Robust H_∞ filtering for a class of nonlinear systems with state delay and parameter uncertainty

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Abstract

This paper deals with the problem of robust H_∞ filtering for a class of state-delayed nonlinear systems with norm-bounded parameter uncertainty appearing in all the matrices of the linear part of the system model. The nonlinearities are assumed to satisfy the global Lipschitz conditions and appear in both the state and measured output equations. Attention is focused on the design of a nonlinear filter which ensures both the robust stability and a prescribed H_∞ performance of the filtering error dynamics for all admissible uncertainties. A sufficient condition for the existence of such a filter is given in terms of a linear matrix inequality (LMI). When this LMI is feasible, the expression of a desired H_∞ filter is also presented. A numerically example is provided to demonstrate the applicability of the proposed approach.

Keywords: H_∞ filtering, uncertain systems, time-delay systems, nonlinear systems, linear matrix inequality, robust filtering.

1 Introduction

State estimation is a subject of great practical and theoretical importance which has received much attention in the past years. The Kalman filtering approach is one of the most popular ways to deal with this topic. This approach provides an optimal estimation of some desired variables of a dynamic system from available measurements in the sense that the covariance of the estimation errors is minimized [1]. It should be pointed out that the Kalman filtering approach is based on the assumptions that the system under consideration is exactly known and its disturbances are stationary Gaussian noises with known statistics. In practical applications, however, the statistics of the noise sources may not be exactly known and the system uncertainties are unavoidable in modelling, which limit the application scope of the Kalman filtering approach.

To deal with the estimation problem for systems without exact knowledge of the statistics of the noise signals and with modelling uncertainties, H_∞ filtering has been introduced as an alternative [2, 10, 12]. The purpose is to design an estimator which guarantees that the $\mathcal{L}_2$-induced gain from the noise signals to the estimation error is below a prescribed level. It is worth noting that in the context of H_∞ filtering, the noise sources are assumed to be arbitrary signals with bounded energy, or bounded average power. It is known that the H_∞ filtering approach provides not only a guaranteed noise attenuation level but also robustness against

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unmodeled dynamics [13]. When parameter uncertainty appears in a system model, the robust H_∞ filtering problem has been investigated and many results have been reported. For example, for continuous systems with norm-bounded uncertainty, the robust H_∞ filtering problem was considered in [17] and a sufficient condition for the solvability was presented in terms of two Riccati equations. By using a similar technique, the results in [17] were extended to a class of uncertain nonlinear systems in [5]. The corresponding results for uncertain discrete-time systems with nonlinearities can be found in [18].

On the other hand, it is well-known that time delay arises quite naturally in propagation phenomena, population dynamics or engineering systems such as chemical processes, long transmission lines in pneumatic systems, and so on [7, 9]. A number of estimation and control problems relating to time-delay systems have been addressed by many researchers [8, 14, 19, 20]. More recently, attention has been focused on the problem of H_∞ filtering for time-delay systems. In [11], H_∞ filtering for systems with a single time delay in the measurements has been developed. For systems with time-delays in both the states and measurements, the same problem was considered in [6] and a Riccati based method was proposed. The robust H_∞ filtering problem for time-delay systems was addressed in [4], where an LMI approach was adopted, while in [16] a Riccati-like approach was developed. However, for time-delay systems in the simultaneous presence of parameter uncertainties and nonlinearities, the problem of robust H_∞ filtering has not been fully investigated, which is more involved and still open.

In this paper, we consider the problem of robust H_∞ filtering for a class of state-delayed nonlinear systems with parameter uncertainty. The class of systems under consideration is described by a linear delayed state space model with the addition of known nonlinearities which depend on state as well as delayed state and satisfy the global Lipschitz conditions. The nonlinearities and time delay appear in both the state and measured output equations. The parameter uncertainties are assumed to be time-varying norm-bounded, and appear in all the matrices of the linear part of the system model. The problem we address is the design of a nonlinear filter such that the filtering error dynamics are robustly stable and the $\mathcal{L}_2$-induced gain from the noise signals to the estimation error is less than a prescribed level for all admissible uncertainties. A sufficient condition for the solvability of this problem is obtained in terms of an LMI. The desired filter can be constructed through a convex optimization problem that can be efficiently implemented using standard numerical algorithms [3].

