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Systems & Control Letters 50 (2003) 371–381
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

Computing the zeros of periodic descriptor systems

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Received 17 October 2002; received in revised form 16 May 2003; accepted 17 May 2003

Abstract

In this paper, we give a numerically reliable algorithm to compute the zeros of a periodic descriptor system. The algorithm is a variant of the staircase algorithm applied to the system pencil of an equivalent lifted time-invariant state-space system and extracts a low-order pencil which contains the zeros (both finite and infinite) as well as the Kronecker structure of the periodic descriptor system. The proposed algorithm is efficient in terms of complexity by exploiting the structure of the pencil and is exclusively based on orthogonal transformations, which ensures some form of numerical stability.
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Keywords: Periodic systems; Descriptor systems; Zeros; Poles; Numerical methods

1. Introduction

Zeros of transfer functions play an important role in the analysis and design of multi-variable linear systems: besides characterizing when the system is minimum phase or not, the zeros provide information on several structural properties of a system. Also reachability/stabilizability and observability/detectability is defined in terms of the zeros of particular systems without outputs or inputs, and even the poles can be seen as zeros of a system without inputs and outputs. For periodic systems there was no reliable numerical algorithm available for computing zeros until recently [29] a stable and efficient algorithm was presented for periodic systems in standard state space form. In the present paper, we pursue the same idea and present a more general method which handles the periodic descriptor case as well. Such a tool is very important since it provides a way to evaluate the transfer-function matrix of a general periodic system [25] and fills a gap in the set of stable and efficient algorithms for periodic systems of the most general class [28].

We consider here periodic time-varying descriptor systems of the form

$$E_k x(k+1) = A_k x(k) + B_k u(k), \qquad y(k) = C_k x(k) + D_k u(k), \tag{1}$$

where the matrices $E_k \in \mathbb{R}^{\mu_{k+1}\times n_{k+1}}$, $A_k \in \mathbb{R}^{\mu_{k+1}\times n_k}$, $B_k \in \mathbb{R}^{\mu_{k+1}\times m}$, $C_k \in \mathbb{R}^{p\times n_k}$, $D_k \in \mathbb{R}^{p\times m}$ are periodic with period $K \geqslant 1$, and the dimensions fulfill the condition $\sum_{k=1}^{K} \mu_k = \sum_{k=1}^{K} n_k$. For the computation of zeros, it

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doi:10.1016/S0167-6911(03)00194-4

is important to consider the more general case of time-varying dimensions. Since the transmission zeros of a standard system are defined in terms of a minimal realization, a similar definition is appropriate also for the zeros of a periodic system (see for example [15]). However, the minimal realization theory of standard periodic systems (i.e., $E_k = I_{n_{k+1}}$) revealed (see for example [6,9]) that minimal order (i.e., reachable and observable) state-space realizations of periodic systems have, in general, time-varying state dimensions. It follows immediately that the minimal realization of a periodic descriptor system computed, for example, via a forward–backward decomposition [22], leads in general to rectangular descriptor matrices E_k as well. Note that standard periodic systems with time-varying dimensions have been already considered earlier in [13,9]. But only recently, numerically reliable algorithms for systems with time-varying dimensions have been developed. Notable examples are the recent algorithms for the computation of a minimal realization [23] and the evaluation of the transfer-function matrix of a periodic system [25]. Finally, the development of general algorithms able to address the case of time-varying dimensions, is one of the requirements formulated for a *satisfactory* numerical algorithm for periodic systems [28].

The definition of zeros of a periodic descriptor system can be introduced starting from the stacked forms of time-invariant lifted reformulations [18,8,13]. These zeros have a nice interpretation in terms of periodic blocking property of exponentially periodic input signals [2,5]. The computation of zeros using lifted reformulations leads to large order standard or descriptor system representations with sparse and highly structured matrices. While the direct application of the numerically stable methods of [7,17] to these representations is certainly possible, the computational complexity for large-order systems is very high. Assuming constant dimensions $\mu_i = n_i = n$, such an algorithm has a complexity of $O((Kn)^3)$, instead of an expected complexity of $O(Kn^3)$ for a satisfactory algorithm [28]. In the case that E_k is square and invertible, we can always multiply the first equation of (1) by E_k^{-1} from the left to reduce it to a standard periodic system. One can then also use the lifting approach proposed by Meyer and Burrus [16] as basis for computing zeros. However, using this second lifting to compute zeros involves now forming products of up to K matrices. Apart from being computationally expensive, the explicit computation of this lifted reformulation can also lead to severe numerical difficulties. In passing, we note that alternative approaches like those based on the manipulation of polynomial matrices (e.g., to compute zeros via the Smith-form of the lifted system pencil) are out of discussion because of their well-known lack of numerical stability [21].

