

Imposing Stability in Subspace Identification by Regularization

Tony Van Gestel¹, Johan Suykens¹,
Paul Van Dooren² and Bart De Moor^{1, 3}

Abstract –In subspace methods for system identification, the system matrices are usually estimated by least squares, based on estimated Kalman filter state sequences and the observed inputs and outputs. For an infinite number of data points and a correct choice of the system order, this least squares estimate of the system matrices is unbiased. However, when using subspace identification on a finite number of data points, the estimated model can become unstable, for a given deterministic system which is known to be stable. In this paper, stability of the estimated model is imposed by adding a regularization term to the least squares cost function. The regularization term used here is the trace of a matrix which involves the dynamical system matrix and a positive (semi-)definite weighting matrix. The amount of regularization needed can be determined by solving a generalized eigenvalue problem. It is shown that the so-called data augmentation method proposed by Chui and Maciejowski corresponds to adding regularization terms with specific choices for the weighting matrix.

Keywords: stability, subspace identification, regularization, ridge regression

1 Introduction

The linear combined deterministic-stochastic identification problem is concerned with systems and models of the form¹:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + w_k & (1a) \\ y_k = Cx_k + Du_k + v_k & (1b) \end{cases}$$

with

$$\mathbf{E}\left(\begin{bmatrix} w_p \\ v_p \end{bmatrix} \begin{bmatrix} w_p \\ v_p \end{bmatrix}^T\right) = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \delta_{pq} \geq 0. \quad (2)$$

The vectors $u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^l$ with discrete time index k denote the m inputs and l outputs of the system respectively. The n states at the time index k of the system with order n are denoted by the state vector $x_k \in \mathbb{R}^n$. The process noise $w_k \in \mathbb{R}^n$ and the measurement noise $v_k \in \mathbb{R}^l$ are assumed to be zero mean, white Gaussian with covariance matrices as given by (2). The model matrices A, B, C, D and the covariance matrices Q, S, R have appropriate dimensions. Both the deterministic and stochastic identification problem are special cases of the combined identification problem, without noise inputs $w_k = 0$ and $v_k = 0$ in (1)-(2) and no deterministic inputs $u_k = 0$, respectively.

The system identification problem is stated as follows: *given* a large number of observations of the input u_k and the corresponding output y_k generated by the unknown system (1)-(2), *determine* the system order \hat{n} of the unknown system (1)-(2) and the estimates $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ up to within a similarity transformation together with the estimated noise covariances matrices $\hat{Q}, \hat{S}, \hat{R}$ so that the second order statistics of the output of the model *and* of the given output are equal (or equivalent in the sense of Faurre [3]).

In the last decade, so-called subspace identification methods [7] have been developed.

¹ \mathbf{E} denotes the expected value operator and δ_{pq} the Kronecker delta. It is assumed that the process is stationary and ergodic: $\mathbf{E}[a_k b_k^T] = \lim_{j \rightarrow \infty} \frac{1}{j} \sum_{i=0}^j a_i b_i^T$.

¹Katholieke Universiteit Leuven, Dept. of Electrical Engineering ESAT-SISTA, Kard. Mercierlaan 94, B-3001 Leuven, Belgium. T. Van Gestel is a Research Assistant with the Fund for Scientific Research-Flanders (FWO-Vlaanderen). J. Suykens is a Postdoctoral Researcher with the FWO-Vlaanderen.

²P. Van Dooren is Professor at the Catholic University of Louvain, Dept. of Mathematical Engineering, Av. Georges Lemaître 4, B-1348 Louvain-la-Neuve, Belgium.