Notation. Throughout this paper, the notation $X \geq Y$ (respectively, $X > Y$) for symmetric matrices X and Y means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite); M^T represents the transpose of the matrix M; $\mathcal{L}_2[0, \infty)$ is the space of square-integrable vector functions over $[0, \infty)$; The notation $\|\cdot\|$ refers to the Euclidean vector norm, while $\|\cdot\|_2$ stands for the usual $L_2[0, \infty)$ norm.

2 Problem Formulation

Consider the following class of uncertain nonlinear time-delay systems:

$$
\begin{aligned}
(\Sigma): \qquad \dot{x}(t) &= (A + \Delta A(t))\, x(t) + (A_d + \Delta A_d(t))\, x(t-\tau) + Gg(x(t), x(t-\tau)) + B\omega(t) && (1)\\
y(t) &= (C + \Delta C(t))\, x(t) + (C_d + \Delta C_d(t))\, x(t-\tau) + Hh(x(t), x(t-\tau)) + D\omega(t) && (2)\\
z(t) &= (E + \Delta E(t))\, x(t) + (E_d + \Delta E_d(t))\, x(t-\tau) && (3)\\
x(t) &= \phi(t), \ \forall t \in [-\tau, 0], && (4)
\end{aligned}
$$

where $x(t) \in \mathbb{R}^n$ is the system state, $y(t) \in \mathbb{R}^m$ is the measurement, $\omega(t) \in \mathbb{R}^s$ is the noise signal which belongs to $\mathcal{L}_2[0,\infty)$, $z(t) \in \mathbb{R}^l$ is a linear combination of state variables to be estimated, $g(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n_g}$ and $h(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n_h}$ are known nonlinear functions, A, A_d, B, C, C_d, D, E, E_d, G and H are known real constant matrices, $\phi(t)$ is a real-valued continuous initial function on $[-\tau, 0]$, $\tau > 0$ is a known time delay of the system, $\Delta A(t)$, $\Delta A_d(t)$, $\Delta C(t)$, $\Delta C_d(t)$, $\Delta E(t)$ and $\Delta E_d(t)$ are unknown matrices representing time-varying parameter uncertainties, and are assumed to be of the form

$$\begin{bmatrix} \Delta A(t) & \Delta A_d(t) \\ \Delta C(t) & \Delta C_d(t) \\ \Delta E(t) & \Delta E_d(t) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} F(t) \begin{bmatrix} N_1 & N_2 \end{bmatrix} \tag{5}$$

where M_1, M_2, M_3, N_1 and N_2 are known real constant matrices and $F(\cdot) : \mathbb{R} \to \mathbb{R}^{k \times j}$ is an unknown real-valued time-varying matrix satisfying

$$F(t)^T F(t) \leq I, \ \forall t \tag{6}$$

It is assumed that all the elements of $F(t)$ are Lebesgue measurable. $\Delta A(t)$, $\Delta A_d(t)$, $\Delta C(t)$, $\Delta C_d(t)$, $\Delta E(t)$ and $\Delta E_d(t)$ are said to be admissible if both (5) and (6) hold.

Throughout the paper, we make the following assumption on the nonlinear functions in system (Σ).

Assumption 1. *(Lipschitz condition)*

(I) $g(0,0) = 0$;

(II) $\|g(x_1, x_2) - g(y_1, y_2)\| \leq \|S_{1g}(x_1 - y_1)\| + \|S_{2g}(x_2 - y_2)\|$,
$\|h(x_1, x_2) - h(y_1, y_2)\| \leq \|S_{1h}(x_1 - y_1)\| + \|S_{2h}(x_2 - y_2)\|$,
for all x_1, x_2, y_1, $y_2 \in \mathbb{R}^n$, where S_{1g}, S_{2g}, S_{1h} and S_{2h} are known real constant matrices.

