Model Reduction of Interconnected Systems

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Summary. We consider a particular class of structured systems that can be modelled as a set of input/output subsystems that interconnect to each other, in the sense that outputs of some subsystems are inputs of other subsystems. Sometimes, it is important to preserve this structure in the reduced order system. Instead of reducing the entire system, it makes sense to reduce each subsystem (or a few of them) by taking into account its interconnection with the other subsystems in order to approximate the entire system in a so-called structured manner. The purpose of this paper is to present both Krylov-based and Gramian-based model reduction techniques that preserve the structure of the interconnections. Several structured model reduction techniques existing in the literature appear as special cases of our approach, permitting to unify and generalize the theory to some extent.

1 Introduction

Specialized model reduction techniques have been developed for various types of structured problems such as weighted model reduction, controller reduction and second order model reduction. Interconnected systems, also called aggregated systems, have been studied in the eighties [FB87] in the model reduction framework, but they have not received a lot of attention lately. This is in contrast with controller and weighted SVD-based model reduction techniques, which have been extensively studied [AL89, Enn84]. Controller reduction Krylov techniques have also been considered recently in [GBAG04]. It turns out that many structured systems can be modelled as particular cases of more general *interconnected* systems defined below (the behavioral approach [PW98] for interconnected systems is not considered here).

In this paper, we define an *interconnected system* as a linear system $G(s)$ composed of an interconnection of k sub-systems $T_i(s)$. Each subsystem is assumed to be a linear MIMO transfer function. Subsystem $T_i(s)$ has α_i inputs denoted by the vector a_i and β_i outputs denoted by the vector b_i :

$$b_i(s) = T_i(s)a_i(s). \tag{1}$$

Note that these inputs and outputs can also be viewed as internal variables of the interconnected system. The input $a_i(s)$ of each subsystem is a linear combination of the outputs of all subsystems and of the external input $u(s) \in \mathbb{R}^m(s)$:

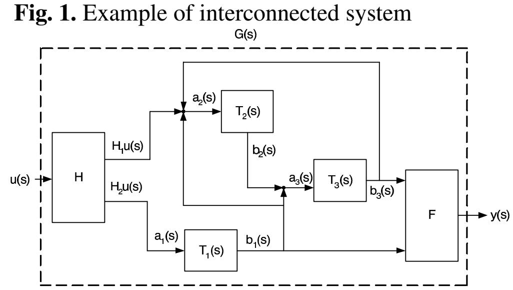
$$a_i(s) = H_i u(s) + \sum_{j=1}^{k} K_{i,j} b_j(s), \tag{2}$$

where $H_i \in \mathbb{R}^{\alpha_i \times m}$. The output $y(s) \in \mathbb{R}^p(s)$ of $G(s)$ is a linear function of the outputs of the subsystems:

$$y(s) = \sum_{i=1}^{k} F_i b_i(s), \tag{3}$$

with $F_i \in \mathbb{R}^{p \times \beta_i}$. Figure 1 gives an example of an interconnected system $G(s)$ composed of three subsystems.

Fig. 1. Example of interconnected system



We now introduce some notation in order to rewrite this in a block form. The matrix I_n denotes the identity matrix of size n and the matrix $0_{p,q}$ the $p \times q$ zero matrix. If $M_1, \ldots M_k$ is a set of matrices, then the matrix $diag\{M_1, \ldots, M_k\}$ denotes the block diagonal matrix

$$\begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_k \end{bmatrix}.$$

We also define $\alpha := \sum_{i=1}^{k} \alpha_i$ and $\beta := \sum_{i=1}^{k} \beta_i$. If the transfer functions $T_i(s) \in \mathbb{R}^{\beta_i \times \alpha_i}(s)$ are rational matrix function with real coefficients, then (1) can be rewritten as $b(s) = T(s)a(s)$, where

$$b(s) := \begin{bmatrix} b_1(s) \\ \vdots \\ b_k(s) \end{bmatrix}, \quad T(s) := diag\{T_1(s), \ldots, T_k(s)\}, \quad a(s) := \begin{bmatrix} a_1(s) \\ \vdots \\ a_k(s) \end{bmatrix}, \tag{4}$$

are respectively in $\mathbb{R}^{\beta}(s)$, $\mathbb{R}^{\beta\times\alpha}(s)$ and $\mathbb{R}^{\alpha}(s)$. If we also define $F \in \mathbb{R}^{p\times\beta}$, $K \in \mathbb{R}^{\alpha\times\beta}$ and $H \in \mathbb{R}^{\alpha\times m}$, as follows :

$$F := \begin{bmatrix} F_1 \dots F_k \end{bmatrix}, \quad K := \begin{bmatrix} K_{1,1} & \dots & K_{1,k} \\ \vdots & \ddots & \vdots \\ K_{k,1} & \dots & K_{k,k} \end{bmatrix}, \quad H := \begin{bmatrix} H_1 \\ \vdots \\ H_k \end{bmatrix}, \tag{5}$$

then (2),(3) can then be rewritten as follows :

$$a(s) = Hu(s) + Kb(s), \quad y(s) = Fb(s), \tag{6}$$

from which it easily follows that

$$y(s) = F(I_\beta - T(s)K)^{-1}T(s)Hu(s). \tag{7}$$

We assume that the Mc Millan degree of $T_i(s)$ is n_i and that (A_i, B_i, C_i, D_i) is a minimal state space realization of $T_i(s)$. If we define $n := \sum_{i=1}^{k} n_i$, then a realization for $T(s)$ is given by $C(sI_n - A)^{-1}B + D$ with

$$\begin{aligned} A &:= diag\{A_1, \dots, A_k\}\ , \quad B := diag\{B_1, \dots, B_k\}, \\ C &:= diag\{C_1, \dots, C_k\}\ , \quad D := diag\{D_1, \dots, D_k\}. \end{aligned} \tag{8}$$

