Identification of Stable Models in Subspace Identification by Using Regularization

T. Van Gestel, J. A. K. Suykens, P. Van Dooren, and B. De Moor

***Abstract*—In subspace identification methods, the system matrices are usually estimated by least squares, based on estimated Kalman filter state sequences and the observed inputs and outputs. For a finite number of data points, the estimated system matrix is not guaranteed to be stable, even when the true linear system is known to be stable. In this note, stability is imposed by using regularization. The regularization term used here is the trace of a matrix which involves the dynamical system matrix and a positive (semi) definite weighting matrix. The amount of regularization can be determined from a generalized eigenvalue problem. The data augmentation method of Chui and Maciejowski is obtained by using specific choices for the weighting matrix in the regularization term.**

***Index Terms*—Regularization, stability, subspace identification.**

I. Introduction

The linear combined deterministic–stochastic identification problem is concerned with systems and models of the form[1]

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1}$$

$$y_k = Cx_k + Du_k + v_k \tag{2}$$

with

$$\mathbf{E}\left(\begin{bmatrix} w_p \\ v_p \end{bmatrix}\begin{bmatrix} w_q^T & v_q^T \end{bmatrix}\right) = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}\delta_{pq} \geq 0. \tag{3}$$

The vectors $u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^l$ with discrete-time index k denote the m inputs and l outputs of the system respectively. The n states at the time index k of the system with order n are denoted by the state vector $x_k \in \mathbb{R}^n$. The process noise $w_k \in \mathbb{R}^n$ and the measurement noise $v_k \in \mathbb{R}^l$ are assumed to be zero mean, white Gaussian with covariance matrices as given by (3). The model matrices A, B, C, D and the covariance matrices Q, S, R have appropriate dimensions. Both the deterministic and stochastic identification problem are special cases of the combined identification problem, without noise inputs $w_k = 0$ and $v_k = 0$ in (1)–(3) and no deterministic inputs $u_k = 0$, respectively.

In the last decade, so-called subspace identification methods [13] have been developed to determine the system order $\hat{n}$ of the unknown system (1)–(3) and the estimates $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ (up to a similarity transformation) together with the estimated noise covariances matrices $\hat{Q}$, $\hat{S}$, $\hat{R}$, from a large number of observations of the input u_k and the corresponding output y_k generated by the unknown system (1)–(3). Typically, in a first step, Kalman filter state sequences $\hat{X}_i \in \mathbb{R}^{\hat{n}\times j}$ and $\hat{X}_{i+1} \in \mathbb{R}^{\hat{n}\times j}$ of the system are estimated using geometric operations of subspaces spanned by the column or row vectors of block Hankel matrices formed by input–output data. The system order $\hat{n}$ typically has to be estimated by the user from a singular value plot, where j is related to the number of data points, see [13] for details.

The system matrices $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of the combined stochastic–deterministic identification problem are identified in the second step by a least squares problem

$$\min_{\hat{A},\hat{B},\hat{C},\hat{D}} \left\| \begin{bmatrix} \hat{X}_{i+1} \\ Y_{i|i} \end{bmatrix} - \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \cdot \begin{bmatrix} \hat{X}_i \\ U_{i|i} \end{bmatrix} \right\|_F^2 \tag{4}$$

with the input sequence $U_{i|i} = [u_i, u_{i+1}, \ldots, u_{i+j-1}] \in \mathbb{R}^{m\times j}$ and the output sequence $Y_{i|i} = [y_i, y_{i+1}, \ldots, y_{i+j-1}] \in \mathbb{R}^{l\times j}$ [13]. Motivated by consistency results [13] for $j \to \infty$, this optimization problem is solved in a least squares sense. However, when identifying on a finite number of data points (finite j), the least squares estimate does not always yield a stable system [2], while often the true linear system is known to be stable. For a finite number of data points, this may, e.g., occur in the presence of high noise levels or when overparameterizing the system by overestimating the system order. A discrete time linear system is called stable when it has all its poles inside the unit disk or when there exists a positive–definite matrix $\hat{P} = \hat{P}^T > 0$ of appropriate dimension such that $\hat{A}\hat{P}\hat{A}^T - \hat{P} < 0$.

In this note, stability of the system is imposed by adding a regularization term to (4). Usually, regularization is obtained by adding the norm $\|\theta\|_2$ of the parameter vector θ to the cost function $J(\theta)$ of the optimization problem [1], [4], [10], [11]. The resulting θ is then obtained as the solution of the minimization problem $\min_\theta J(\theta) + c\|\theta\|_2$. In this note, a new and specific type of regularization term is used: stability is obtained by adding the trace of the product of the system matrix, a positive–semidefinite matrix $W = W^T \geq 0$, and the transpose of the system matrix, $\mathrm{Tr}(\hat{A}W\hat{A}^T)$, to the cost function (4). Upper bounds for the spectral radius are obtained and it is shown that the calculation of the amount of regularization needed to obtain a specific spectral radius boils down to solving a generalized eigenvalue problem.

Also for the stochastic identification problem, the solution $\hat{A}$ of the least squares problem

$$\min_{\hat{A},\hat{C}} \left\| \left[\hat{X}_{i+1}^T \; Y_{i|i}^T\right]^{\mathrm{T}} - [\hat{A}\;\hat{C}]^{\mathrm{T}} \cdot \hat{X}_i \right\|_F^2$$

is not always stable. Since the stochastic identification least squares estimate is a special case of the combined estimate (4) with $U_{i|i} = 0$, we restrict ourselves to the more general case of stability of (4).