In this paper we are concerned with obtaining an estimate, $\hat{z}(t)$, of $z(t)$ by a causal filter $\mathcal{F}$ using the measurement $\mathcal{Y}_t = \{y(\tau) : 0 \leq \tau \leq t\}$. More specifically, given a prescribed level of noise attenuation $\gamma > 0$, our objective is the design of a causal filter $\mathcal{F}$ such that the filtering error dynamics are asymptotically stable and $\|\hat{z}(t) - z(t)\|_2^2 < \gamma \|\omega(t)\|_2^2$ under zero-initial conditions for any nonzero $\omega(t) \in \mathcal{L}_2[0,\infty)$ and all admissible uncertainties.

3 Main Results

In this section, an LMI approach is developed to solve the robust H_∞ filtering problem formulated in the previous section. We first give the performance analysis result of the system (1) and (3).

Lemma 1. *Consider the system (1) and (3), that is,*

$$(\Sigma_1): \quad \dot{x}(t) = (A + \Delta A(t))\, x(t) + (A_d + \Delta A_d(t))\, x(t-\tau) + Gg(x(t), x(t-\tau)) + B\omega(t) \tag{7}$$

$$z(t) = (E + \Delta E(t))\, x(t) + (E_d + \Delta E_d(t))\, x(t-\tau) \tag{8}$$

$$x(t) = \phi(t), \ \forall t \in [-\tau, 0], \tag{9}$$

Suppose Assumption 1 holds and $\gamma > 0$ is a given constant scalar, then the system (Σ_1) is asymptotically stable and $\|z(t)\|_2^2 < \gamma \|\omega(t)\|_2^2$ under zero-initial conditions for any nonzero $\omega(t) \in \mathcal{L}_2[0,\infty)$ if there exist scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$ and $\epsilon_3 > 0$ and matrices $P > 0$ and $Q > 0$ such that

$$\begin{bmatrix} \Omega_1(\epsilon_1,\epsilon_2,\epsilon_3,P,Q) & PA_d + E^T E_d + (\epsilon_1+\epsilon_3)N_1^T N_2 & P\Lambda & E^T M_3 \\ A_d^T P + E_d^T E + (\epsilon_1+\epsilon_3)N_2^T N_1 & \Omega_2(\epsilon_1,\epsilon_2,\epsilon_3,Q) & 0 & E_d^T M_3 \\ \Lambda^T P & 0 & \Omega_3(\epsilon_1,\epsilon_2) & 0 \\ M_3^T E & M_3^T E_d & 0 & M_3^T M_3 - \epsilon_3 I \end{bmatrix} < 0 \quad (10)$$

where

$$\begin{aligned} \Omega_1(\epsilon_1,\epsilon_2,\epsilon_3,P,Q) &= A^T P + PA + (\epsilon_1+\epsilon_3)N_1^T N_1 + E^T E + 2\epsilon_2 S_{1g}^T S_{1g} + Q \\ \Omega_2(\epsilon_1,\epsilon_2,\epsilon_3,Q) &= E_d^T E_d + (\epsilon_1+\epsilon_3)N_2^T N_2 + 2\epsilon_2 S_{2g}^T S_{2g} - Q \\ \Omega_3(\epsilon_1,\epsilon_2) &= diag(-\gamma I, -\epsilon_1 I, -\epsilon_2 I) \\ \Lambda &= \begin{bmatrix} B & M_1 & G \end{bmatrix} \end{aligned}$$

To prove this Lemma, we need the following result.

Lemma 2. [15]*Let $\mathcal{A}$, $\mathcal{D}$, $\mathcal{S}$, F and P be real matrices of appropriate dimensions with $P > 0$ and F satisfying $F^T F \leq I$. Then the following statements hold:*

(a) For any scalar $\epsilon > 0$ and vectors x, $y \in \mathbb{R}^n$,

$$2x^T \mathcal{D} F \mathcal{S} y \leq \epsilon^{-1} x^T \mathcal{D}\mathcal{D}^T x + \epsilon y^T \mathcal{S}^T \mathcal{S} y$$

(b) For any scalar $\epsilon > 0$ satisfying $\epsilon I - \mathcal{D}^T P \mathcal{D} > 0$,

$$(\mathcal{A} + \mathcal{D}F\mathcal{S})^T P (\mathcal{A} + \mathcal{D}F\mathcal{S}) \leq \mathcal{A}^T P \mathcal{A} + \mathcal{A}^T P \mathcal{D}(\epsilon I - \mathcal{D}^T P \mathcal{D})^{-1}\mathcal{D}^T P \mathcal{A} + \epsilon \mathcal{S}^T \mathcal{S}$$