Although the lifting techniques are useful for their theoretical insight, their sparsity and structure may not be suited for numerical computations. This is why, in parallel to the theoretical developments, numerical methods have been developed that try to exploit this structure. The purpose of this paper is to propose a numerical approach to compute the zeros of the periodic system (1) which meets the requirements of generality, speed and accuracy for a good numerical algorithm for periodic systems as formulated in [28]. This goal is mainly achieved by exploiting the sparse structure of the associated lifted system pencil by performing locally row compressions to extract a low-order pencil (of the order of $\max_i \{n_i\}$) which contains the zeros (both finite and infinite) as well as the Kronecker structure of the periodic system. For the low-order pencil, standard methods can be employed to determine the zeros and the Kronecker structure (e.g., [17]). The new algorithm belongs to the family of fast, structure exploiting algorithms and relies exclusively on using orthogonal transformations. This is why, for the overall zeros computation a certain form of numerical stability can be ensured. The proposed algorithm solves the zeros computation problem for descriptor periodic systems in its most general setting, being a generalization of the algorithm developed by the authors for standard periodic systems [29].

2. Zeros and poles of periodic systems

To define the zeros and poles of periodic system (1), we define first the *transfer-function matrix* (TFM) corresponding to the associated *stacked lifted representation* [13], which is a time-invariant descriptor system representation of the form

$$L_k^{\mathrm{S}} x_k^{\mathrm{S}}(h+1) = F_k^{\mathrm{S}} x_k^{\mathrm{S}}(h) + G_k^{\mathrm{S}} u_k^{\mathrm{S}}(h), \qquad y_k^{\mathrm{S}}(h) = H_k^{\mathrm{S}} x_k^{\mathrm{S}}(h) + J_k^{\mathrm{S}} u_k^{\mathrm{S}}(h), \tag{2}$$

where $G_k^{\mathrm{S}} = \mathrm{diag}(B_k, B_{k+1}, \ldots, B_{k+K-1})$, $H_k^{\mathrm{S}} = \mathrm{diag}\,(C_k, C_{k+1}, \ldots, C_{k+K-1})$, $J_k^{\mathrm{S}} = \mathrm{diag}\,(D_k, D_{k+1}, \ldots, D_{k+K-1})$, and

$$F_k^{\mathrm{S}} - zL_k^{\mathrm{S}} = \begin{bmatrix} A_k & -E_k & O & \cdots & O \\ O & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -E_{k+K-3} & O \\ O & & \ddots & A_{k+K-2} & -E_{k+K-2} \\ -zE_{k+K-1} & O & \cdots & O & A_{k+K-1} \end{bmatrix}. \tag{3}$$

Assuming the square pencil (3) is regular (i.e. $\det(F_k^{\mathrm{S}} - zL_k^{\mathrm{S}})$ is not identically 0), the TFM of the stacked lifted system is

$$W_k^{\mathrm{S}}(z) = H_k^{\mathrm{S}}(zL_k^{\mathrm{S}} - F_k^{\mathrm{S}})^{-1} G_k^{\mathrm{S}} + J_k^{\mathrm{S}} \tag{4}$$

and the associated system pencil is defined as

$$S_k^{\mathrm{S}}(z) = \begin{bmatrix} F_k^{\mathrm{S}} - zL_k^{\mathrm{S}} & G_k^{\mathrm{S}} \\ H_k^{\mathrm{S}} & J_k^{\mathrm{S}} \end{bmatrix}, \tag{5}$$

which both depend on the sampling time k. Obviously, $W_{k+K}^{\mathrm{S}}(z) = W_k^{\mathrm{S}}(z)$ and the TFMs at two successive values of k are related by the following relation [11]:

$$W_{k+1}^{\mathrm{S}}(z) = \begin{bmatrix} 0 & I_{p(K-1)} \\ zI_p & 0 \end{bmatrix} W_k^{\mathrm{S}}(z) \begin{bmatrix} 0 & z^{-1}I_m \\ I_{m(K-1)} & 0 \end{bmatrix}.$$

It follows from this relation that poles and zeros of the TFMs for different sampling times, can only differ at $z = 0$ and ∞.