This note is organized as follows. In Section II, the use of regularization to impose stability is explained. In the companion paper [12], the use of regularization is motivated by simulation results which are compared with the results of [2]. The following notation is used. The eigenvalues and singular values of a square matrix $A \in \mathbb{R}^{n\times n}$ are denoted by $\lambda_i(A)$ and $\sigma_i(A)$ respectively, for $i = 1, \ldots, n$. The spectral radius of A is denoted by $\rho(A) = \max_{i=1,\ldots,n} |\lambda_i(A)|$. The minimal and maximal singular value of A is denoted by $\underline{\sigma}(A)$ and $\overline{\sigma}(A)$, respectively. The trace of the matrix A is denoted by $\mathrm{Tr}(A)$ and the matrix norm $\overline{\sigma}(MAM^{-1})$, with $M \in \mathbb{R}^{n\times n}$ a nonsingular matrix of appropriate dimensions, is denoted by $\|A\|_M$. The Kronecker product of two matrices X and Z is denoted by $X \otimes Z$.

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T. Van Gestel, J. A. K. Suykens, and B. De Moor are with the Katholieke Universiteit Leuven, Department of Electrical Engineering ESAT-SISTA, B-3001 Leuven, Belgium.

P. Van Dooren is with the Catholic University of Louvain, Department of Mathematical Engineering, B-1348 Louvain-la-Neuve, Belgium.

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[1] $\mathbf{E}$ denotes the expected value operator and δ_{pq} the Kronecker delta. It is assumed that the process is stationary and ergodic: the equality $\mathbf{E}[a_k b_k^T] = \lim_{j\to\infty}\left[1/j\sum_{i=0}^{j} a_i b_i^T\right]$ holds with probability 1.

II. Stability by Using Regularization

The estimation problem that we consider is the following: given the matrices $\hat{X}_{i+1}, Y_{i|i}, \hat{X}_i$ and $U_{i|i}$ from the first step, estimate the model matrices $\hat{A}, \hat{B}, \hat{C}, \hat{D}$. The least squares estimate (4) does not guarantee a stable $\hat{A}$ matrix for given finite data, while often the true linear model (1)–(3) is known to be stable.

The least squares problem (4) can be separated into two least squares problems, with solutions

$$\begin{aligned}[\hat{A} \quad \hat{B}] &= \hat{X}_{i+1} \cdot [\hat{X}_i^T \quad U_{i|i}^T] \cdot \hat{\Sigma}_{XU}^{-1} \\ [\hat{C} \quad \hat{D}] &= Y_{i|i} \cdot [\hat{X}_i^T \quad U_{i|i}^T] \cdot \hat{\Sigma}_{XU}^{-1}\end{aligned} \tag{5}$$

where

$$\hat{\Sigma}_{XU} = \begin{bmatrix} \hat{X}_i \hat{X}_i^T & \hat{X}_i U_{i|i}^T \\ U_{i|i} \hat{X}_i^T & U_{i|i} U_{i|i}^T \end{bmatrix}. \tag{6}$$

For a finite number of data points, the estimates of $\hat{X}_i$ and $\hat{X}_{i+1}$ may result in an unstable estimate of the system matrix $\hat{A}$ in (5).

Stability of the model can be imposed by adding a regularization term to the cost function from which (5) is determined, i.e.,

$$\min_{\tilde{A}\tilde{B}} J(\tilde{A}, \tilde{B}) = J_1(\tilde{A}\tilde{B}) + cJ_2(\tilde{A}) \tag{7}$$

with

$$J_1(\tilde{A}, \tilde{B}) = \left\| \hat{X}_{i+1} - [\tilde{A} \quad \tilde{B}] \cdot [\hat{X}_i^T \quad U_{i|i}^T]^{\mathrm{T}} \right\|_F^2 \tag{8}$$

$$J_2(\tilde{A}) = \|\tilde{A}Q\|_F^2 = \mathrm{Tr}\left(\tilde{A} W \tilde{A}^T\right). \tag{9}$$

The amount of regularization is characterized by the positive real scalar c and by the positive semidefinite matrix $W = QQ^T \geq 0$. The optimal solution to (7) is then given by[2]

$$\begin{aligned}[\tilde{A}_c \quad \tilde{B}_c] &= [\hat{X}_{i+1} \hat{X}_i^T \quad \hat{X}_{i+1} U_{i|i}^T] (\hat{\Sigma}_{XU} + cW_e)^{-1} \\ &= [\hat{A} \quad \hat{B}] \hat{\Sigma}_{XU} (\hat{\Sigma}_{XU} + cW_e)^{-1}\end{aligned} \tag{10}$$

where $\hat{\Sigma}_{XU}$ is defined in (6) and W_e as

$$W_e = \begin{bmatrix} W & 0_{\hat{n} \times m} \\ 0_{m \times \hat{n}} & 0_m \end{bmatrix}. \tag{11}$$

In [3] and [4], regularization is used to obtain a reduction in the variance of the estimate, while allowing for a (small) bias. The regularization parameter c is chosen in such a way that a cross-validation weighted square error is minimized. In subspace identification, this approach is less appropriate, since the state sequences are calculated by QR-decompositions and an SVD. Therefore, the assumptions on the noise distribution used in [3], [4] may not hold. A second motivation is that stability is not proven. In this note, the regularization parameter c is chosen such that a stable system matrix is obtained, i.e., $\rho(\tilde{A}_c) < 1$ or such that $\tilde{A}_c$ has a spectral radius $\rho(\tilde{A}_c)$ smaller than γ or

$$\tilde{A}_c \tilde{P} \tilde{A}_c^T - \gamma^2 \tilde{P} < 0 \tag{12}$$

with $\tilde{P} = \tilde{P}^T > 0$ and γ a positive-real constant.