Proof of Lemma 1. Under the conditions of the lemma, we first establish the asymptotic stability of the system (Σ_1) with $\omega(t) \equiv 0$. To this end, we define the following Lyapunov-Krasovskii function candidate:

$$V(x_t) = x(t)^T P x(t) + \int_{t-\tau}^{t} x(s)^T Q x(s) ds \quad (11)$$

where

$$x_t = x(t+\beta), \quad \beta \in [-\tau, 0].$$

The time-derivative of $V(x_t)$ along the solution of (7) with $\omega(t) \equiv 0$ is then given by

$$\begin{aligned} \dot{V}(x_t) &= 2x(t)^T P \left[(A + \Delta A(t))\, x(t) + (A_d + \Delta A_d(t))\, x(t-\tau) + Gg(x(t), x(t-\tau)) \right] \\ &\quad + x(t)^T Q x(t) - x(t-\tau)^T Q x(t-\tau). \end{aligned} \quad (12)$$

Using Assumption 1, we have

$$\|g(x(t), x(t-\tau))\| \leq \|S_{1g} x(t)\| + \|S_{2g} x(t-\tau)\|,$$

and hence

$$\|g(x(t), x(t-\tau))\|^2 \leq 2\|S_{1g} x(t)\|^2 + 2\|S_{2g} x(t-\tau)\|^2. \quad (13)$$

Considering this and (5) and using statement (a) in Lemma 2, we can deduce that for any scalars $\epsilon_1 > 0$ and $\epsilon_2 > 0$,

$$\begin{aligned} & 2x(t)^T P\left[\Delta A(t)x(t) + \Delta A_d(t)x(t-\tau)\right] = 2x(t)^T P M_1 F(t) \left[\begin{array}{cc} N_1 & N_2 \end{array}\right] \left[\begin{array}{cc} x(t)^T & x(t-\tau)^T \end{array}\right]^T \\ \leq\ & \epsilon_1^{-1} x(t)^T P M_1 M_1^T P x(t) + \epsilon_1 \left[N_1 x(t) + N_2 x(t-\tau)\right]^T \left[N_1 x(t) + N_2 x(t-\tau)\right] \end{aligned} \tag{14}$$

by setting

$$\mathcal{D} = PM_1,\ \mathcal{S} = \left[\begin{array}{cc} N_1 & N_2 \end{array}\right],\ y^T = \left[\begin{array}{cc} x(t)^T & x(t-\tau)^T \end{array}\right]$$

and

$$\begin{aligned} 2x(t)^T PGg(x(t), x(t-\tau)) &\leq \epsilon_2^{-1} x(t)^T PGG^T P x(t) + \epsilon_2 g(x(t), x(t-\tau))^T g(x(t), x(t-\tau)) \\ &\leq \epsilon_2^{-1} x(t)^T PGG^T P x(t) + 2\epsilon_2 \left[x(t)^T S_{1g}^T S_{1g} x(t) + x(t-\tau)^T S_{2g}^T S_{2g} x(t-\tau)\right] \end{aligned} \tag{15}$$

by setting

$$\mathcal{D} = PG,\ \mathcal{S} = I.$$

Thus

$$\dot{V}(x_t) \leq \left[\begin{array}{cc} x(t)^T & x(t-\tau)^T \end{array}\right] W \left[\begin{array}{c} x(t) \\ x(t-\tau) \end{array}\right] \tag{16}$$

where

$$W = \left[\begin{array}{cc} A^T P + PA + \epsilon_1 N_1^T N_1 + P(\epsilon_1^{-1} M_1 M_1^T + \epsilon_2^{-1} GG^T)P + 2\epsilon_2 S_{1g}^T S_{1g} + Q & PA_d + \epsilon_1 N_1^T N_2 \\ A_d^T P + \epsilon_1 N_2^T N_1 & \epsilon_1 N_2^T N_2 + 2\epsilon_2 S_{2g}^T S_{2g} - Q \end{array}\right]. \tag{17}$$