This lifting technique uses the input–output behavior of the system over time intervals of length K, rather then 1. For a given sampling time k, the corresponding mK-dimensional input vector, pK-dimensional output vector and $(\sum_{k=1}^{K} n_k)$-dimensional state vector are

$$u_k^{\mathrm{S}}(h) = [u^{\mathrm{T}}(k+hK) \cdots u^{\mathrm{T}}(k+hK+K-1)]^{\mathrm{T}}, \qquad y_k^{\mathrm{S}}(h) = [y^{\mathrm{T}}(k+hK) \cdots y^{\mathrm{T}}(k+hK+K-1)]^{\mathrm{T}},$$

$$x_k^{\mathrm{S}}(h) = [x^{\mathrm{T}}(k+hK) \cdots x^{\mathrm{T}}(k+hK+K-1)]^{\mathrm{T}}.$$

In order to define poles and zeros of periodic system (1), we need minimality of the system and of realization (2). This is equivalent to the notion of reachability and observability at finite and infinite eigenvalues of pencil (5), as introduced in [30]. If we assume that system (1) is minimal in that sense (this implies time-varying state dimensions and rectangular descriptor matrices) then we have the following definitions of poles, zeros and minimal indices of transfer function (4) based on system matrix (5) of the stacked lifted system with TFM (4).

Definition 1. The transmission zeros of the transfer function $W_k^{\mathrm{S}}(z)$ of the minimal periodic system (1) are the invariant zeros of the associated stacked system pencil (5).

Definition 2. The left and right minimal indices of the transfer function $W_k^{\mathrm{S}}(z)$ of the minimal periodic system (1) are those of the associated stacked system pencil (5).

Definition 3. The poles of the transfer function $W_k^{\mathrm{S}}(z)$ of the minimal periodic system (1) are the zeros of the associated stacked pole pencil $F_k^{\mathrm{S}} - zL_k^{\mathrm{S}}$ defined in (3).

Let us now introduce another lifting, which requires the matrices E_k to be invertible. This is the lifted system introduced in [16] and corresponds to the time-lifted system discussed in [3]. This lifting uses again the input–output behavior of the system over time intervals of length K, rather than 1. For a given sampling time k, the corresponding mK-dimensional input and pK-dimensional output vectors are the same as for the stacked system but an n_k-dimensional state vector is defined as

$$x_k^{\mathrm{L}}(h) := x(k + hK).$$

To define the lifted system we denote the $n_j \times n_i$ transition matrix of system (1) as $\Phi(j,i) = E_{j-1}^{-1}A_{j-1}E_{j-2}^{-1}A_{j-2}\cdots E_i^{-1}A_i$, where $\Phi(i,i) := I_{n_i}$. Then, the lifted system has the form

$$x_k^{\mathrm{L}}(h+1) = F_k^{\mathrm{L}}x_k^{\mathrm{L}}(h) + G_k^{\mathrm{L}}u_k^{\mathrm{S}}(h), \qquad y_k^{\mathrm{S}}(h) = H_k^{\mathrm{L}}x_k^{\mathrm{L}}(h) + L_k^{\mathrm{L}}u_k^{\mathrm{S}}(h), \tag{6}$$

where

$$F_k^{\mathrm{L}} = \Phi(k+K,k),$$

$$G_k^{\mathrm{L}} = [\Phi(k+K,k+1)E_k^{-1}B_k \;\; \Phi(k+K,k+2)E_{k+1}^{-1}B_{k+1} \cdots E_{k+K-1}^{-1}B_{k+K-1}],$$

$$H_k^{\mathrm{L}} = \begin{bmatrix} C_k \\ C_{k+1}\Phi(k+1,k) \\ \vdots \\ C_{k+K-1}\Phi(k+K-1,k) \end{bmatrix}, \quad J_k^{\mathrm{L}} = \begin{bmatrix} D_k & 0 & \cdots & 0 \\ J_{k,2,1} & D_{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J_{k,K,1} & J_{k,K,2} & \cdots & D_{k+K-1} \end{bmatrix}$$

with

$$J_{k,i,j} = C_{k+i-1}\Phi(k+i-1,k+j)E_{k+j-1}^{-1}B_{k+j-1}$$

for $i = 2,\ldots,K,\ j = 1,2,\ldots,K-1$, and $i > j$.