In others words, $G(s) = F(I_\beta - T(s)K)^{-1}T(s)H$ and a state space realization of $G(s)$ is given by (A_G, B_G, C_G, D_G) (see for instance [ZDG96]), where

$$\begin{aligned} A_G &:= A + BK(I_\beta - DK)^{-1}C\ , \quad B_G := B(I_\alpha - KD)^{-1}H, \\ C_G &:= F(I_\beta - DK)^{-1}C\ , \quad D_G := FD(I_\alpha - KD)^{-1}H. \end{aligned} \tag{9}$$

If all the transfer functions are strictly proper, i.e. $D = 0$, the state space realization (9) of $G(s)$ reduces to :

$$A_G = A + BKC, \quad B_G = BH, \quad C_G = FC, \quad D_G = 0.$$

Let us finally remark that if all systems are connected in parallel, i.e. $K = 0$, then $G(s) = FT(s)H$.

The problem of *interconnected systems model reduction* proposed here consists in reducing some (e.g. one) of the subsystems $T_i(s)$ in order to approximate the global mapping from $u(s)$ to $y(s)$ and not the internal mappings from $a_i(s)$ to $b_i(s)$.

This paper is organized as follows. After some preliminary results, a Balanced Truncation framework for interconnected systems is derived in Section 2. Krylov model reduction techniques for interconnected systems are presented in Section 3. In Section 4, several connections with existing model reduction techniques for structured systems are given, and Section 5 contains some concluding remarks.

2 Interconnected Systems Balanced Truncation

We first recall the well-known Balanced Truncation method and emphasize their energetic interpretation. We then show how to extend Balanced Truncation to the so-called *Interconnected System Balanced Truncation*.

We consider a general transfer function $T(s) := C(sI_n - A)^{-1}B + D$ which corresponds to the linear system

$$\mathcal{S}\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t). \end{cases} \tag{10}$$

If the matrix A is Hurwitz, the controllability and observability Gramians, denoted respectively by P and Q are the unique solutions of the following equations

$$AP + PA^T + BB^T = 0\ , \quad A^TQ + QA + C^TC = 0.$$

If we apply an input $u(.) \in \mathcal{L}_2[-\infty, 0]$ to the system (10) for $t < 0$, the position of the state at time $t = 0$ (by assuming the zero initial condition $x(-\infty) = 0$) is a linear function of $u(t)$ given by the convolution

$$x(0) = \mathcal{C}_o(u(t)) := \int_{-\infty}^{0} e^{-At}Bu(t)dt.$$

By assuming that a zero input is applied to the system for $t > 0$, then for all $t \geq 0$, the output $y(.) \in \mathcal{L}_2[0, +\infty]$ of the system (10) is a linear function of $x(0)$, given by

$$y(t) = \mathcal{O}_b(x(0)) := Ce^{At}x(0).$$

The so-called controllability operator $\mathcal{C}_o : \mathcal{L}_2[-\infty, 0] \mapsto \mathbb{R}^n$ (mapping past inputs $u(.)$ to the present state) and observability operator $\mathcal{O}_b : \mathbb{R}^n \mapsto \mathcal{L}_2[0, +\infty]$ (mapping the present state to future outputs $y(.)$) have dual operators, respectively denoted by $\mathcal{C}_o^*$ and $\mathcal{O}_b^*$ (see [Ant05]).

A physical interpretation of the Gramians is the following. The controllability matrix arises from the following optimization problem. Let

$$J(v(t), a, b) := \int_a^b v(t)^T v(t)dt$$

be the *energy* of the vector function $v(t)$ in the interval $[a, b]$. Then [Glo84]

$$\min_{\mathcal{C}_0 u(t) = x_0} J(u(t), -\infty, 0) = x_0^T P^{-1} x_0, \tag{11}$$

and, by duality, we have that

$$\min_{\mathcal{O}_b^* y(t) = x_0} J(y(t), -\infty, 0) = x_0^T Q^{-1} x_0. \tag{12}$$

Essential properties of the Gramians P and Q are as follows. First, under a coordinate transformation $x(t) = S\bar{x}(t)$, the new Gramians $\bar{P}$ and $\bar{Q}$ corresponding to the state-space realization $(\bar{A}, \bar{B}, \bar{C}) = (S^{-1}AS, S^{-1}B, CS)$ undergo the following (so-called *contragradient*) transformation :

$$\bar{P} = S^{-1}PS^{-T}, \quad \bar{Q} = S^TQS. \tag{13}$$

This implies that there exists a state-space realization $(A_{bal}, B_{bal}, C_{bal})$ of $T(s)$ such that the corresponding Gramians are equal and diagonal $\bar{P} = \bar{Q} = \Sigma$ [ZDG96]. Secondly, because these Gramians appear in the solutions of the optimization problems (11) and (12), they tell something about the energy that goes through the system, and more specifically, about the distribution of this energy among the state variables. The idea of the Balanced Truncation model reduction framework is to perform a state space transformation that yields equal and diagonal Gramians and to keep only the most controllable and observable states. If the original transfer function is stable, the reduced order transfer function is guaranteed to be stable and an a priori global error bound between both systems is available [Ant05].