On the other hand, from (10) it is easy to show that

$$\left[\begin{array}{cccc} \Omega_1(\epsilon_1, \epsilon_2, \epsilon_3, P, Q) & PA_d + E^T E_d + (\epsilon_1 + \epsilon_3) N_1^T N_2 & PM_1 & PG \\ A_d^T P + E_d^T E + (\epsilon_1 + \epsilon_3) N_2^T N_1 & \Omega_2(\epsilon_1, \epsilon_2, \epsilon_3, Q) & 0 & 0 \\ M_1^T P & 0 & -\epsilon_1 I & 0 \\ G^T P & 0 & 0 & -\epsilon_2 I \end{array}\right] < 0. \tag{18}$$

By Schur complement, we can show that (18) is equivalent to

$$\left[\begin{array}{cccc|cc} A^T P + PA + \epsilon_1 N_1^T N_1 + 2\epsilon_2 S_{1g}^T S_{1g} + Q & PA_d + \epsilon_1 N_1^T N_2 & PM_1 & PG & N_1^T & E^T \\ A_d^T P + \epsilon_1 N_2^T N_1 & \epsilon_1 N_2^T N_2 + 2\epsilon_2 S_{2g}^T S_{2g} - Q & 0 & 0 & N_2^T & E_d^T \\ M_1^T P & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ G^T P & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \hline N_1 & N_2 & 0 & 0 & -\epsilon_3^{-1} I & 0 \\ E & E_d & 0 & 0 & 0 & -I \end{array}\right] < 0.$$

Therefore

$$\left[\begin{array}{cccc} A^T P + PA + \epsilon_1 N_1^T N_1 + 2\epsilon_2 S_{1g}^T S_{1g} + Q & PA_d + \epsilon_1 N_1^T N_2 & PM_1 & PG \\ A_d^T P + \epsilon_1 N_2^T N_1 & \epsilon_1 N_2^T N_2 + 2\epsilon_2 S_{2g}^T S_{2g} - Q & 0 & 0 \\ M_1^T P & 0 & -\epsilon_1 I & 0 \\ G^T P & 0 & 0 & -\epsilon_2 I \end{array}\right] < 0$$

which by Schur complement implies that $W < 0$, and hence $\dot{V}(x_t) < 0$, which guarantees the asymptotic stability of the system (Σ_1) with $\omega(t) \equiv 0$.

Next, we shall show $\|z(t)\|_2^2 < \gamma \|\omega(t)\|_2^2$ under zero initial condition. To this end, we introduce

$$J = \int_0^\infty \left[z(t)^T z(t) - \gamma\omega(t)^T \omega(t)\right] dt. \tag{19}$$

Considering the stability of the system and the zero initial condition, we have that for all nonzero $\omega(t) \in \mathcal{L}_2[0,\infty)$,

$$J \le \int_0^\infty \left[z(t)^T z(t) - \gamma\omega(t)^T \omega(t) + \dot{V}(x_t)\right] dt \tag{20}$$

where $V(x_t)$ is given in (11). From (10), it is easy to have that

$$\Psi =: \epsilon_3 I - M_3^T M_3 > 0$$

Thus, by statement (b) in Lemma 2 for

$$\mathcal{A} = \bar{E},\ \mathcal{D} = M_3,\ \mathcal{S} = \bar{N}$$

it follows that

$$\begin{aligned} z(t)^T z(t) &= \begin{bmatrix} x(t)^T & x(t-\tau)^T \end{bmatrix} \left(\bar{E} + M_3 F(t)\bar{N}\right)^T \left(\bar{E} + M_3 F(t)\bar{N}\right) \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \\ &\le \begin{bmatrix} x(t)^T & x(t-\tau)^T \end{bmatrix} \left[\bar{E}^T\bar{E} + \bar{E}^T M_3 \Psi^{-1} M_3^T \bar{E} + \epsilon_3 \bar{N}^T \bar{N}\right] \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \end{aligned} \tag{21}$$

where

$$\bar{E} = \begin{bmatrix} E & E_d \end{bmatrix},\ \bar{N} = \begin{bmatrix} N_1 & N_2 \end{bmatrix}.$$