System (6) is called the *standard lifted system* at time k of the given K-periodic system (1). The associated TFM $W_k^{\mathrm{L}}(z)$ is

$$W_k^{\mathrm{L}}(z) = H_k^{\mathrm{L}}(zI_{n_k} - F_k^{\mathrm{L}})^{-1}G_k^{\mathrm{L}} + J_k^{\mathrm{L}} \tag{7}$$

and depends again on the sampling time k. Let us now define the zeros and poles of periodic system (1) which we assume to be minimal, i.e. completely reachable and completely observable. It follows from [2] that lifted system (6) is minimal too and the converse is also true. Consider the system and pole pencils of the *standard lifted system*

$$S_k^{\mathrm{L}}(z) = \begin{bmatrix} F_k^{\mathrm{L}} - zI_{n_k} & G_k^{\mathrm{L}} \\ H_k^{\mathrm{L}} & J_k^{\mathrm{L}} \end{bmatrix}, \qquad [F_k^{\mathrm{L}} - zI_{n_k}]. \tag{8}$$

Definition 4. The transmission zeros of the transfer function $W_k^{\mathrm{L}}(z)$ of minimal periodic system (1) with all E_k invertible, are the invariant zeros of associated standard lifted system pencil (8).

Definition 5. The left and right minimal indices of the transfer function $W_k^{\mathrm{L}}(z)$ of the minimal periodic system (1) with all E_k invertible, are the left and right minimal indices of associated standard lifted system pencil (8).

Definition 6. The poles of the transfer function $W_k^{\mathrm{L}}(z)$ of minimal periodic system (1) with all E_k invertible, are the zeros of the associated standard lifted pole pencil $F_k^{\mathrm{L}} - zI_{n_k}$, or equivalently, the eigenvalues of the monodromy matrix F_k^{L}.

These definitions completely rely on the Kronecker structure of pencils (8). However, since these pencils involve forming matrix products, they are certainly not suited for reliable computations.

The relation of these pencils with those of the stacked lifted system is explained by the following lemma, which is easily proven by standard elimination.

Lemma 1. *If all E_k are invertible, then there exist invertible matrices T_ℓ and T_{r} and matrices X and Y such that*

$$\left[\begin{array}{cc|c} I_{N_k} & O & O \\ O & F_k^{\mathrm{L}} - zI_{n_k} & G_k^{\mathrm{L}} \\ \hline O & H_k^{\mathrm{L}} & L_k^{\mathrm{L}} \end{array}\right] = \left[\begin{array}{c|c} T_\ell & O \\ \hline X & I \end{array}\right] \left[\begin{array}{c|c} F_k^{\mathrm{S}} - zL_k^{\mathrm{S}} & G_k^{\mathrm{S}} \\ \hline H_k^{\mathrm{S}} & L_k^{\mathrm{S}} \end{array}\right] \left[\begin{array}{c|c} T_{\mathrm{r}} & Y \\ \hline O & I \end{array}\right]$$

and hence also

$$\begin{bmatrix} I_{N_k} & O \\ O & F_k^{\mathrm{L}} - zI_{n_k} \end{bmatrix} = T_\ell [F_k^{\mathrm{S}} - zL_k^{\mathrm{S}}] T_{\mathrm{r}},$$

where $N_k := n_{k+1} + \cdots + n_{k+K-1}$.

By taking a Schur complement, it is now easy to see that $W_k^{\mathrm{S}}(z) = W_k^{\mathrm{L}}(z)$, that is, the TFMs of the *stacked* and *standard lifted systems* are the same. Moreover, both system pencils are essentially equivalent since the transformations of the above lemma only eliminated the non-dynamical part of the systems matrix $S_k^{\mathrm{S}}(z)$ [30]. We have the following immediate results.

Lemma 2. *The zeros and left and right minimal indices of the system pencils* (5) *and* (8) *are identical. The zeros of the pole pencils $F_k^{\mathrm{S}} - zL_k^{\mathrm{S}}$ and $F_k^{\mathrm{L}} - zI_{n_k}$ are identical.*

It follows from this discussion that Definitions 1–3 introduced for general matrices E_k based on the stacked lifted, coincide with Definitions 4–6 when the matrices E_k are invertible. But Definitions 1–3 clearly apply as well to the case where any of the matrices E_k is singular or even rectangular. These are also the definitions which we will use in the next section.