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Typically, in a first step, Kalman filter state sequences $\hat{X}_i \in \mathbb{R}^{\hat{n} \times j}$ and $\hat{X}_{i+1} \in \mathbb{R}^{\hat{n} \times j}$ of the system are estimated directly from input-output data using geometric operations of subspaces spanned by the column or row vectors of block Hankel matrices formed by input-output data. The output data block Hankel matrix $Y_{0|i-1}$ is constructed from the observations $y_0, y_1, \dots, y_{i+j-2}$ as follows:

$$Y_{0|i-1} = \begin{bmatrix} y_0 & y_1 & \dots & y_{j-1} \\ y_1 & y_2 & \dots & y_j \\ \dots & \dots & \dots & \dots \\ y_{i-1} & y_i & \dots & y_{i+j-2} \end{bmatrix},$$

where i is a user defined dimension with $i \ll j$ and the subscript $0|i-1$ denotes the time indices of the outputs in the first column of $Y_{0|i-1}$. The input data block Hankel matrix $U_{0|i-1}$ is constructed in a similar way. The calculations are performed in a numerically reliable way, based on the singular value decomposition (SVD) and QR-decomposition (see [7] for details).

After the estimation of the Kalman filter states sequences \hat{X}_i and \hat{X}_{i+1} in the first step, the system matrices $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of the combined stochastic-deterministic identification problem are identified in the second step:

$$\min_{\hat{A}, \hat{B}, \hat{C}, \hat{D}} \left\| \begin{bmatrix} \hat{X}_{i+1} \\ Y_{i|i} \end{bmatrix} - \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \cdot \begin{bmatrix} \hat{X}_i \\ U_{i|i} \end{bmatrix} \right\|_F^2. \quad (3)$$

Motivated by consistency results [7] for $j \rightarrow \infty$, this optimization problem is solved in a least squares sense. However, when identifying on a finite number of observations (finite j), the least squares estimate does not always yield a stable system [2], while often the real system is known to be stable. For a finite number of data points, this may e.g. occur in the presence of high noise levels or when overparameterizing the system by choosing a too high system order. A discrete time linear system is called stable when it has all its poles inside the unit disk or when there exists a positive definite matrix $\hat{P} = \hat{P}^T > 0$ of appropriate dimension such that

$$\hat{A}\hat{P}\hat{A}^T - \hat{P} < 0. \quad (4)$$

In this paper, stability of the system is imposed by adding a regularization term to (3). Usually, regularization is done by adding the

norm $\|\theta\|_2$ of the parameter vector θ to the cost function $J(\theta)$ of the optimization problem [1, 4, 6]. The resulting θ is then obtained as the solution of the minimization problem $\min_{\theta} J(\theta) + c\|\theta\|_2$. In this paper, a new and specific type of regularization term is used: stability is obtained by adding the trace of product of the system matrix, a positive semi-definite matrix $W \geq 0$, and the transpose of the system matrix, $\text{Tr}(\hat{A}W\hat{A}^T)$, to the cost function (3). Upper bounds for the spectral radius are obtained and it is shown that the calculation of the amount of regularization needed to obtain a specific spectral radius boils down to solving a Generalized Eigenvalue problem.

Also for the stochastic identification problem, the least squares estimate

$$\min_{\hat{A}, \hat{C}} \left\| \begin{bmatrix} \hat{X}_{i+1} \\ Y_{i|i} \end{bmatrix} - \begin{bmatrix} \hat{A} \\ \hat{C} \end{bmatrix} \cdot \hat{X}_i \right\|_F^2 \quad (5)$$

does not always yield a stable system. Since the stochastic identification least squares estimate (5) is a special case of the combined estimate (3) with $U_{i|i} = 0$, in the sequel of the paper, the more general case of stability of (3) is discussed.

This paper is organized as follows. In Section 2, the use of regularization to impose stability is explained. In Section 3, the use of regularization is motivated by simulation results. This paper is a shortened version of the manuscript [8], which is submitted for publication.

The eigenvalues and singular values of a square matrix $A \in \mathbb{R}^{n \times n}$ are denoted by $\lambda_i(A)$ and $\sigma_i(A)$ respectively, for $i = 1, \dots, n$. The spectral radius of A is denoted by $\rho(A) = \max_{i=1, \dots, n} |\lambda_i(A)|$. The minimal and maximal singular value of A is denoted by $\underline{\sigma}(A)$ and $\bar{\sigma}(A)$, respectively. The trace of the matrix A is denoted by $\text{Tr}(A)$ and the matrix norm $\bar{\sigma}(MAM^{-1})$, with $M \in \mathbb{R}^{n \times n}$ a non-singular matrix of appropriate dimensions, is denoted by $\|A\|_M$. The Kronecker product of two matrices X and Z is denoted by $X \otimes Z$.