Using (21) together with (14), (15) and (20), we obtain

$$J \le \int_0^\infty \left(\begin{bmatrix} x(t)^T & x(t-\tau)^T & \omega(t)^T \end{bmatrix} \Gamma \begin{bmatrix} x(t) \\ x(t-\tau) \\ \omega(t) \end{bmatrix} \right) dt$$

where

$$\begin{aligned} \Gamma &= \begin{bmatrix} \bar{E}^T\bar{E} + \bar{E}^T M_3 \Psi^{-1} M_3^T \bar{E} + \epsilon_3 \bar{N}^T \bar{N} + W & \Gamma_{21} \\ \Gamma_{21}^T & -\gamma I \end{bmatrix} \\ \Gamma_{21} &= \begin{bmatrix} PB \\ 0 \end{bmatrix} \end{aligned}$$

and W is given in (17). By Schur complement we can verify that the LMI (10) ensures that $\Gamma < 0$, and hence $J < 0$ for any nonzero $\omega(t) \in \mathcal{L}_2[0,\infty)$, which implies that $\|z(t)\|_2^2 < \gamma \|\omega(t)\|_2^2$. This completes the proof. □

Now we are in a position to provide a solution to the robust H_∞ filtering problem for the system (Σ).

Theorem 3.1. *Consider the uncertain nonlinear system* (Σ) *satisfying Assumption 1. If there exist scalars* $\epsilon_1 > 0$, $\epsilon_2 > 0$ *and* $\epsilon_3 > 0$ *and matrices* $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$ *and* $Q_2 > 0$ *and* Z *such that the following LMI holds:*

$$\begin{bmatrix} \Phi_1(\epsilon_1,\epsilon_2,\epsilon_3,P_1,P_2,Q_1,Q_2,Z) & \Upsilon_1(\epsilon_1,\epsilon_3,P_1,P_2,Z) & \Upsilon_2(P_1,P_2,Z) & \Xi_1 \\ \Upsilon_1(\epsilon_1,\epsilon_3,P_1,P_2,Z)^T & \Phi_2(\epsilon_1,\epsilon_2,\epsilon_3,Q_1,Q_2) & 0 & \Xi_2 \\ \Upsilon_2(P_1,P_2,Z)^T & 0 & \Phi_3(\epsilon_1,\epsilon_2) & 0 \\ \Xi_1^T & \Xi_2^T & 0 & M_3^T M_3 - \epsilon_3 I \end{bmatrix} < 0 \tag{22}$$

where

$$\begin{aligned}
\Phi_1(\epsilon_1,\epsilon_2,\epsilon_3,P_1,P_2,Q_1,Q_2,Z) &= diag\left(\Phi_{11}(\epsilon_1,\epsilon_2,\epsilon_3,P_1,Q_1), \Phi_{22}(\epsilon_2,P_2,Q_2,Z)\right) && (23)\\
\Phi_{11}(\epsilon_1,\epsilon_2,\epsilon_3,P_1,Q_1) &= A^T P_1 + P_1 A + (\epsilon_1+\epsilon_3) N_1^T N_1 + 2\epsilon_2 S_{1g}^T S_{1g} + Q_1 && (24)\\
\Phi_{22}(\epsilon_2,P_2,Q_2,Z) &= A^T P_2 + P_2 A - ZC - C^T Z^T + E^T E + 2\epsilon_2 S_1^T S_1 + Q_2 && (25)\\
\Phi_2(\epsilon_1,\epsilon_2,\epsilon_3,P_1,P_2,Q_1,Q_2) &= diag\Big((\epsilon_1+\epsilon_3) N_2^T N_2 + 2\epsilon_2 S_{2g}^T S_{2g} - Q_1 \ , \ E_d^T E_d + 2\epsilon_2 S_2^T S_2 - Q_2 \Big) && (26)\\
\Upsilon_1(\epsilon_1,\epsilon_3,P_1,P_2,Z) &= diag\Big(P_1 A_d + (\epsilon_1+\epsilon_3) N_1^T N_2 \ , \ P_2 A_d - Z C_d + E^T E_d \Big) && (27)\\
\Upsilon_2(P_1,P_2,Z) &= \begin{bmatrix} P_1 B & P_1 M_1 & P_1 G & 0 & 0 \\ P_2 B - ZD & P_2 M_1 - Z M_2 & 0 & P_2 G & -ZH \end{bmatrix}, && (28)\\
\Phi_3(\epsilon_1,\epsilon_2) &= diag(-\gamma I, -\epsilon_1 I, -\epsilon_2 I, -\epsilon_2 I, -\epsilon_2 I), && (29)\\
\Xi_1 &= \begin{bmatrix} 0 \\ E^T M_3 \end{bmatrix}, \ \Xi_2 = \begin{bmatrix} 0 \\ E_d^T M_3 \end{bmatrix} && (30)\\
S_1 &= \begin{bmatrix} S_{1g} \\ S_{1h} \end{bmatrix}, \ S_2 = \begin{bmatrix} S_{2g} \\ S_{2h} \end{bmatrix} && (31)
\end{aligned}$$