3. Computational approach

In this section, we propose an efficient computational approach to determine the zeros of the *stacked lifted system* (2) at $k = 1$. The zeros for other time moments $k = 2, \ldots, K$ can be computed in a similar manner by just permuting the order of the underlying matrices. To simplify the notation for the case $k = 1$, we drop the index used for the sampling time in the lifted system matrices. Before starting our developments, we discuss shortly possible approaches relying on existing algorithms for standard systems.

For a standard periodic system, a straightforward approach to compute the zeros of the $pK \times mK$ TFM $W^{\mathrm{L}}(z)$ is to apply the algorithm of [7] to system matrix (8) and to extract additional information on zeros and Kronecker structure using the results of [20]. However, because the construction of the *standard lifted system* involves matrix multiplications, this approach is certainly not recommended for numerical computation. To avoid matrix multiplications, we can employ the general approach of [17] to the *stacked lifted system* and compute the system zeros as the invariant zeros of system matrix (5). This approach is numerically reliable because it exclusively uses orthogonal transformations. But since it ignores the structure of the problem, the computational complexity of this approach is too high. To compute the zeros, the computational complexity is, in the worst case, of the order of $\mathrm{O}((N+Kp)(N+Km)N)$ operations, where $N=\sum_{i=1}^{K} n_i$. For example, in the case of a periodic system with constant dimensions $\mu_i = n_i = n$, the computational complexity is $\mathrm{O}(K^3(n+p)(n+m)n)$ instead of a complexity of $\mathrm{O}(K(n+p)(n+m)n)$ which—as reported in [28]—would be more satisfactory for periodic systems. In what follows, we show that such a computational complexity can indeed be achieved by exploiting the problem structure.

A *fast* numerical algorithm to compute eigenvalues of products of square matrices (introduced in [22]) can be used to compute the poles of periodic systems with constant dimensions by deflating the $(K-1)n$ simple eigenvalues at infinity of the pencil $F^{\mathrm{S}} - zL^{\mathrm{S}}$ by applying $(K-1)$ orthogonal transformations on low-order submatrices of this pencil. This approach is an orthogonal version of the technique employed by Luenberger [14] and is equivalent to the "swapping" technique described in [1]. The extension of this algorithm to the time-varying case is relatively straightforward and is a particular case of the approach proposed in this paper. In what follows, we show how this idea can be applied to deflate a part of system pencil (5) which corresponds to $\sum_{i=2}^{K} \mu_i$ simple eigenvalues at infinity. Since the multiplicity of eigenvalues by definition exceeds the multiplicity of infinite zeros by one [30], this deflation will not affect the computation of both finite and infinite zeros.

Instead of $S^{\mathrm{S}}(z)$ in (5), we consider an equivalent pencil $S(z)$ with permuted block rows and columns

$$S(z) = S - \lambda T = \begin{bmatrix} S_1 & -T_1 & O & \cdots & O \\ O & S_2 & -T_2 & \cdots & O \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & & & S_{K-1} & -T_{K-1} \\ -zT_K & O & \cdots & O & S_K \end{bmatrix}, \tag{9}$$

where for $i = 1, \ldots, K$

$$S_i := \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \qquad T_i := \begin{bmatrix} E_i & O \\ O & O \end{bmatrix}.$$

Consider the $(\mu_2+\mu_3+2p)$th-order orthogonal transformation matrix U_1 compressing the rows of the $(\mu_2+\mu_3+2p)\times(n_2+m)$ matrix $\left[\begin{smallmatrix} -T_1 \\ S_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} R_1 \\ O \end{smallmatrix}\right]$, where R_1 is an $r_1 \times (n_2+m)$ matrix of full row rank r_1 with $n_2 \leqslant r_1 \leqslant \min(\mu_2+\mu_3+p, n_2+m)$ (the lower bound follows from the minimality of the system). Applying U_1 to the first two blocks rows of $S(z)$ we obtain for the nonzero elements

$$U_1 \begin{bmatrix} S_1 & -T_1 & O \\ O & S_2 & -T_2 \end{bmatrix} = \begin{bmatrix} \tilde{S}_1 & R_1 & -\tilde{T}_1 \\ \hat{S}_2 & O & -\hat{T}_2 \end{bmatrix},$$