2 Stability by using Regularization

The estimation problem that we consider is the following: given the matrices \hat{X}_{i+1} , $Y_{i|i}$, \hat{X}_i and $U_{i|i}$ from the first step, estimate the

model \hat{A} , \hat{B} , \hat{C} , \hat{D} . The least squares estimates (3) and (5) do not guarantee a stable \hat{A} matrix for given finite data, while often the true linear model (1)-(2) is known to be stable.

The least squares solution of (3) for $[\hat{A} \hat{B}]$ is given by:

$$\begin{bmatrix} \hat{A} & \hat{B} \end{bmatrix} = \hat{X}_{i+1} \cdot \begin{bmatrix} \hat{X}_i^T & U_{i|i}^T \end{bmatrix} \cdot \begin{bmatrix} \hat{X}_i \hat{X}_i^T & \hat{X}_i U_{i|i}^T \\ U_{i|i} \hat{X}_i^T & U_{i|i}^T U_{i|i} \end{bmatrix}^{-1}. \quad (6)$$

For a finite number of data points, the estimates of \hat{X}_i and \hat{X}_{i+1} may result in an unstable estimate of the system matrix \hat{A} in (6). Stability of the model can be imposed by adding a regularization term to the cost function from which (6) is determined, i.e.:

$$\min_{\tilde{A}, \tilde{B}} J(\tilde{A}, \tilde{B}) = J_1(\tilde{A}, \tilde{B}) + cJ_2(\tilde{A}), \quad (7)$$

with

$$J_1(\tilde{A}, \tilde{B}) = \left\| \hat{X}_{i+1} - [\tilde{A} \tilde{B}] \cdot \begin{bmatrix} \hat{X}_i^T & U_{i|i}^T \end{bmatrix}^T \right\|_F^2 \quad (8)$$

$$J_2(\tilde{A}) = \left\| \tilde{A} Q \right\|_F^2 = \text{Tr}(\tilde{A} W \tilde{A}^T). \quad (9)$$

The amount of regularization is characterized by the positive real scalar c and by the positive semidefinite matrix $W = Q Q^T \geq 0$. The optimal solution to (7) is then given by²:

$$[\tilde{A} \tilde{B}] = [\hat{A} \hat{B}] \hat{\Sigma}_{XU} (\hat{\Sigma}_{XU} + cW_e)^{-1} \quad (10)$$

with

$$\hat{\Sigma}_{XU} = \begin{bmatrix} \hat{X}_i \hat{X}_i^T & \hat{X}_i U_{i|i}^T \\ U_{i|i} \hat{X}_i^T & U_{i|i} U_{i|i}^T \end{bmatrix} \quad (11)$$

$$W_e = \begin{bmatrix} W & 0_{\hat{n} \times m} \\ 0_{m \times \hat{n}} & 0_{m \times m} \end{bmatrix}. \quad (12)$$

In [4], regularization is used to obtain a reduction in the variance of the estimate, while allowing for a (small) bias. The regularization parameter c is chosen in such a way that a cross-validation weighted square error is minimized. In subspace identification, this approach is less appropriate, since the state vector sequences are calculated by QR-decompositions and an SVD. Therefore, the

²In the sequel of this paper, estimates from (6) are denoted by \hat{A} and estimates from (7) by \tilde{A} .

assumptions on the noise distribution used in [4] may not hold. A second motivation is that stability is not proven. In this paper, the regularization parameter c is chosen such that a stable system matrix is obtained, i.e. $\rho(\tilde{A}) < 1$ or such that \tilde{A} has a spectral radius $\rho(\tilde{A})$ smaller than γ or

$$\tilde{A} \tilde{P} \tilde{A}^T - \gamma^2 \tilde{P} < 0, \quad (13)$$

with $\tilde{P} = \tilde{P}^T > 0$ and γ a positive real constant.