If the standard balanced truncation technique is applied to the state space realization (A, B, C) (8) of an interconnected system, the structure of the subsystems is lost in the resulting reduced order transfer function. We show then how to preserve the structure in the balancing process. We first recall a basic lemma that will be used in the sequel.

Lemma 1. *Let $x_i \in \mathbb{R}^{n_i}$ and $M_{i,j} \in \mathbb{R}^{n_i \times n_j}$ for $1 \leq i \leq k$ and define*

$$x := \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}, \quad M := \begin{bmatrix} M_{1,1} & \dots & M_{1,k} \\ \vdots & \ddots & \vdots \\ M_{k,1} & \dots & M_{k,k} \end{bmatrix}.$$

Assume M to be positive definite and consider the product

$$J(x, M) := x^T M^{-1} x.$$

Then, for any fixed $x_i \in \mathbb{R}^{n_i \times n_i}$,

$$J(x, M)_{x_j=0, j\neq i} = x_i^T \left(M_{i,i} - M_{i,j} M_{j,j}^{-1} M_{j,i}\right)^{-1} x_i, \tag{14}$$

and

$$\min_{x_j, j\neq i} J(x, M) = x_i^T M_{i,i}^{-1} x_i. \tag{15}$$

Proof. Without loss of generality, let us assume that $i = 1$. For ease of notation, define $y := \left[x_2^T \dots x_k^T\right]^T$ and $\begin{bmatrix} N_{1,1} & N_{1,2} \\ N_{1,2}^T & N_{2,2} \end{bmatrix} = N := M^{-1}$ with $N_{1,1} \in \mathbb{R}^{n_1 \times n_1}$. We obtain the following expression

$$J(x, M) = x_1^T N_{1,1} x_1 + 2x_1^T N_{1,2} y + y^T N_{2,2} y. \tag{16}$$

For $y = 0$ and using the Schur complement formula for the inverse of a matrix, we retrieve (14). In order to prove (15) we note that N is positive definite since M is positive definite. This implies that $N_{1,1}$ and $N_{2,2}$ are positive definite. $J(x, M)$ is a quadratic form and the Hessian of $J(x, M)$ with respect to y is equal to $N_{2,2}$. The minimum is then obtained by annihilating the gradient :

$$\frac{tialJ(x,M)}{tialy} = 2N_{1,2}^T x_1 + 2N_{2,2} y,$$

which is obtained for $y = -N_{2,2}^{-1} N_{1,2}^T x_1$ and yields

$$\min_y J(x,M) = x_1^T N_{1,1} x_1 - x_1^T N_{1,2} N_{2,2}^{-1} N_{1,2}^T x_1 = x_1^T M_{1,1}^{-1} x_1.$$

The last equality is again obtained by using the Schur complement formula. □

Let us now consider the controllability and observability Gramians of $G(s)$:

$$A_G P_G + P_G A_G^T + B_G B_G^T = 0, \quad A_G^T Q_G + Q_G A_G + C_G^T C_G = 0, \tag{17}$$

and let us partition them as follows :

$$P_G = \begin{bmatrix} P_{1,1} & \dots & P_{1,k} \\ \vdots & \ddots & \vdots \\ P_{k,1} & \dots & P_{k,k} \end{bmatrix}, \quad Q_G = \begin{bmatrix} Q_{1,1} & \dots & Q_{1,k} \\ \vdots & \ddots & \vdots \\ Q_{k,1} & \dots & Q_{k,k} \end{bmatrix}, \tag{18}$$

where $P_{i,j} \in \mathbb{R}^{n_i \times n_j}$. If we perform a state space transformation S_i to the state $x_i(t) = S_i \bar{x}_i(t)$ of each interconnected transfer function $T_i(s)$, we actually perform a state space transformation

$$S := diag\{S_1, \dots, S_k\}$$

to the realization $(\bar{A}, \bar{B}, \bar{C}, \bar{D}) = (S^{-1}AS, S^{-1}B, CS, D)$ of $T(s)$. This, in turn, implies that $(\bar{A}_G, \bar{B}_G, \bar{C}_G, \bar{D}_G) = (S^{-1}A_G S, S^{-1}B_G, C_G S, D_G)$ and

$$(\bar{P}_G, \bar{Q}_G) = (S^{-1} P_G S^{-T}, S^T Q_G S),$$

i.e. they undergo a contragradient transformation. This implies that $(\bar{P}_{i,i}, \bar{Q}_{i,i}) = (S_i^{-1} P_{i,i} S_i^{-T}, S_i^T Q_{i,i} S_i)$, which is a contra-gradient transformation that only depends on the state space transformation on x_i, i.e. on the state space associated to $T_i(s)$.

Let us recall that the minimal past energy necessary to reach $x_i(0) = x_i$ for each $1 \le i \le k$ with the pair (A_G, B_G) is given by the expression

$$\begin{bmatrix} x_1^T & \dots & x_k^T \end{bmatrix} P_G^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}. \tag{19}$$

The following result is then a consequence of Lemma 1.

