. We can thus perform a bloc diagonal transformation in order to make the gramians $P_{i,i}$ and $Q_{i,i}$ both equal and diagonal :$P_{ii} = Q_{ii} = \Sigma_i$. This suggests then that we can truncate each subsystem $T_i(s)$ by deleting the states corresponding to the smallest eigenvalues of Σ_i.

If one balance and then project via the Schur complement of $P_{i,i}$ and $Q_{i,i}$, the state-space of each system C_i, A_i, B_i is sorted with respect to the optimization problem $\min_u \parallel u(t) \parallel^2$ such that $x_i(0) = x_0$ and $x_j = 0$ for $j \neq i$.

4 Krylov techniques for interconnected systems

Krylov techniques for structured systems have already been considered in the literature. See for instance [7] in the controller reduction framework, or [8] in the second-order model reduction framework. This last case has been revisited recently in [9] and [10].

The problem is the following. If one projects the state-space realizations (C_i, A_i, B_i) of the interconnected transfer functions $T_i(s)$ with projecting matrices Z_i, V_i containing Krylov subspaces, giving rise to reduced-order transfer functions $\hat{T}_i(s)$ that satisfy interpolation conditions with respect to $T_i(s)$, what are the resulting relations between $\hat{G}(s)$ and $G(s)$?

If one chooses the same interpolation conditions for each subsystem then $\hat{T}(s)$ automatically satisfies the same interpolation conditions with respect to $T(s)$. Let us investigate what this implies for $G(s)$ and $\hat{G}(s)$. Let us assume that

$$(\hat{C}, \hat{A}, \hat{B}) = (CV, Z^T A V, Z^T B)$$

such that $Z^T V = I$ and

$$\mathcal{K}_k\left((\lambda I - A)^{-1}, (\lambda I - A)^{-1} B\right) \in Im(V).$$

It is well known that $\hat{T}(s) \doteq \hat{C}(sI - \hat{A})^{-1}\hat{B}$ interpolates $T(s)$ at $s = \lambda$ up to the k first derivatives. Concerning $G(s)$, the matrices F, K, D and H are unchanged. One obtains

$$\hat{G}(s) = C_G V(sI - Z^T A_G V)^{-1} Z^T B_G + D_G.$$

In general there is no reason for V to contain the subspace $\mathcal{K}_k\left((\lambda I - A_G)^{-1}, (\lambda I - A_G)^{-1}B_G\right)$, except for the case of interpolation at infinity. Indeed, it can easily be proven recursively that

$$\mathcal{K}_k\left(A + BK(I - DK)^{-1}C, B(I - KD)^{-1}H\right) \subseteq \mathcal{K}_k\left(A, B\right).$$

It is possible to perform an *interconnected structure preserving* Krylov technique for an arbitrary point λ in the complex plane as follows.

Lemma 4.1 *Define*

$$V \in \mathbb{C}^{n \times r} \doteq \begin{bmatrix} V_1^T & \dots & V_k^T \end{bmatrix}^T,$$

such that $V_i \in \mathbb{C}^{n_i \times r}$. *Assume that*

$$\mathcal{K}_k\left((\lambda I - A_G)^{-1}, (\lambda I - A_G)^{-1}B_G\right) \subseteq Im(V).$$

Construct left projecting matrices $Z_i \in \mathbb{C}^{n_i \times r}$ *such that* $Z_i^T V_i = I_r$. *Project each subsystem as follows :*

$$(\hat{C}_i, \hat{A}_i, \hat{B}_i) \doteq (CV_i, Z_i^T A_i V_i, Z_i^T B_i).$$

Then, $\hat{G}(s)$ *interpolates* $G(s)$ *at* λ *up to the first* k *derivatives.*

Proof :
The preceding operation correspond to projecting C_G, A_G, B_G with

$$\mathcal{Z} \doteq \begin{bmatrix} Z_1 & & \\ & \ddots & \\ & & Z_k \end{bmatrix} \quad , \quad \mathcal{V} \doteq \begin{bmatrix} V_1 & & \\ & \ddots & \\ & & V_k \end{bmatrix}.$$

This implies that $\mathcal{Z}^T \mathcal{V} = I$ and $Im(V) \subseteq Im(\mathcal{V})$. This concludes the proof. □

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