The use of regularization to obtain a stable system matrix estimate $\tilde{A}_c$ is motivated by the following inequality:

$$\begin{aligned}\|\tilde{A}_c W^{1/2}\|_F^2 &= \mathrm{Tr}\left(\tilde{A}_c W \tilde{A}_c^T\right) = \sum_{i=1}^{n} \left(\sigma_i(\tilde{A}_c W^{1/2})\right)^2 \\ &\geq \sum_{i=1}^{n} \underline{\sigma}(W) \left(\sigma_i(\tilde{A}_c)\right)^2 \geq \underline{\sigma}(W) \sum_{i=1}^{n} |\lambda_i(\tilde{A}_c)|^2.\end{aligned}$$

[2]In the remainder of this note, estimates from (5) are denoted by $\hat{A}$ and estimates from (7) by $\tilde{A}_c$. The subscript c is used to denote the dependency of the solution $\tilde{A}_c$ as a function of c.

Let $\tilde{A}_c = \hat{A}\hat{\Sigma}_{XU}(\hat{\Sigma}_{XU} + cW_e)^{-1}$ denote the solution $\tilde{A}_c$ of (7) for a given c, it follows from the optimality of the least squares estimate (10) that $J_2(\tilde{A}_{c_2}) \leq J_2(\tilde{A}_{c_1})$ for $c_1 \leq c_2$, i.e., the regularization term evaluated at the corresponding solutions for c is a nonincreasing function of c. However, the spectral radius $\rho(\tilde{A}_c)$ is not always a monotonically decreasing function for increasing c.

In what follows, this behavior is characterized in more detail by deriving an upper bound for $\rho(\tilde{A}_c)$ and by calculating the amount of regularization needed to obtain $\rho(\tilde{A}_c) = \gamma < 1$. The relation between $\hat{A}$ and $\tilde{A}_c$ is given by (10). By defining the (extended) square matrices

$$\tilde{A}_e = \begin{bmatrix} \tilde{A}_c & \tilde{B}_c \\ 0_{m \times \hat{n}} & 0_m \end{bmatrix} \quad \hat{A}_e = \begin{bmatrix} \hat{A} & \hat{B} \\ 0_{m \times \hat{n}} & 0_m \end{bmatrix} \tag{13}$$

we obtain that the eigenvalues $\lambda(\tilde{A}_c)$ and $\lambda(\hat{A})$ of $\tilde{A}_c$ and $\hat{A}$ are equal to the $\hat{n}$ eigenvalues with largest modulus of $\tilde{A}_e$ and $\hat{A}_e$, respectively. Given (10), the influence of regularization on the eigenvalues of $\tilde{A}_c$ is given by the relation $\tilde{A}_e = \hat{A}_e \hat{\Sigma}_{XU}(\hat{\Sigma}_{XU} + cW_e)^{-1}$. However, the increase in dimensionality from $\hat{n}$ to $\hat{n} + m$ can be avoided.

Lemma 1: Let $\hat{\Sigma}_s$ be defined as

$$\hat{\Sigma}_s = \hat{X}_i \hat{X}_i^T - \hat{X}_i U_{i|i}^T \left(U_{i|i} U_{i|i}^T\right)^{-1} U_{i|i} \hat{X}_i^T \tag{14}$$

then the eigenvalues of $\tilde{A}_c$ are equal to

$$\lambda(\tilde{A}_c) = \lambda\left(\hat{A}\hat{\Sigma}_s(\hat{\Sigma}_s + cW)^{-1}\right). \tag{15}$$

Proof: The $\hat{n}$ largest eigenvalues of $\tilde{A}_e$ are the eigenvalues of $\tilde{A}_c$. The eigenvalues $\lambda(\tilde{A}_e)$ are also the eigenvalues of the generalized eigenvalue problem $\lambda(\hat{A}_e \hat{\Sigma}_{XU}, \hat{\Sigma}_{XU} + cW_e)$. By applying a transformation

$$\bar{T}_s = \begin{bmatrix} I_{\hat{n}} & -T_s \\ 0 & I_m \end{bmatrix}$$

with $T_s = \hat{X}_i U_{i|i}^T (U_{i|i} U_{i|i}^T)^{-1}$ we obtain

$$\begin{aligned}&\lambda(\hat{A}_e \hat{\Sigma}_{XU}, \hat{\Sigma}_{XU} + cW_e) \\ &\quad = \lambda\left(\bar{T}_s \hat{A}_e \hat{\Sigma}_{XU} \bar{T}_s^T \bar{T}_s (\hat{\Sigma}_{XU} + cW_e) \bar{T}_s^T\right) \\ &\quad = \lambda\left(\begin{bmatrix} \hat{A} & \hat{B} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\Sigma}_s & \hat{X}_i U_{i|i}^T \\ 0 & U_{i|i} U_{i|i}^T \end{bmatrix} \begin{bmatrix} \hat{\Sigma}_s + cW & 0 \\ 0 & U_{i|i} U_{i|i}^T \end{bmatrix}\right).\end{aligned}$$