Lemma 2. *With the preceding notation, the minimal past input energy*

$$J := \int_{-\infty}^{0} u(t)^T u(t) dt$$

needed to apply to the interconnected transfer function $G(s)$ in order that for subsystem i at time $t = 0$, $x_i(0) = x_i$ over all initial input condition $x_j(0), j \neq i$, is given by

$$x_i^T P_{i,i}^{-1} x_i.$$

Moreover, the minimal input needed in order that for subsystem i at time $t = 0$, $x_i(0) = x_i$ and that for all the other subsystems, $x_j(0) = 0, j \neq i$, is given by

$$x_i^T (P_G^{-1})_{i,i} x_i,$$

where $(P_G^{-1})_{i,i}$ is the i,i block of the inverse of P_G, and this block is equal to the inverse of the Schur complement of $P_{i,i}$.

Finally,

$$0 < P_{i,i}^{-1} \leq (P_G^{-1})_{i,i}. \tag{20}$$

Proof. The two first results are direct consequences of Lemma 1. Let us prove (20). For any nonzero vector x_i, the minimum energy necessary for subsystem i at time $t = 0$ to reach $x_i(0) = x_i$ over all initial input conditions $x_j(0), j \neq i$, cannot be larger than by imposing $x_j(0) = 0, j \neq i$. This implies that for any nonzero vector v,

$$v^T \left((P_G^{-1})_{i,i} - P_{i,i}^{-1} \right) v \geq 0.$$

□

Similar energy interpretations hold for the diagonal blocks of the observability matrix Q_G and of its inverse.

Because of Lemma 2, it makes sense to truncate the part of the state x_i of each subsystem $T_i(s)$ corresponding to the smallest eigenvalues of the product $P_{i,i}Q_{i,i}$. We can thus perform a block diagonal transformation in order to make the Gramians $P_{i,i}$ and $Q_{i,i}$ both equal and diagonal : $P_{i,i} = Q_{i,i} = \Sigma_i$. Then, we can truncate each subsystem $T_i(s)$ by deleting the states corresponding to the smallest eigenvalues of Σ_i. This is resumed in the following *Interconnected Systems Balanced Truncation* (ISBT) Algorithm. Let $(A_G, B_G, C_G, D_G) \sim G(s)$, where $G(s)$ is an interconnection of k subsystems

$$(A_i, B_i, C_i, D_i) \sim T_i(s),$$

of order n_i. In order to construct a reduced order system $\widehat{G}(s)$ while preserving the interconnections, proceed as follows.

ISBT Algorithm

1. Compute the Gramians P_G and Q_G satisfying (17).
2. For each subsystem $T_i(s)$ requiring an order reduction, perform the contragradient transformation S_i in order to make the Gramians $P_{i,i}$ and $Q_{i,i}$ equal and diagonal.

3. For each subsystem (A_i, B_i, C_i, D_i), keep only the space of states corresponding to the largest eigenvalues of $P_{i,i} = Q_{i,i} = \Sigma_i$, giving the reduced subsystems $\widehat{T}_i(s)$.
4. Define
$$\widehat{G}(s) = F(I_\beta - \widehat{T}(s)K)^{-1}\widehat{T}(s)H,$$
with $\widehat{T}(s) := diag\{\widehat{T}_i(s)\}$.

Remark 1. A variant of the ISBT Algorithm consists in performing a *balance and truncate* procedure for each subsystem $T_i(s)$ with respect to the Schur complements of $P_{i,i}$ and $Q_{i,i}$ instead of $P_{i,i}$ and $Q_{i,i}$. From Lemma 2, this corresponds to sorting the state-space of each system (A_i, B_i, C_i) with respect to the optimization problem $\min_u allelu(t)allel^2$ such that $x_i(0) = x_i$ and $x_j(0) = 0$ for $j \neq i$. Mixed strategies are also possible (see for instance [VA03] in the Controller Order Reduction framework).

It should be mentioned that a related balanced truncation approach for second order systems can be found in [MS96, CLVV06].

A main criticism concerning the ISBT Algorithm is that the reduced order system is not guaranteed to be stable. If all the subsystems $T_i(s)$ are stable, it is possible to impose all the subsystems $\widehat{T}_i(s)$ to remain stable by following a technique similar to that described in [WSL99]. Let us consider the $(1,1)$ block of P_G and Q_G, i.e. $P_{1,1}$ and $Q_{1,1}$. These Gramians are positive definite because P_G and Q_G are assumed to be positive definite (here, $G(s)$ is assumed stable and (A_G, B_G, C_G, D_G) is a minimal realization). From (17), $P_{1,1}$ and $Q_{1,1}$ satisfy the Lyapunov equation

$$A_1 P_{1,1} + P_{1,1} A_1 + X = 0, \quad A_1^T Q_{1,1} + Q_{1,1} A_1 + Y = 0$$

where the symmetric matrices X and Y are not necessary positive definite. If one modifies X and Y to positive semi-definite matrices $\bar{B}\bar{B}^T$ and $\bar{C}^T\bar{C}$, one is guaranteed to obtain a stable reduced system $\widehat{T}_1(s)$. The main criticism about this technique is that the energetic interpretation of the modified Gramians is lost.