The use of regularization to obtain a stable system matrix estimate \tilde{A} is motivated by the following inequality:

$$\begin{aligned} \|\tilde{A} W^{\frac{1}{2}}\|_F^2 &= \text{Tr}(\tilde{A} W \tilde{A}^T) = \sum_{i=1}^n \left(\sigma_i(\tilde{A} W^{\frac{1}{2}}) \right)^2 \\ &\geq \sum_{i=1}^n \underline{\sigma}(W) \left(\sigma_i(\tilde{A}) \right)^2 \geq \underline{\sigma}(W) \sum_{i=1}^n \left| \lambda_i(\tilde{A}) \right|^2. \end{aligned}$$

Let $\tilde{A}_c = \hat{A} \hat{\Sigma}_{XU} (\hat{\Sigma}_{XU} + cW_e)^{-1}$ denote the solution \tilde{A} of (7) for a given c , it follows from the optimality of the least squares estimate (10) that $J_2(\tilde{A}_{c_2}) \leq J_2(\tilde{A}_{c_1})$ for $c_1 \leq c_2$, i.e., the regularization term evaluated at the corresponding solutions for c is a non-increasing function of c . However, the spectral radius $\rho(\tilde{A}_c)$ is not always a monotonically decreasing function for increasing c .

In the sequel of this Section, this behaviour is characterized in more detail by deriving an upper bound for $\rho(\tilde{A})$ and by calculating the amount of regularization needed to obtain $\rho(\tilde{A}) = \gamma < 1$. The relation between \hat{A} and \tilde{A} is given by (10). The eigenvalues $\lambda(\tilde{A})$ of \tilde{A} are equal to the \hat{n} eigenvalues of largest modulus of the (extended) square matrix

$$\tilde{A}_e = \begin{bmatrix} \hat{A} & \hat{B} \\ 0_{m \times \hat{n}} & 0_{m \times m} \end{bmatrix}. \quad (14)$$

In a similar way

$$\hat{A}_e = \begin{bmatrix} \hat{A} & \hat{B} \\ 0_{m \times \hat{n}} & 0_{m \times m} \end{bmatrix}$$

is defined. Given (10), the influence of regularization on the eigenvalues of \tilde{A} is given by the relation $\tilde{A}_e = \hat{A}_e \hat{\Sigma}_{XU} (\hat{\Sigma}_{XU} + cW_e)^{-1}$. However, the increase in dimensionality from \hat{n} to $\hat{n} + m$ can be avoided:

Lemma 1 *Let $\hat{\Sigma}_s$ be defined as*

$$\hat{\Sigma}_s = \hat{X}_i \hat{X}_i^T - \hat{X}_i U_{i|i}^T (U_{i|i} U_{i|i}^T)^{-1} U_{i|i} \hat{X}_i^T, \quad (15)$$

then the eigenvalues of \tilde{A} are equal to

$$\lambda(\tilde{A}) = \lambda(\hat{A}\hat{\Sigma}_s(\hat{\Sigma}_s + cW)^{-1}). \quad (16)$$

We refer to [8] for the proof. By using the QR-decomposition

$$\left[\begin{array}{c|c} U_{|i}^T & \hat{X}_i^T \end{array} \right] = \left[\begin{array}{c|c} Q_1 & Q_2 \end{array} \right] \left[\begin{array}{cc} R_{11} & R_{12} \\ 0 & R_{22} \end{array} \right],$$

it is easily shown that $\hat{\Sigma}_s = R_{22}^T R_{22}$ or $\hat{\Sigma}_s = \hat{X}_s \hat{X}_s^T$, with $\hat{X}_s = \hat{X}_i - \hat{X}_i U_{|i}^T (U_{|i} U_{|i}^T)^{-1} U_{|i}$.

An upper bound for the spectral radius $\rho(\tilde{A})$ can be formulated as follows:

Theorem 1 *Let $W > 0$, then the following upper bound holds for $\rho(\tilde{A}_c)$:*

$$\rho(\tilde{A}_c) \leq \frac{\bar{\sigma}(R_{22}^{-T} \hat{A} R_{22}^T)}{1 + c \underline{\sigma}(R_{22}^{-T} W R_{22}^{-1})}. \quad (17)$$

From this upper bound, it follows that

$$\rho(\tilde{A}_c) \leq \gamma \quad \text{for} \quad c \geq \frac{\left(\frac{\bar{\sigma}(R_{22}^{-T} \hat{A} R_{22}^T)}{\gamma} - 1 \right)}{\underline{\sigma}(R_{22}^{-T} W R_{22}^{-1})}.$$

We refer to [8] for the proof. In the next Theorem, conditions are derived to obtain $\rho(\tilde{A}_c) \leq \gamma$, with $\gamma > 0$ a positive real scalar.