which defines the new matrices $\hat{S}_2$ and $\hat{T}_2$. These matrices clearly have $v_3 + p$ rows, where

$$v_3 := \mu_3 + \mu_2 + p - r_1.$$

Then construct the $(v_{i+1} + \mu_{i+2} + 2p)$th order orthogonal transformations U_i for $i = 2, \ldots, K-1$ such that

$$U_i \begin{bmatrix} \hat{S}_i & -\hat{T}_i & O \\ O & S_{i+1} & -T_{i+1} \end{bmatrix} = \begin{bmatrix} \tilde{S}_i & R_i & -\tilde{T}_i \\ \hat{S}_{i+1} & O & -\hat{T}_{i+1} \end{bmatrix},$$

where R_i are matrices of full row rank r_i. This recursively defines the new matrices $\hat{S}_{i+1}$ and $\hat{T}_{i+1}$ which have $v_{i+2} + p$ rows, where

$$v_{i+2} := \mu_{i+2} + v_{i+1} + p - r_i.$$

Applying the transformations U_i successively to the ith and $(i+1)$th block rows of the transformed pencil $S(z)$, we finally obtain the reduced pencil

$$\left[\begin{array}{c|cccc} \tilde{S}_1 & R_1 & -\tilde{T}_1 & O & O \\ \tilde{S}_2 & O & R_2 & \ddots & O \\ \vdots & \vdots & \vdots & \ddots & -\tilde{T}_{K-2} \\ \tilde{S}_{K-1} - z\tilde{T}_{K-1} & O & O & \cdots & R_{K-1} \\ \hline \hat{S}_K - z\hat{T}_K & O & O & \cdots & O \end{array}\right], \tag{10}$$

which is orthogonally similar to $S(z)$ and hence also to the original system pencil $S^{\mathrm{S}}(z)$ in (5). Since the matrices R_i have full row rank, the subpencil

$$\hat{S}_K - z\hat{T}_K \tag{11}$$

will contain all finite zeros of the original pencil. The Kronecker structure and the infinite zeros of $S^{\mathrm{S}}(z)$ are essentially those of the subpencil

$$\begin{bmatrix} \tilde{S}_{K-1} - z\tilde{T}_{K-1} & R_{K-1} \\ \hat{S}_K - z\hat{T}_K & O \end{bmatrix}. \tag{12}$$

The number of rows of this reduced pencil equals

$$v_{K+1} := \sum_{i=1}^{K} (n_i + p) - \sum_{i=1}^{K-2} r_i = \sum_{i=1}^{K-2} (n_{i+1} + p - r_i) + n_K + n_1 + 2p.$$

Generically, all submatrices of the pencil have maximal ranks $r_i = n_{i+1} + \min(p, m)$. If $p \leqslant m$ then $v_{K+1} = n_K + n_1 + 2p$, while if $p > m$ it follows that $v_{K+1} = n_K + n_1 + 2p + (K-2)(p-m)$. It is therefore recommended to then work on the transposed of pencil (9).

To compute the finite zeros and Kronecker structure of the periodic system, we can now apply to resulting reduced-order pencils (11) and (12) a general algorithm to compute the eigenvalues and the Kronecker structure of a system matrix of a particular descriptor system [17].

The proposed algorithm to compute zeros can be applied to compute the poles as well by defining

$$S_i := A_i, \qquad T_i := E_i.$$

In a similar way, with

$$S_i := [A_i \quad B_i], \qquad T_i := [E_i \quad O]$$

or

$$S_i := \begin{bmatrix} A_i \\ C_i \end{bmatrix}, \qquad T_i := \begin{bmatrix} E_i \\ O \end{bmatrix}$$

the zeros algorithm can be used to compute the input decoupling and output decoupling zeros, respectively [12]. It is also obvious that it applies as well to systems in standard form, i.e. where $E_k = I_{n_{k+1}}$.