At least m eigenvalues are zero and the $\hat{n}$ eigenvalues of largest modulus are the eigenvalues of $\lambda\left(\hat{A}\hat{\Sigma}_s(\hat{\Sigma}_s + cW)^{-1}\right)$. □

By using the QR-decomposition

$$\begin{bmatrix} U_{i|i}^T \\ \hat{X}_i^T \end{bmatrix} = [Q_1 | Q_2] \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$$

it is easily shown that $\hat{\Sigma}_s = R_{22}^T R_{22}$ or $\hat{\Sigma}_s = \hat{X}_s \hat{X}_s^T$, with $\hat{X}_s = \hat{X}_i - \hat{X}_i U_{i|i}^T (U_{i|i} U_{i|i}^T)^{-1} U_{i|i}$ and where we assume that R_{11} is nonsingular. This assumption is related to a form of persistent excitation in subspace methods, see [7] for a discussion. An upper bound for the spectral radius $\rho(\tilde{A}_c)$ can be formulated as follows.

Theorem 1: Let $W = QQ^T > 0$, then the following upper bound holds for $\rho(\tilde{A}_c)$:

$$\rho(\tilde{A}_c) \leq \frac{\overline{\sigma}\left(R_{22}^{-T} \hat{A} R_{22}^T\right)}{1 + c\underline{\sigma}\left(R_{22}^{-T} W R_{22}^{-1}\right)}. \tag{16}$$

From this upper bound, it follows that:

$$\rho(\tilde{A}_c) \leq \gamma \text{ for } c \geq c_u = \frac{\left(\frac{\overline{\sigma}\left(R_{22}^{-T} \hat{A} R_{22}^T\right)}{\gamma} - 1\right)}{\underline{\sigma}\left(R_{22}^{-T} W R_{22}^{-1}\right)}. \tag{17}$$

Proof: By use of Lemma 1, we have that

$$\begin{aligned}\rho(\tilde{A}_c) =&\rho(\hat{A}\hat{\Sigma}_s(\hat{\Sigma}_s+cW)^{-1})\\ =&\rho\left(\hat{A}R_{22}^T R_{22}\left(R_{22}^T\left(I_{\hat{n}}+cR_{22}^{-T}WR_{22}^{-1}\right)R_{22}\right)^{-1}\right).\end{aligned}$$

Since $\rho(\tilde{A}_c)=\rho\left(R_{22}^{-T}\tilde{A}_cR_{22}^T\right)$ and $\rho(M)\leq\overline{\sigma}(M)$ for any square matrix M, we also have that

$$\begin{aligned}\rho(\tilde{A}_c) =&\rho\left(R_{22}^{-T}\hat{A}R_{22}^T\left(I_{\hat{n}}+cR_{22}^{-T}WR_{22}^{-1}\right)^{-1}\right)\\ \leq&\overline{\sigma}\left(R_{22}^{-T}\hat{A}R_{22}^T\right)\overline{\sigma}\left(\left(I_{\hat{n}}+cR_{22}^{-T}WR_{22}^{-1}\right)^{-1}\right)\\ =&\frac{\overline{\sigma}\left(R_{22}^{-T}\hat{A}R_{22}^T\right)}{\underline{\sigma}\left(I_{\hat{n}}+cR_{22}^{-T}WR_{22}^{-1}\right)}=\frac{\overline{\sigma}\left(R_{22}^{-T}\hat{A}R_{22}^T\right)}{1+c\underline{\sigma}\left(R_{22}^{-T}WR_{22}^{-1}\right)}.\end{aligned}$$

The proof of (17) follows directly by putting this upper bound equal to γ. □

In Theorem 2, conditions are derived to obtain $\rho(\tilde{A}_c)\leq\gamma$, with $\gamma>0$ a positive-real scalar.

Theorem 2: Given $\hat{X}_{i+1},\hat{X}_i,U_{i|i},Y_{i|i}$ and W. Let the matrix $\hat{\Sigma}_s$ be defined by (14). Let $\gamma\geq 0$ be a positive real scalar and let $\hat{A}$ and $\tilde{A}_c$ be estimated as in (4) and (10). Let the matrices P_2,P_1 and $P_0\in\mathbb{R}^{\hat{n}^2\times\hat{n}^2}$ be defined as follows:

$$\begin{aligned}&P_2=-\gamma^2W\otimes W,\quad P_1=-\gamma^2W\otimes\hat{\Sigma}_s-\gamma^2\hat{\Sigma}_s\otimes W\\ &P_0=\hat{A}\hat{\Sigma}_s\otimes\hat{A}\hat{\Sigma}_s-\gamma^2\hat{\Sigma}_s\otimes\hat{\Sigma}_s.\end{aligned}\tag{18}$$

Define the set of eigenvalues ϑ of the generalized eigenvalue problem

$$\vartheta=\lambda\left(\begin{bmatrix}0_{\hat{n}^2} & -I_{\hat{n}^2}\\ P_0 & P_1\end{bmatrix},-\begin{bmatrix}I_{\hat{n}^2} & 0_{\hat{n}^2}\\ 0_{\hat{n}^2} & P_2\end{bmatrix}\right).\tag{19}$$

Then $\rho(\tilde{A}_c)\leq\gamma$, for $c\geq c_m=\max_{i|\vartheta_i\in\mathbb{R}^+}\vartheta_i$, with $\rho(\tilde{A}_{c_m})=\gamma$.