Theorem 2 *Given \hat{X}_{i+1} , \hat{X}_i , $U_{|i}$, $Y_{|i}$ and W . Let the matrix $\hat{\Sigma}_s$ be defined by (15). Let $\gamma \geq 0$ be a positive real scalar and let \tilde{A}_c and \tilde{A}_c be estimated as in (3) and (10). Let the matrices P_2 , P_1 and $P_0 \in \mathbb{R}^{\hat{n} \times \hat{n}}$ be defined as follows:*

$$P_2 = -\gamma W \otimes \gamma W, \quad (18)$$

$$P_1 = -\gamma W \otimes \hat{\Sigma}_s - \gamma \hat{\Sigma}_s \otimes W, \quad (19)$$

$$P_0 = \hat{A} \hat{\Sigma}_s \otimes \hat{A} \hat{\Sigma}_s - \gamma \hat{\Sigma}_s \otimes \gamma \hat{\Sigma}_s. \quad (20)$$

Define the set of eigenvalues ϑ of the Generalized Eigenvalue problem

$$\vartheta = \lambda \left(\left[\begin{array}{cc} 0 & -I \\ P_0 & P_1 \end{array} \right], - \left[\begin{array}{cc} I & 0 \\ 0 & P_2 \end{array} \right] \right) \quad (21)$$

Then $\rho(\tilde{A}_c) \leq \gamma$, for $c \geq c_m = \max_{i|\vartheta_i \in \mathbb{R}^+} \vartheta_i$, with $\rho(\tilde{A}_{c_m}) = \gamma$.

The reader is referred to [8] for the proof. For high order systems, a large Generalized Eigenvalue Problem (21) of dimension $2\hat{n}^2$ has to be solved, hence requiring $O(\hat{n}^6)$ operations. When \hat{n} is large, an iterative algorithm

to determine the value of c_m is proposed in [8]. The algorithm requires $O(\hat{n}^3)$ operations per iteration step and the simulation results indicate linear convergence. The problem can also be formulated as a real stability radius problem with one repeated block for which a fast algorithm exists [5], but this algorithm requires more complex computations (a Linear Matrix Inequality Problem) per iteration step and faster convergence has not been proven.

Now, we show that the so-called data augmentation method of Chui and Maciejowski [2] corresponds to adding regularization terms to the least squares cost function with specific choices for the weighting matrix W . In the method of [2], the non-steady Kalman filter state sequences \hat{X}_i, \hat{X}_{i+1} are iteratively augmented by appending $\sqrt{c_p} V_p$ and $0_{\hat{n} \times 1}$ ($\sqrt{c_q} [V_q \ V_q^*]$ and $0_{\hat{n} \times 2}$) respectively, for each unstable pole λ_p (pole pair (λ_q, λ_q^*)) with corresponding normalized right eigenvector V_p (right eigenvector V_q and its conjugate V_q^*). The inputs $U_{|i}$ and outputs $Y_{|i}$ are extended by appending null vectors of appropriate dimensions. The constant c_p (c_q for a complex pole pair) is determined such that the magnitude of the stabilized pole is M , where $M \geq 0$ is chosen by the user. The other eigenvalues are not changed.

For the case of one real unstable pole, this method corresponds to minimizing the following cost function in least squares sense:

$$\min_{\tilde{A}_{CM}, \tilde{B}_{CM}} \|\hat{X}_{i+1} - \tilde{A}_{CM} \hat{X}_i - \tilde{B}_{CM} U_{|i}\|_F^2 + c_p \|\tilde{A}_{CM} V\|_F^2,$$

with the regularization term $c_p \|\tilde{A}_{CM} V\|_F^2 = c_{CM} \text{Tr}(\tilde{A}_{CM} V V^T \tilde{A}_{CM}^T)$. More regularization terms are iteratively added for each other pole or pole pair with amplitude larger than γ . Hence, the data augmentation method of Chui and Maciejowski [2] corresponds to the following weighting matrix in the regularization term (7):