4. Algorithmic aspects

The reduction of $S(z)$ can be done by computing successively $K-1$ rank revealing QR decompositions (with column pivoting) of $(v_i + \mu_{i+1} + 2p) \times (n_i + m)$ matrices and applying the transformation to two sub-blocks of dimensions $(v_i + \mu_{i+1} + 2p) \times (n_{i-1} + m)$ and $(v_i + \mu_{i+1} + 2p) \times n_{i+1}$. Assuming constant dimensions $\mu_i = n_i = n,\ p \leqslant m$ and generic ranks (i.e., $v_i = n_i$), the reduction step has a computational complexity of $\mathrm{O}((K-1)(n+p)(n+m)n)$. Since the last step, the computation of zeros of the reduced pencil, has a complexity of $\mathrm{O}((n+p)(n+m)n)$, it follows that the overall computational complexity of the proposed approach corresponds to what is expected for a satisfactory algorithm for periodic systems. In fact, when this approach is employed to compute the poles, it is even more efficient than the standard algorithm based on the periodic real Schur form [4]. This is why, the proposed algorithm belong to the family of *fast* algorithms [22], being more efficient than an algorithm based on eigenvalue computation (if this is applicable, as for example, in the case when all D_k are invertible).

Since the main reduction consists of successive QR-decompositions, it can be shown [10] that the matrices of the computed reduced pencil $\bar{S} - \lambda\bar{T}$ satisfy

$$\|UXV - \bar{X}\|_2 \leqslant \varepsilon_{\mathrm{M}} f(N)\|X\|_2, \quad X = S, T,$$

where U and V are the matrices of accumulated left and right orthogonal transformations, ε_{M} is the relative machine precision, and $f(N)$ is a quantity of order of N. The subsequent zeros computation step is performed using the algorithm of [17] and is also based exclusively on orthogonal transformations. This second step is numerically stable as well. Overall, we have thus guaranteed that the computed zeros are exact for a slightly perturbed system pencil. It follows that the proposed algorithm to compute zeros is *numerically backward stable*.

Since the structure of the perturbed pencil is not preserved in the reduction, we cannot say however that the computed zeros are exact for a slightly perturbed original system (i.e., the algorithm is not *strongly* stable). In spite of this weaker type of stability, the proposed algorithm is the first numerically reliable procedure able to compute zeros of a periodic system with an acceptable computational effort. First results to develop a strongly stable algorithm to compute the finite zeros of a periodic system are reported in [26].

5. Numerical experiments

We give two examples that illustrate the capabilities of the proposed approach. The computations have been performed using MATLAB-based implementations relying, among others, on the generalized zeros computation tools available in the DESCRIPTOR TOOLBOX [24]. [1]

[1] http://www.robotic.dlr.de/control/num/desctool.html.

Example 1. This example analyzes the minimality of periodic descriptor systems by computing appropriate types of zeros. Consider the 2-periodic single-input single-output system described by the following matrices:

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \eta \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_1 = 1, \quad D_1 = 0,$$

$$E_2 = 2, \quad A_2 = [0 \quad 1], \quad B_2 = 2, \quad C_2 = [1 \quad 0], \quad D_2 = 0,$$

where the state-space dimensions are $n_1 = 1$, $n_2 = 2$. We computed first the zeros at time $k = 1$. For $\eta = 2$, this system has no input or output decoupling zeros, thus is minimal (i.e., reachable and observable). The system has a pole at $\rho = 0.25$ and a zero at $\mu = \infty$. However, for $\eta = 0$, the descriptor periodic system is non-minimal, having an input/output decoupling zero (i.e., unreachable/unobservable pole) at ∞. At time $k = 2$, the system is minimal. For $\eta = 2$ the system has the poles $\rho_1 = 0.25$, $\rho_2 = \infty$ and the zeros $\mu_1 = 0$, $\mu_2 = \infty$. For $\eta = 0$ the system has a pole in $\rho = 0$ and a zero at $\mu = \infty$.

Example 2. This example illustrates the use of the proposed approach to periodic systems with relatively large periods. Consider a discrete-time periodic system originating from a continuous-time periodic model of a spacecraft pointing and attitude system described in [19]. This system has state, input and output dimensions $n = 4$, $m = 1$, $p = 2$, respectively. The continuous-time linearized state-space model of the spacecraft system is described by the matrices

$$A = \begin{bmatrix} 0 & 0 & 0.05318064 & 0 \\ 0 & 0 & 0 & 0.05318064 \\ -0.001352134 & 0 & 0 & -0.07099273 \\ 0 & -0.0007557182 & 0.03781555 & 0 \end{bmatrix},$$