Proof: First we will show that $\rho(\tilde{A}_{c_m})=\gamma$, for $c_m=\max_{i|\vartheta_i\in\mathbb{R}^+}\vartheta_i$. We start from Lemma 1: $\rho(\tilde{A}_c)=\rho\left(\hat{A}\hat{\Sigma}_s(\hat{\Sigma}_s+cW)^{-1}\right)$. All solutions of c for which the mapping $\tilde{P}\rightarrow\tilde{A}_c\tilde{P}\tilde{A}_c^T-\gamma^2\tilde{P}$ is singular, correspond to solutions with a pole on the circle with radius γ or a pole pair that is symmetric with respect to the circle with radius γ (a symplectic pole pair if $\gamma=1$) [8]. By applying the vec-operation and using the property $\mathrm{vec}(XYZ^T)=(Z\otimes X)\mathrm{vec}(Y)$[8], we obtain

$$\mathrm{vec}\left(\tilde{A}_c\tilde{P}\tilde{A}_c^T-\gamma^2\tilde{P}\right)=\left(\tilde{A}_c\otimes\tilde{A}_c-\gamma^2I_{\hat{n}}\otimes I_{\hat{n}}\right)\mathrm{vec}(\tilde{P}).$$

Therefore the mapping is $\tilde{P}\rightarrow\tilde{A}_c\tilde{P}\tilde{A}_c^T-\gamma^2\tilde{P}$ singular iff $\det\left(\tilde{A}_c\otimes\tilde{A}_c-\gamma^2I_{\hat{n}}\otimes I_{\hat{n}}\right)=0$. By using (15) and $(X\otimes Y)(Z\otimes T)=XZ\otimes YT$, this is equivalent to

$$\begin{aligned}&\det\Big(\hat{A}\hat{\Sigma}_s(\hat{\Sigma}_s+cW)^{-1}\otimes\hat{A}\hat{\Sigma}_s(\hat{\Sigma}_s+cW)^{-1}\\ &\quad-\gamma^2I_{\hat{n}^2}\Big)=0\\ &\Leftrightarrow\det\Big((\hat{A}\hat{\Sigma}_s\otimes\hat{A}\hat{\Sigma}_s)\Big((\hat{\Sigma}_s+cW)^{-1}\otimes(\hat{\Sigma}_s+cW)^{-1}\Big)\\ &\quad-\gamma^2I_{\hat{n}^2}\Big)=0.\end{aligned}$$

Since $(X\otimes X)^{-1}=X^{-1}\otimes X^{-1}$ (when X is invertible) and $\det(\hat{\Sigma}_s+cW)\neq 0$, we obtain

$$\begin{aligned}&\det\left(\hat{A}\hat{\Sigma}_s\otimes\hat{A}\hat{\Sigma}_s-\gamma^2\left((\hat{\Sigma}_s+cW)\otimes(\hat{\Sigma}_s+cW)\right)\right)=0\\ &\Leftrightarrow\det\Big(-c^2(\gamma W\otimes\gamma W)-c(\gamma^2W\otimes\hat{\Sigma}_s+\gamma^2\hat{\Sigma}_s\otimes W)\\ &\quad+(\hat{A}\hat{\Sigma}_s\otimes\hat{A}\hat{\Sigma}_s-\gamma\hat{\Sigma}_s\otimes\gamma\hat{\Sigma}_s)\Big)=0.\end{aligned}\tag{20}$$

We now use the following equivalence relations $\left(P_2,P_1,P_0\in\mathbb{R}^{\hat{n}^2\times\hat{n}^2}\right)$:

$$\begin{aligned}&\det(c^2P_2+cP_1+P_0)=0\\ \Leftrightarrow\ &\det(c(cP_2+P_1)+P_0)=0\\ \Leftrightarrow\ &\det\left(c\begin{bmatrix}I_{\hat{n}^2} & 0_{\hat{n}^2}\\ 0_{\hat{n}^2} & P_2\end{bmatrix}+\begin{bmatrix}0_{\hat{n}^2} & -I_{\hat{n}^2}\\ P_0 & P_1\end{bmatrix}\right)=0.\end{aligned}$$

Then by identifying P_2, P_1 and P_0 from (20), we obtain (18). Hence, the solutions ϑ of the generalized eigenvalue problem (19) correspond to values for c for which $\tilde{A}_c$ has a pole on the circle with radius γ or have symmetric pole pair with respect to this circle. This is true for all real and positive ϑ_i. Hence, by a continuity argument it follows that $\rho(\tilde{A}_{c_m})=\gamma$, with $c_m=\max_{i|\vartheta_i\in\mathbb{R}^+}\vartheta_i$ and that $\rho(\tilde{A}_c)<\gamma$ for $c>c_m$.