$$c \text{Tr}(\tilde{A} W \tilde{A}^T) = \sum_{\text{unstable poles } p} c_p \text{Tr}(\tilde{A} V_p V_p^T \tilde{A}^T) + \sum_{\text{unstable pole pairs } q} c_q \text{Tr}(\tilde{A} [V_q \ V_q^*] [V_q \ V_q^*]^T \tilde{A}^T),$$

or $cW = \sum_p c_p V_p V_p^T + \sum_q c_q [V_q \ V_q^*] [V_q \ V_q^*]^T$, which can be considered as a special case of $cJ_2(\tilde{A})$, with $J_2(\tilde{A})$ from (9).

3 Simulation results

In this Section, the use of regularization in order to impose stability of the estimated system is illustrated by means of a simulation example. The original system has order $n = 3$ and the effects of high noise levels for both process and measurement noise, overestimating the system order and a small number of given data points are illustrated. In [8], results on a second example using a model of an industrial dryer [2] with order $n = 10$ are reported.

The original system is given by:

$$A = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 0.65 & 0.70 \\ 0 & -0.70 & 0.65 \end{bmatrix} \quad B = \begin{bmatrix} 10 \\ 1 \\ 1 \end{bmatrix},$$
$$C = [1 \quad 1 \quad 1] \quad D = 1.$$

The known input u_k has a standard normal distribution. The influence of the noise level of covariance matrices Q , R , S (2) is illustrated. For the weighting matrix W , the identity matrix $W = I_{\hat{n}}$ is chosen. The evolution of the spectral radius $\rho(\tilde{A}_c)$ and the different parts of the cost function (7) as a function of c is depicted in Figure 1 for an estimated unstable system with system order $\hat{n} = 3$. The influence of the regularization on the distribution of the poles is illustrated in Fig. 2, 3, 4 and 5. The effect of a finite number of data points, overparameterization and high noise levels is illustrated by comparison of Fig. 3 with Fig. 2, 4 and 5, respectively.

4 Conclusions

Subspace methods for the identification of linear time-invariant systems are known to be asymptotically efficient. The system matrices are usually estimated in least squares sense, based on estimated Kalman filter state sequences and the observed inputs and outputs. However, for a finite number of data points, it is not guaranteed that the least square estimate yields a stable system, even when it is known that the true linear system is stable. Such situations may also occur when the estimated system is overparameterized due to a too high system order estimate. In this paper, we have shown that stability of the estimated system can be imposed by adding a

regularization term to the least squares cost function. The regularization term used here is the trace of a matrix which involves the dynamical system matrix and a positive (semi-)definite weighting matrix. The amount of regularization is calculated by solving a generalized eigenvalue problem. The so-called data augmentation method proposed by Chui and Maciejowski can be interpreted as iteratively applying regularization with specific choices for the weighting matrix.

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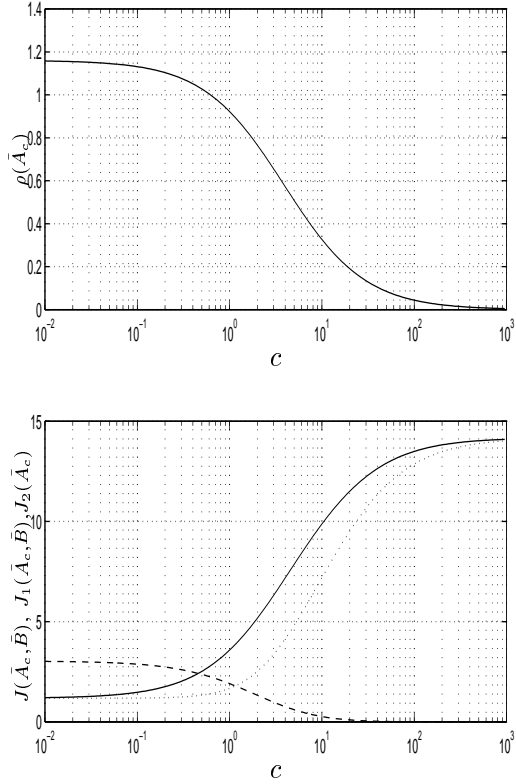


Figure 1: Top: evolution of $\rho(\tilde{A}_c)$ as a function of the regularization parameter c . Bottom: evolution of the two parts of the cost function (7) (for the optimal solution) as a function of c : $J_1(\tilde{A}_c, \tilde{B})$ (dashed-dotted line), $J_2(\tilde{A}_c)$ (dotted line) and the total cost function $J(\tilde{A}_c, \tilde{B})$ (full line).