3 Krylov techniques for interconnected systems

Krylov subspaces appear naturally in interpolation-based model reduction techniques. Let us recall that for any matrix M, $Im(X)$ is the space spanned by the columns of M.

Definition 1. *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. The Krylov matrix $K_k(A,B) \in \mathbb{R}^{n\times km}$ is defined as follows*

$$K_k(A,B) := \begin{bmatrix} B \ AB \ \dots \ A^{k-1}B \end{bmatrix}.$$

The subspace spanned by the columns of $K_k(A,B)$ is denoted by $\mathcal{K}_k(A,B)$.

Krylov techniques have already been considered in the literature for particular cases of structured systems. See for instance [SA86] in the controller reduction framework, or [SC91] in the second-order model reduction framework. This last case has been revisited recently in [Fre05] and [VV04]. But, to our knowledge, it is the first time they are studied in the general framework of *Interconnected Systems*.

The problem is the following. If one projects the state-space realizations (A_i, B_i, C_i) of the interconnected transfer functions $T_i(s)$ with projecting matrices Z_i, V_i derived from Krylov subspaces, this yields reduced-order transfer functions $\widehat{T}_i(s)$ that satisfy interpolation conditions with respect to $T_i(s)$; what are then the resulting relations between $\widehat{G}(s)$ and $G(s)$?

If one imposes the same interpolation conditions for every pair of subsystems $T_i(s)$ and $\widehat{T}_i(s)$, then the same interpolation conditions hold between the block diagonal transfer functions $T(s)$ and $\widehat{T}(s)$ as well. Let us investigate what this implies for $G(s)$ and $\widehat{G}(s)$. Assume that

$$(\widehat{A}, \widehat{B}, \widehat{C}) = (Z^T AV, Z^T B, CV)$$

such that $Z^T V = I$ and

$$\mathcal{K}_k\left((\lambda I - A)^{-1}, (\lambda I - A)^{-1} B\right) \subseteq Im(V).$$

In such a case, it is well known that [VS87, Gri97] $\widehat{T}(s) := \widehat{C}(sI - \widehat{A})^{-1}\widehat{B} + D$ interpolates $T(s) := C(sI - A)^{-1}B + D$ at $s = \lambda$ up to the k first derivatives. Concerning $G(s)$, the matrices F, K, D and H are unchanged, from which it easily follows that

$$\widehat{G}(s) = C_G V(sI - Z^T A_G V)^{-1} Z^T B_G + D_G.$$

It can easily be proved recursively that

$$\mathcal{K}_k\left(A_G, B_G\right) = \mathcal{K}_k\left(A + BK(I - DK)^{-1}C, B(I - KD)^{-1}H\right) \subseteq \mathcal{K}_k\left(A, B\right),$$

and it turns out that such a result holds for arbitrary interpolation points in the complex plane, as shown in the following lemma.

Lemma 3. *Let $\lambda \in \mathbb{C}$ be a point that is neither an eigenvalue of A nor an eigenvalue of A_G (defined in* (9)*). Then*

$$\mathcal{K}_k\left((\lambda I - A_G)^{-1}, (\lambda I - A_G)^{-1} B_G\right) \subseteq \mathcal{K}_k\left((\lambda I - A)^{-1}, (\lambda I - A)^{-1} B\right), \quad (21)$$

$$\mathcal{K}_k\left((\lambda I - A_G)^{-T}, (\lambda I - A_G)^{-T} C_G^T\right) \subseteq \mathcal{K}_k\left((\lambda I - A)^{-T}, (\lambda I - A)^{-T} C^T\right). \quad (22)$$

Proof. Only (21) will be proved. An analog proof can be given for (22). First, let us prove that the column space of $(\lambda I - A_G)^{-1} B_G$ is included in the column space of $(\lambda I - A)^{-1}B$. In order to simplify the notation, let us define the following matrices

$$M := (\lambda I_n - A)^{-1}B, \quad X := K(I_\beta - DK)^{-1}C, \quad G := (I_\alpha - KD)^{-1}H. \quad (23)$$

From the identity $(I - MX)^{-1}M = M(I - XM)^{-1}$, it then follows that

$$(\lambda I - A_G)^{-1} B_G = (\lambda I - A - BX)^{-1} BG = (I - MX)^{-1} MG = M (I - XM)^{-1} G.$$

This clearly implies that the column space of $(\lambda I - A_G)^{-1} B_G$ is included in the column space of $M = (\lambda I - A)^{-1} B$. Let us assume that

$$\mathcal{K}_{k-1}\left((\lambda I - A_G)^{-1}, (\lambda I - A_G)^{-1} B_G\right) \subseteq \mathcal{K}_{k-1}\left((\lambda I - A)^{-1}, (\lambda I - A)^{-1} B\right),$$

and prove that this implies that

$$\mathcal{K}_{k}\left((\lambda I - A_G)^{-1}, (\lambda I - A_G)^{-1} B_G\right) \subseteq \mathcal{K}_{k}\left((\lambda I - A)^{-1}, (\lambda I - A)^{-1} B\right). \quad (24)$$

Since the image of $(\lambda I - A_G)^{-k+1} B_G$ belongs to $\mathcal{K}_{k-1}\left((\lambda I - A)^{-1}, (\lambda I - A)^{-1} B\right)$, there exists a matrix Y such that

$$(\lambda I - A_G)^{-k+1} B_G = K_{k-1}\left((\lambda I - A)^{-1}, (\lambda I - A)^{-1} B\right) Y.$$