$$B(t) = \begin{bmatrix} 0 \\ 0 \\ 0.1389735 \times 10^{-6} \sin(\omega_0 t) \\ -0.3701336 \times 10^{-7} \cos(\omega_0 t) \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $\omega_0 = 0.00103448$ rad/s is the orbital frequency. Notice that A is a constant matrix with all its eigenvalues on the imaginary axis. The matrix $B(t)$ is however a time-dependent periodic matrix with the period $2\pi/\omega_0$. The discretized system for different sampling periods K has been used in [27] to design periodic output feedback controller for this system. For a given K, the corresponding sampling period is $T = 2\pi/(\omega_0 K)$. The matrices of the discrete-time periodic system can be computed explicitly as $E_k = I$, $A_k = \exp(AT)$, $B_k = \int_{(k-1)T}^{kT} e^{A[kT-\tau]} B(\tau)\, d\tau$. To show the applicability of our algorithm to periodic systems with large periods, we computed the zeros of the TFM for $K = 120$. Note that the corresponding TFM is a 240×120 rational matrix.

For reference purposes we give the matrices of the discretized periodic model which results for $K = 120$ and $T = 50.61468$ s

$$A_k = \begin{bmatrix} 0.9506860 & 0.0429866 & 0.4827320 & -2.5564383 \\ -0.0409684 & 0.9721628 & 1.3617328 & 0.5081454 \\ -0.0122736 & 0.0363280 & -0.8671394 & -0.6014295 \\ -0.0346225 & -0.0072209 & 0.3203622 & -0.8456626 \end{bmatrix},$$

Table 1
Computational times for zeros determination

K	40	80	120	240
T_{fast} (s)	0.05	0.06	0.11	0.17
T_{lifted} (s)	0.33	2.31	7.74	61.46

$$B_k = 10^{-5}\begin{bmatrix} 0.2220925 \\ -0.1300536 \\ 0.1877217 \\ -0.0271167 \end{bmatrix}\cos\theta_k + 10^{-5}\begin{bmatrix} 0.5035620 \\ 0.4241087 \\ 0.1218290 \\ 0.3583826 \end{bmatrix}\sin\theta_k,$$

where $\theta_k = 2\pi(k-1)/K$.

The periodic system for $K = 120$ has a zero at $\mu = \infty$ and has poles at

$$\rho_1 = 0.7626 + 0.6469\mathrm{i}, \quad \rho_2 = 0.7626 - 0.6469\mathrm{i},$$
$$\rho_3 = 0.9942 + 0.1077\mathrm{i}, \quad \rho_4 = 0.9942 - 0.1077\mathrm{i}.$$

Note that the order of the *stacked lifted system* is 480. Although the direct application of the zeros algorithm of [17] to this system is still feasible, it is certainly too expensive to solve this problem.

In Table 1, computational times are given to determine the zeros of the associated TFM for different values of K. The values for T_{fast} represent computational times for the proposed fast method, while the values for T_{lifted} are the times when applying the algorithm of [17] directly to the *stacked lifted system*. The computations have been done on a 866 MHz PC running MATLAB 6.1 under Windows ME. For the computation of zeros of both reduced pencil (12) and lifted pencil (9), the `szero` function of the DESCRIPTOR TOOLBOX [24] has been used. This function relies on the robust Fortran implementation of the algorithm of [17] available in the SLICOT library. [2]

It is easy to see that the computational time for the fast method varies almost linearly with K, and this confirms our claim for a computational complexity of $O(Kn^3)$ of the proposed approach. In contrast, when applying the algorithm of [17] to the *stacked lifted system*, the resulting times clearly indicates a computational complexity of $O(K^3n^3)$.

6. Conclusion

In this paper, we presented a numerically backward stable algorithm to compute the generalized eigenstructure of a stacked system matrix of a periodic system. This algorithm can be applied to find the zeros, poles and decoupling zeros of the system matrix and the left and right null space structures of the corresponding lifted transfer function. The algorithm works for matrices of varying dimension and exploits the block cyclic structure of the pencil to yield a complexity which is linear in the period K and cubic in the maximum dimension of the blocks.

Acknowledgements

Part of the work of the first author has been performed in the framework of the Swedish Strategic Research Foundation Grant "Matrix Pencil Computations in Computer-Aided Control System Design: Theory, Algorithms and Software Tools". The work of the second author has been performed as part of the Belgian

[2] http://www.win.tue.nl/niconet/niconet.html.

Programme on Interuniversity Attraction Poles, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with its authors.

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