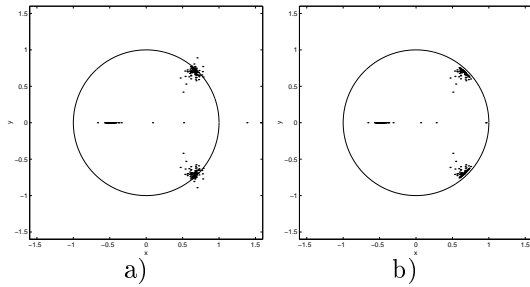


Figure 2: Monte Carlo simulation of the distribution of the poles $p_i = x_i + jy_i$ ($i = 1, \dots, \hat{n}$) of the estimated system with $\hat{n} = 3$: a) no regularization, b) with regularization. The original system is simulated with $Q = \text{diag}([1 \ 1 \ 1])$, $S = 0$, $R = 1$. The number of data points is 40, $\gamma = 0.96$.

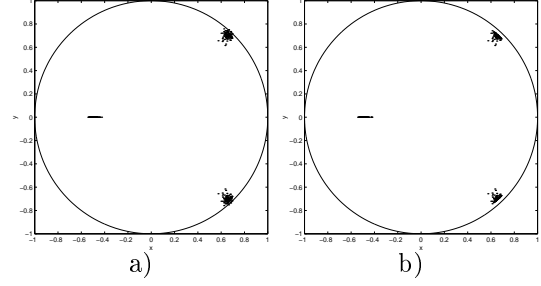


Figure 3: Monte Carlo simulation of the distribution of the poles of the estimated system with order $\hat{n} = 3$: a) no regularization, b) with regularization. The original system is simulated with $Q = \text{diag}([1 \ 1 \ 1])$, $S = 0$, $R = 1$. The number of data points is 80, $\gamma = 0.96$.

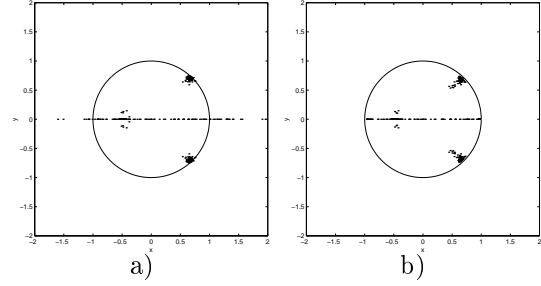


Figure 4: Monte Carlo simulation of the distribution of the poles of the estimated system with order $\hat{n} = 4$: a) no regularization, b) with regularization. The original system is simulated with $Q = \text{diag}([1 \ 1 \ 1])$, $S = 0$, $R = 1$. The number of data points is 80, $\gamma = 0.96$.

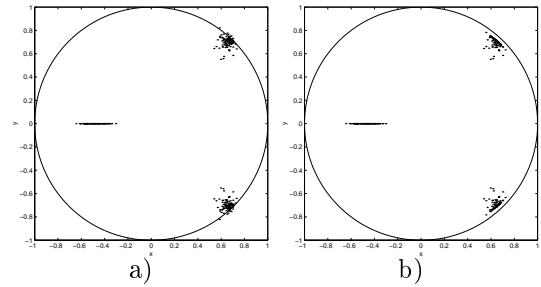


Figure 5: Monte Carlo simulation of the distribution of the poles of the estimated system with order $\hat{n} = 3$: a) no regularization, b) with regularization. The original system is simulated with $Q = \text{diag}([10 \ 10 \ 10])$, $S = 0$, $R = 10$. The number of data points is 80, $\gamma = 0.96$.