Indeed, we already know from Theorem 1 that for $c\geq c_u$, with c_u defined by (17), we have that $\rho(A_c)\leq\gamma$. For $c=c_m$, there are two possibilities: 1) $\tilde{A}_{c_m}$ has a pole pair symmetric to the circle with radius γ, and 2) $\tilde{A}_{c_m}$ has at least one pole on the circle with radius γ. We will now show that 1) is impossible. Observe that assuming 1) implies that $\rho(\tilde{A}_{c_m})>\gamma$. A continuity argument and the definition of c_m being the largest real eigenvalue ϑ_i of (19), tell us that that for $c>c_m$, there is no crossing of the poles $\lambda(\tilde{A}_c)$ of $\tilde{A}_c$ with the circle with radius γ. This implies that for $c>c_u$, $\rho(\tilde{A}_c)>\gamma$, which is in contradiction with Theorem 1. Hence, $\tilde{A}_{c_m}$ has at least one pole on the circle with radius γ. Also $\rho(\tilde{A}_c)<\gamma$ for $c>c_m$, because the contradiction implies another crossing of the circle with radius γ, which is again impossible by the definition of c_m. □

Observe that $\rho(\tilde{A}_c)=\rho\left(\hat{A}\hat{\Sigma}_s(\hat{\Sigma}_s+cW)^{-1}\right)$ is a nonlinear function of c. Theorem 2 says that the value c_m is determined by the generalized eigenvalue problem (19) such that $\rho(\tilde{A}_c)\leq\gamma$ for $c\geq c_m$, with equality for $c=c_m$. However, the Theorem gives no further information on the evolution of $\rho(\tilde{A}_c)$, which may be a nonmonotonous function of c. Theorem 2 only tells us that the eigenvalues $\lambda(\tilde{A}_c)$ remain inside the disk with radius γ for $c>c_m$.

For high-order systems, a large generalized eigenvalue problem (19) of dimension $2\hat{n}^2$ has to be solved, hence requiring $O(\hat{n}^6)$ operations. When $\hat{n}$ is large, we propose instead an iterative algorithm to determine the value of c_m. From (17) of Theorem 1, a starting value $c_0=c_u$ is obtained such that $\rho(\tilde{A}_c)\leq\gamma$ for $c\geq c_0$. From this starting value c_0, we will now decrease c in each iteration step until $\rho(\tilde{A}_c)=\gamma$. This is done by using the following three relations between the spectral radius and matrix norms [6]: 1) $\rho(A)\leq\|A\|_M$, for any regular matrix M; 2) for a diagonalizable matrix A with eigenvalue decomposition $A=VDV^{-1}$, we have $\rho(A)=\|A\|_V$; for a nondiagonalizable matrix A, there exists a nonsingular matrix V such that $\rho(A)=\|A\|_V+\epsilon_V$, with ϵ_V arbitrarily small; 3) let $\|A\|_M\leq 1$, then $\|(I-A)^{-1}\|_M\leq(1-\|A\|_M)^{-1}$.

Let $\bar{A}_c$ be defined by $\bar{A}_c=\hat{A}\hat{\Sigma}_s(\hat{\Sigma}_s+cW)^{-1}$, then, by use of Lemma 1, we have that $\rho(\tilde{A}_c)=\rho(\bar{A}_c)$. To avoid the calculation of the inverse $(\hat{\Sigma}_s+cW)^{-1}$ for a new value c in each iteration step, $\hat{\Sigma}_s$ and W can be simultaneously diagonalized

$$\hat{\Sigma}_s+cW=Q_d(D_{\hat{\Sigma}_s}+cI_{\hat{n}})Q_d^T\tag{21}$$

where Q_d can be calculated as

$$Q_d=U_WS_W^{1/2}U_{\hat{\Sigma}_s}\tag{22}$$

with the SVD of $W=U_WS_WU_W^T$ and the SVD of $S_W^{-1/2}U_W^T\hat{\Sigma}_sU_WS_W^{-1/2}=U_{\hat{\Sigma}_s}D_{\hat{\Sigma}_s}U_{\hat{\Sigma}_s}^T$. This yields $\rho(\bar{A}_{c_{i+1}})=\rho\left(Q_d^{-1}\bar{A}_{c_{i+1}}Q_d\right)=\rho\left(Q_d^{-1}\bar{A}_{c_i}Q_d(D_{\hat{\Sigma}_s}+c_iI_{\hat{n}})(D_{\hat{\Sigma}_s}+c_{i+1}I_{\hat{n}})^{-1}\right)$.

Let c_i be the value of c after i iteration steps such that $\rho(\bar{A}_{c_i}) \leq \gamma$, with $\bar{A}_c = \hat{A}\hat{\Sigma}_s(\hat{\Sigma}_s + c_i W)^{-1}$, and let the eigenvalue decomposition of $Q_d^{-1}\bar{A}_{c_i}Q_d$ be given by $Q_d^{-1}\bar{A}_{c_i}Q_d = V_i D_i V_i^{-1}$, then we can calculate $\Delta c_i = c_i - c_{i+1} > 0$ such that $\rho(\bar{A}_{i+1}) \leq \gamma$ or

$$\begin{aligned}\rho(\bar{A}_{c_{i+1}}) &\leq \left\|Q_d^{-1}\bar{A}_{c_i}Q_d\left(D_{\hat{\Sigma}_s}+c_iI_{\hat{n}}\right)\left(D_{\hat{\Sigma}_s}+c_{i+1}I_{\hat{n}}\right)^{-1}\right\|_{V_i}\\ &\leq \left\|Q_d^{-1}\bar{A}_{c_i}Q_d\right\|_{V_i}\\ &\quad\times\left\|\left(D_{\hat{\Sigma}_s}+c_iI_{\hat{n}}\right)\left(D_{\hat{\Sigma}_s}+c_{i+1}I_{\hat{n}}\right)^{-1}\right\|_{V_i}\\ &=(\rho(\bar{A}_{c_i})+\epsilon_{V_i})\left\|\left(I_{\hat{n}}-\Delta c_i\left(D_{\hat{\Sigma}_s}+c_iI_{\hat{n}}\right)^{-1}\right)^{-1}\right\|_{V_i}\\ &\leq\frac{\rho(\bar{A}_{c_i})+\epsilon_{V_i}}{1-\Delta c_i\left\|\left(D_{\hat{\Sigma}_s}+c_iI_{\hat{n}}\right)^{-1}\right\|_{V_i}}.\end{aligned}\tag{23}$$