One obtains then that $(\lambda I - A_G)^{-k} B_G$ equals

$$(\lambda I - A_G)^{-1} (\lambda I - A_G)^{-k+1} B_G = \sum_{i=0}^{\infty} (MX)^i (\lambda I - A)^{-1} K_{k-1}\left((\lambda I - A)^{-1}, M\right) Y.$$

Note that

$$Im\left((\lambda I - A)^{-1} K_{k-1}\left((\lambda I - A)^{-1}, M\right)\right) \subseteq \mathcal{K}_k\left((\lambda I - A)^{-1}, M\right).$$

Moreover, for any integer $i > 0$, it is clear that

$$Im\left((MX)^i\right) \in Im(M).$$

This proves that (24) is satisfied. □

Thanks to the preceding lemma, there are at least two ways to project the subsystems $T_i(s)$ in order to satisfy a set of interpolation conditions using Krylov subspaces as follows.

Lemma 4. *Let $\lambda \in \mathbb{C}$ be neither a pole of $T(s)$ nor a pole of $G(s)$. Define*

$$V := \begin{bmatrix} V_1 \\ \vdots \\ V_k \end{bmatrix} \in \mathbb{C}^{n \times r},$$

such that $V_i \in \mathbb{C}^{n_i \times r}$. Assume that either

$$\mathcal{K}_k\left((\lambda I - A_G)^{-1}, (\lambda I - A_G)^{-1} B_G\right) \subseteq Im(V). \quad (25)$$

or

$$\mathcal{K}_k\left((\lambda I - A)^{-1}, (\lambda I - A)^{-1} B\right) \subseteq Im(V). \quad (26)$$

Construct matrices $Z_i \in \mathbb{C}^{n_i \times r}$ such that $Z_i^T V_i = I_r$. Project each subsystem as follows :

$$(\widehat{A}_i, \widehat{B}_i, \widehat{C}_i) := (Z_i^T A_i V_i, Z_i^T B_i, C_i V_i). \quad (27)$$

Then, $\widehat{G}(s)$ interpolates $G(s)$ at λ up to the first k derivatives.

Proof. First note that (26) implies (25) because of Lemma 3, and that (27) amounts to projecting (A, B, C) to $(\widehat{A}, \widehat{B}, \widehat{C}) := (\mathcal{Z}^T A\mathcal{V}, \mathcal{Z}^T B, C\mathcal{V})$ with

$$\mathcal{Z} := diag\{Z_1, \ldots, Z_k\}, \quad \mathcal{V} := diag\{V_1, \ldots, V_k\} \tag{28}$$

and hence also (A_G, B_G, C_G) to $(\widehat{A}_G, \widehat{B}_G, \widehat{C}_G) := (\mathcal{Z}^T A_G\mathcal{V}, \mathcal{Z}^T B_G, C_G\mathcal{V})$. The interpolation property then follows from $\mathcal{Z}^T\mathcal{V} = I$ and

$$\mathcal{K}_k\left((\lambda I - A_G)^{-1}, (\lambda I - A_G)^{-1}B_G\right) \subseteq Im(V) \subseteq Im(\mathcal{V}), \tag{29}$$

which concludes the proof. □

In some contexts, such as controller reduction or weighted model reduction, one does not construct a reduced order transfer function $\widehat{G}(s)$ by projecting the state spaces of all the subsystems (A_i, B_i, C_i) but one may choose to project only some or one of the subsystems. Let us consider this last possibility.

Corollary 1. *Under the assumptions* (26) *or* (25) *of Lemma 4, $\widehat{G}(s)$ interpolates $G(s)$ at λ up to the first k derivatives even if only one subsystem i is projected according to* (27) *and all the other subsystems are kept unchanged.*

Proof. This corresponds to $(\widehat{A}_G, \widehat{B}_G, \widehat{C}_G) := (\mathcal{Z}^T A_G\mathcal{V}, \mathcal{Z}^T B_G, C_G\mathcal{V})$ with

$$\mathcal{Z} := diag\{I_{\sum_{j=1}^{i-1} n_j}, Z_i, I_{\sum_{j=i+1}^{k} n_j}\}, \quad \mathcal{V} := diag\{I_{\sum_{j=1}^{i-1} n_j}, V_i, I_{\sum_{j=i+1}^{k} n_j}\} \tag{30}$$

Again we have $\mathcal{Z}^T\mathcal{V} = I$ and $Im(V) \subseteq Im(\mathcal{V})$, which concludes the proof. □