Putting this upper bound (23) for $\rho(\bar{A}_{c_{i+1}})$ equal to γ yields

$$\Delta c_i = \frac{\frac{1-(\rho(\bar{A}_{c_i})+\epsilon_{V_i})}{\gamma}}{\left\|\left(D_{\hat{\Sigma}_s}+c_iI_{\hat{n}}\right)^{-1}\right\|_{V_i}}.\tag{24}$$

Since $\rho(\bar{A}_{c_i}) < \gamma$ and ϵ_{V_i} can be made arbitrarily small, it is easily verified that for this value of $\Delta c_i > 0$ the last step in (23) is valid. Also notice that $\rho(\bar{A}_{c_{i+1}}) \leq \gamma$. This yields the following iterative algorithm:

Input: $\hat{A}$, $\hat{\Sigma}_s$, W, γ and tolerance $\epsilon_{\text{tol}} > 0$.
Output: c_m.
Initialization: $i = 0$, determine c_0 from (17) in Theorem 1, calculate $D_{\hat{\Sigma}_s}$ and Q_d as described in (21) and (22), compute $\hat{A}_{Q_d} = Q_d^{-1}\hat{A}Q_d$ and $Q_d^{-1}\bar{A}_{c_0}Q_d = \hat{A}_{Q_d}\left(I_{\hat{n}} + c_0 D_{\hat{\Sigma}_s}^{-1}\right)^{-1}$; and compute the eigenvalue decomposition $Q_d^{-1}\bar{A}_{c_0}Q_d = V_0 D_0 V_0^{-1}$ (or a V_0 such that $\epsilon_{V_0} \ll \gamma - \rho(Q_d^{-1}\bar{A}_{c_0}Q_d)$)
while $\gamma - \rho(D_i) > \epsilon_{\text{tol}}$
determine Δc_i from (24) and put $c_{i+1} = c_i - \Delta c_i$ and $c_m = c_{i+1}$
compute $Q_d^{-1}\bar{A}_{c_{i+1}}Q_d = \hat{A}_{Q_d}\left(I_{\hat{n}} + c_{i+1}D_{\hat{\Sigma}_s}^{-1}\right)^{-1}$ and compute the eigenvalue decomposition $Q_d^{-1}\bar{A}_{c_{i+1}}Q_d = V_{i+1}D_{i+1}V_{i+1}^{-1}$ (or a V_{i+1} such that $\epsilon_{V_{i+1}} \ll \gamma - \rho(Q_d^{-1}\bar{A}_{c_{i+1}}Q_d)$)
$i = i + 1$
endwhile

The initialization results in $c_0 \leq c_m$. Since $\Delta c_i \geq 0$ by construction, the algorithm decreases the amount of regularization in each iteration step, i.e., $c_{i+1} \leq c_i$, with equality iff $\rho(\tilde{A}_{c_i}) + \epsilon_{V_i} = \gamma$ and where ϵ_{V_i} can be made arbitrarily small. Hence, c_i converges to the largest c such that $\rho(\tilde{A}_c) = \gamma$. This value is equal to c_m as follows from Theorem 2. The algorithm requires $O(\hat{n}^3)$ operations per iteration step and the simulation results in [12] indicate linear convergence. More advanced algorithms like bisection algorithms may speed up the convergence, but one looses the guaranteed convergence to c_m, as it is not guaranteed that $\rho(\tilde{A}_c)$ is a monotonous function of c. The problem can also be formulated as a real stability radius problem with one repeated block for which a fast algorithm exists [9], but this algorithm requires more complex computations (an LMI problem) per iteration step and faster convergence has not been proven.

Now, we show that the so-called data augmentation method of Chui and Maciejowski [2] corresponds to adding regularization terms to the least squares cost function with specific choices for the weighting matrix W. In [2], the nonsteady Kalman filter state sequences $\hat{X}_i$, $\hat{X}_{i+1}$ are iteratively augmented by appending $\sqrt{c_p}V_p$ and $0_{\hat{n}\times 1}$ ($\sqrt{c_q}[V_q \;\; V_q^*]$ and $0_{\hat{n}\times 2}$) respectively, for each unstable pole λ_p (pole pair (λ_q, λ_q^*)) with corresponding normalized right eigenvector V_p (right eigenvector V_q and its conjugate V_q^*). The inputs $U_{i|i}$ and outputs $Y_{i|i}$ are extended by appending null vectors of appropriate dimensions. The constant c_p (c_q for a complex pole pair) is determined such that the magnitude of the stabilized pole is M, with $M \geq 0$ chosen by the user. The other eigenvalues are not changed.