Remark 2. Krylov techniques have recently been generalized for MIMO systems with the *tangential interpolation* framework [GVV04]. It is also possible to project the subsystems $T_i(s)$ in such a way that the reduced interconnected transfer function $\widehat{G}(s)$ satisfies a set of tangential interpolation conditions with respect to the original interconnected transfer function $G(s)$, but special care must be taken. Indeed, Lemma 3 is generically not true anymore for generalized Krylov subspaces corresponding to tangential interpolation conditions. In other words, the column space of the matrix

$$\mathcal{K}_k\left((\lambda I - A_G)^{-1}B_G, (\lambda I - A_G)^{-1}, Y\right) :=$$
$$\left[(\lambda I - A_G)^{-1}B_G \ldots (\lambda I - A_G)^{-k}B_G\right] \begin{bmatrix} y_0 & \cdots & y_{k-1} \\ & \ddots & \vdots \\ & & y_0 \end{bmatrix}$$

is in general not contained in the column space of the matrix

$$\mathcal{K}_k\left((\lambda I - A)^{-1}B, (\lambda I - A)^{-1}, Y\right) :=$$
$$\left[(\lambda I - A)^{-1}B \ldots (\lambda I - A)^{-k}B\right] \begin{bmatrix} y_0 & \cdots & y_{k-1} \\ & \ddots & \vdots \\ & & y_0 \end{bmatrix}.$$

In such a case, interchanging matrices (A_G, B_G, C_G) by (A, B, C), as done in Lemma 4 and Corollary 1 is not always permitted. Nevertheless, Lemma 4 and Corollary 1 can be extended to the tangential interpolation framework by projecting the state space realizations (A_i, B_i, C_i) with generalized Krylov subspaces of the form $\mathcal{K}_k\left((\lambda I - A_G)^{-1}B_G, (\lambda I - A_G)^{-1}, Y\right)$ and not of the form $\mathcal{K}_k\left((\lambda I - A)^{-1}B, (\lambda I - A)^{-1}, Y\right)$.

4 Examples of Structured Model Reduction Problems

As we will see in this section, many structured systems can be modelled as *interconnected systems*. Three well known structured systems are presented, namely *weighted* systems, *second-order* systems and *controlled* systems. For each of these specific cases one recovers well-known formulas. It turns out that several existing model reduction techniques for structured systems are particular cases of our ISBT Algorithm.

The preceding list is by no means exhaustive. For instance, because linear fractional transforms correspond to making a constant feedback to a part of the state, this can also be described by an interconnected system. Periodic systems are also a typical example of interconnected system that is not considered below.

Weighted Model Reduction

As a first example, let us consider the following *weighted* transfer function :

$$y(s) = W_{out}(s)T(s)W_{in}(s)u(s) := G(s)u(s).$$

Let (A_o, B_o, C_o, D_o), (A, B, C, D) and (A_i, B_i, C_i, D_i) be the state space realizations of respectively $W_{out}(s)$, $T(s)$ and $W_{in}(s)$, of respective order n_o, n and n_i. A state space realization (A_G, B_G, C_G, D_G) of $G(s)$ is given by

$$\left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right] := \left[\begin{array}{ccc|c} A_o & B_oC & B_oDC_i & B_oDD_i \\ 0 & A & BC_i & BD_i \\ 0 & 0 & A_i & B_i \\ \hline C_o & D_oC & D_oDC_i & D_oDD_i \end{array}\right]. \tag{31}$$

The transfer function $G(s)$ corresponds to the *interconnected* system $\mathcal{S}$ with

$$\mathcal{S} \;:\; \begin{cases} b_1(s) = W_o(s)a_1(s),\; b_2(s) = T(s)a_2(s), \\ b_3(s) = W_i(s)a_3(s),\; y(s) = b_1(s), \\ a_1(s) = b_2(s),\; a_2(s) = b_3(s),\; a_3 = u(s) \end{cases},$$

and

$$H = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \quad , \quad K = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad F = \begin{bmatrix} I & 0 & 0 \end{bmatrix}.$$

A frequency weighted balanced reduction method was first introduced by Enns [Enn84, ZDG96]. Its strategy is the following. Note that Enns assumes that $D = 0$ (otherwise D can be added to $\widehat{T}(s)$).

ENNS Algorithm

1. Compute the Gramians P_G and Q_G satisfying (17) with (A_G, B_G, C_G, D_G) defined in (31).

2. Perform a state space transformation on (A, B, C) in order to obtain $P = Q = \Sigma$ diagonal, where P and Q are the diagonal blocs of P_G and Q_G corresponding to the $T(s)$:

$$P = \begin{bmatrix} 0_{n,n_o} \; I_n \; 0_{n,n_i} \end{bmatrix} P_G \begin{bmatrix} 0_{n_o,n} \\ I_n \\ 0_{n_i,n} \end{bmatrix}, \quad Q = \begin{bmatrix} 0_{n,n_o} \; I_n \; 0_{n,n_i} \end{bmatrix} Q_G \begin{bmatrix} 0_{n_o,n} \\ I_n \\ 0_{n_i,n} \end{bmatrix}. \tag{32}$$

3. Truncate (A, B, C) by keeping only the part of the state space corresponding to the largest eigenvalues of Σ.

It is clear the ENNS Algorithm is exactly the same as the ISBT Algorithm applied to weighted systems. As for the ISBT Algorithm, there is generally no known a priori error bound for the approximation error and the reduced order model is not guaranteed to be stable either.

There exists other weighted model reduction techniques. See for instance [WSL99] where an elegant error bound is derived.