For the case of one real unstable pole, this method corresponds to minimizing the following cost function in least squares sense: $\min_{\tilde{A}_{CM},\tilde{B}_{CM}} \left\|\hat{X}_{i+1} - \tilde{A}_{CM}\hat{X}_i - \tilde{B}_{CM}U_{i|i}\right\|_F^2 + c_p\left\|\tilde{A}_{CM}V\right\|_F^2$, with the regularization term $c_p\|\tilde{A}_{CM}V\|_F^2 = c_{CM}\,\text{Tr}(\tilde{A}_{CM}VV^T\tilde{A}_{CM}^T)$. More regularization terms are iteratively added for each other pole or pole pair with amplitude larger than γ. Hence, the data augmentation method of Chui and Maciejowski [2] corresponds to the following weighting matrix in the regularization term (7):

$$\begin{aligned}&c\,\text{Tr}\left(\tilde{A}_cW\tilde{A}_c^T\right)\\ &\quad=\sum_{\text{unstable poles }p} c_p\,\text{Tr}\left(\tilde{A}_cV_pV_p^T\tilde{A}_c^T\right)\\ &\qquad+\sum_{\text{unstable pole pairs }q} c_q\,\text{Tr}\left(\tilde{A}_c\left[V_q \;\; V_q^*\right]\left[V_q \;\; V_q^*\right]^{\text{T}}\tilde{A}_c^T\right)\end{aligned}$$

or $cW = \sum_p c_pV_pV_p^T + \sum_q c_q\left[V_q \;\; V_q^*\right]\left[V_q \;\; V_q^*\right]^{\text{T}}$, which is a special case of $cJ_2(\tilde{A})$.

In the companion paper [12], Monte Carlo simulations were conducted to compare the performances of a stable reduced order model (obtained without regularization) with two full order models, forced to be stable by applying regularization 1) with $W = I_{\hat{n}}$, corresponding to ridge regression [3], [4], and 2) with the data augmentation method [2]. The simulation results, conducted on a finite number of data points in the presence of noise, illustrate that the use of regularization allows to identify stable high order models with better performance than the reduced order model. The main conclusions of the comparison between the two choices for the weighting matrix are that ridge regression generally yields (slightly) better results than data augmentation [2], while ridge regression achieves a much lower variance on the elements of $\tilde{A}_c$. These results can also be understood from regularization theory [1], [3]–[5], where the choice of the identity matrix for the weighting matrix is motivated by the maximum entropy principle, equally penalizing all directions of the solution. We refer to [12] for a detailed discussion of the simulation results.

III. Conclusion

Subspace methods for the identification of linear time-invariant systems are known to be asymptotically unbiased. The system matrices are usually estimated from least squares, based on estimated Kalman filter state sequences and the observed inputs and outputs. However, for a finite number of data points, it is not guaranteed that the least squares estimate yields a stable system, even when it is known that the true linear system is stable. In this note, stability of the estimated system is imposed by adding a regularization term to the least squares cost function. The regularization term used here is the trace of a matrix which

involves the system matrix and a positive (semi) definite weighting matrix. The amount of regularization is calculated by solving a generalized eigenvalue problem. The data augmentation method proposed by Chui and Maciejowski can be interpreted as iteratively applying regularization with specific choices for the weighting matrix. Different choices for the weighting matrix can result into different solutions. In ridge regression the identity matrix is used for the weigthing matrix, which has been motivated by the maximum entropy principle in regularization theory.

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A New Bounded Real Lemma Representation for the Continuous-Time Case

U. Shaked and V. Suplin

***Abstract*—A differential linear matrix inequality (DLMI) approach is introduced for the solution of various linear continuous-time control problems. The proposed method permits the application of linear matrix inequalities (LMIs) to the solution of control design problems under uncertainty. These problems are solved for finite horizon linear systems while considerably reducing the overdesign inherent in previous methods. The new approach also allows for the solution of the output-feedback control problem for systems belonging to a finite set of uncertain plants with hardly any overdesign. Four examples are given to demonstrate the applicability of the new method.**

***Index Terms*—Differential linear matrix inequality (DLMI), recursive linear matrix inequalities (LMIs), robust control.**

I. INTRODUCTION

Linear matrix inequalities (LMIs) are now widely used to solve various linear control and filtering problems. The main advantage of the LMIs lie in their ability to tackle multiple objectives and to deal with polytopic type uncertainties [1]. In the past they were used mainly to solve stationary problems. The algebraic nature of the Riccati equations to which these problems correspond, enables the construction of equivalent LMIs by applying the Schur complements formula [2].

Unfortunately, in cases where the systems involved are time-varying or when the time-horizon is finite, differential or difference LMIs appear. The degree of freedom that is entailed in solving these inequalities at each instant of time should be exploited to derive the best solution that will enable the optimal solution at future instances of time. In the discrete-time case, recursive sets of LMIs are obtained, and the question which was raised in [5] was how to find a solution for a given set of difference LMIs at, say, the k-th instant which enables the best solution to these LMIs at instances $i > k$. The method developed in [5] was used to solve robust control and filtering problems and the relationship between the proposed solutions and the corresponding 'central' solutions was discussed.

In this note we adopt a similar approach for continuous-time systems by discretizing the time scale and developing LMIs that resemble the ones obtained in [5] for the discrete-time case. The results we obtain here enable the solution of the state and output-feedback control problems for time-varying systems over a finite horizon. They also provide an efficient means for solving the control problem for multiple operating points and, by an appropriate gridding of the uncertainty intervals, for the robust control of systems with polytopic uncertainties.

II. SOLUTION OF THE BRL VIA DISCRETIZATION

Given the following system $\mathcal{S}(A, B, C)$:

$$\dot{x} = Ax + Bw \quad z = Cx \quad x(0) = 0 \tag{1}$$

where

$x \in \mathcal{R}^n$	system state vector;
$w \in \mathcal{L}_2^q[0, T]$	exogenous disturbance signal;

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The authors are with the Department of Electrical Engineering-Systems, Tel Aviv University, Tel Aviv 69978, Israel.

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