A generalization of weighted systems are *cascaded systems*. If we assume that the interconnected systems are such that the input of $T_i(s)$ is the output of $T_{i+1}(s)$, we obtain a structure similar than for the weighted case. The matrix K has then the form

$$K = \begin{bmatrix} 0 & I_{\beta_1} & & \\ & \ddots & \ddots & \\ & & \ddots & I_{\beta_{k-1}} \\ & & & 0 \end{bmatrix}.$$

Second-Order systems

Second order systems arise naturally in many areas of engineering (see, for example, [Pre97, Rub70, WJ87]) with the following form :

$$\begin{cases} M\ddot{q}(t) + D\dot{q}(t) + Sq(t) = F_{in}\, u(t), \\ \qquad\qquad\qquad\quad\;\; y(t) = F_{out}\, q(t). \end{cases} \tag{33}$$

We assume that $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, $q(t) \in \mathbb{R}^n$, $F_{in} \in \mathbb{R}^{n\times m}$, $F_{out} \in \mathbb{R}^{p\times n}$, and $M, D, S \in \mathbb{R}^{n\times n}$ with M invertible. For mechanical systems the matrices M, D and S represent, respectively, the *mass* (or *inertia*), *damping* and *stiffness* matrices, $u(t)$ corresponds to the vector of *external* forces, F_{in} is the input distribution matrix, $y(\cdot)$

is the output measurement vector, F_{out} is the output measurement matrix, and $q(t)$ to the vector of *internal generalized coordinates*.

Second-Order systems can be seen as an interconnection of two subsystems as follows. For simplicity, the mass matrix M is assumed equal to the identity matrix. Define $T_1(s)$ and $T_2(s)$ corresponding to the following system :

$$\begin{cases} \dot{x}_1(t) = -Dx_1(t) - Sy_2(t) + F_{in}u(t) \\ y_1(t) = x_1(t) \end{cases},$$
$$\begin{cases} \dot{x}_2(t) = 0x_2(t) + y_1(t) \\ y_2(t) = x_2(t) \end{cases}. \tag{34}$$

From this, $y_1(s) := T_1(s)a_1(s) = (sI_n + D)^{-1}a_1(s)$ with $a_1(s) := u_1(t) - Sy_2(s)$ (with the convention $u_1(t) = F_{in}u(t)$) and $y_2(s) = F_{out}s^{-1}a_2(s) := T_2(s)a_2$ with $a_2(s) = y_1(s)$. Matrices F, H, K are given by

$$F := \begin{bmatrix} 0 & F_{out} \end{bmatrix}, \quad H := \begin{bmatrix} F_{in} \\ 0 \end{bmatrix}, \quad K := \begin{bmatrix} 0 & -S \\ I & 0 \end{bmatrix}.$$

From the preceding definitions, one obtains

$$C = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad A = \begin{bmatrix} -D & 0 \\ & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$
$$C_G = \begin{bmatrix} 0 & F_{out} \end{bmatrix}, A_G = \begin{bmatrix} -D & -K \\ I & 0 \end{bmatrix}, B_G = \begin{bmatrix} F_{in} \\ 0 \end{bmatrix}.$$

The matrices (A_G, B_G, C_G) are clearly a state space realization of $F_{out}(s^2I_n + Ds + S)^{-1}F_{in}$. It turns out that the Second-Order Balanced Truncation technique proposed in [CLVV06] is exactly the same as the Interconnected Balanced Truncation technique applied to $T_1(s)$ and $T_2(s)$. In general, systems of order k can be rewritten as an interconnection of k subsystems by generalizing the preceding ideas.

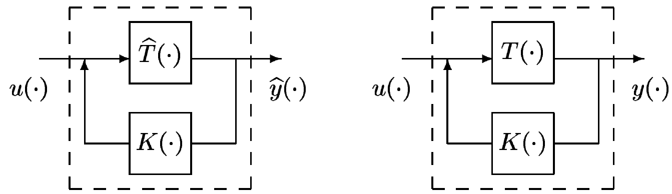
Controller Order Reduction

The Controller Reduction problem introduced by Anderson and Liu [AL89] is the following. Most high-order linear plants $T(s)$ are controlled with a high order linear system $K(s)$. In order to model such *structured* systems by satisfying the computational constraints, it is sometimes needed to approximate either the plant, or the controller, or both systems by reduced order systems, denoted respectively by $\widehat{T}(s)$ and $\widehat{K}(s)$.

The objective of Controller Order Reduction is to find $\widehat{T}(s)$ and/or $\widehat{K}(s)$ that minimize the *structured* error $\|G(s) - \widehat{G}(s)\|$ with

$$G(s) := (I - T(s)K(s))^{-1}T(s), \quad \widehat{G}(s) := (I - \widehat{T}(s)\widehat{K}(s))^{-1}\widehat{T}(s). \tag{35}$$

Fig. 2. Controller Order Reduction



Balanced Truncation model reduction techniques have also been developed for this problem. Again, most of these techniques are very similar to the ISBT Algorithm. See for instance [VA03] for recent results. Depending on the choice of the pair of Gramians, it is possible to develop balancing strategies that ensure the stability of the reduced system, under certain assumptions [LC92].

5 Concluding Remarks

In this paper, general structure preserving model reduction techniques have been developed for interconnected systems, and this for both SVD-based and Krylov-based techniques. Of particular interest, the ISBT Algorithm is a generic tool for performing structured preserving balanced truncation. The advantage of studying model reduction techniques for general interconnected systems is twofold. Firstly, this permits to unify several model reduction techniques developed for weighted systems, controlled systems and second order systems in the same framework. Secondly, our approach permits to extend existing model reduction techniques for a large class of structured systems, namely those that can fit our definition of *interconnected* systems.

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