then the robust H_∞ *filtering problem is solvable. Furthermore, when (22) is satisfied, a suitable nonlinear filter is given as follows:*

$$\begin{aligned}
(\Sigma_f): \quad \dot{\hat{x}}(t) &= A\hat{x}(t) + A_d\hat{x}(t-\tau) + Gg(\hat{x}(t),\hat{x}(t-\tau)) \\
&\quad + L\left[y(t) - C\hat{x}(t) - C_d\hat{x}(t-\tau) - Hh(\hat{x}(t),\hat{x}(t-\tau))\right] && (32)\\
\hat{z}(t) &= E\hat{x}(t) + E_d\hat{x}(t-\tau) && (33)
\end{aligned}$$

where $L = P_2^{-1} Z$.

Proof. Set

$$\tilde{x}(t) = x(t) - \hat{x}(t)$$

Then from (1)–(3), (32) and (33) we obtain

$$\begin{aligned}
\dot{\tilde{x}} &= (A - LC)\tilde{x}(t) + (A_d - LC_d)\tilde{x}(t-\tau) + [\Delta A(t) - L\Delta C(t)]\,x(t) \\
&\quad + [\Delta A_d(t) - L\Delta C_d(t)]\,x(t-\tau) + \bar{G}\xi(x(t),x(t-\tau),\hat{x}(t),\hat{x}(t-\tau)) + \bar{B}\omega(t) && (34)
\end{aligned}$$

where

$$\bar{G} = \begin{bmatrix} G & -LH \end{bmatrix}, \qquad \bar{B} = B - LD \tag{35}$$

$$\xi(x(t),x(t-\tau),\hat{x}(t),\hat{x}(t-\tau)) = \begin{bmatrix} g(x(t),x(t-\tau)) - g(\hat{x}(t),\hat{x}(t-\tau)) \\ h(x(t),x(t-\tau)) - h(\hat{x}(t),\hat{x}(t-\tau)) \end{bmatrix} \tag{36}$$

Defining

$$\eta(t) = \left[\begin{array}{cc} x(t)^T & \tilde{x}(t)^T \end{array} \right]^T, \ \tilde{z}(t) = z(t) - \hat{z}(t)$$

we have the filtering error dynamics as follows

$$\begin{aligned} \dot{\eta}(t) &= (A_c + \Delta A_c(t))\,\eta(t) + (A_{cd} + \Delta A_{cd}(t))\,\eta(t-\tau) + G_c \xi_c(x(t), x(t-\tau), \hat{x}(t), \hat{x}(t-\tau)) + B_c \omega(t) \quad (37) \\ \tilde{z}(t) &= (E_c + \Delta E_c(t))\,\eta(t) + (E_{cd} + \Delta E_{cd}(t))\,\eta(t-\tau) \quad (38) \end{aligned}$$

where

$$\begin{aligned} A_c &= \left[\begin{array}{cc} A & 0 \\ 0 & A - LC \end{array} \right], \quad \Delta A_c(t) = \left[\begin{array}{cc} \Delta A(t) & 0 \\ \Delta A(t) - L\Delta C(t) & 0 \end{array} \right], \quad A_{cd} = \left[\begin{array}{cc} A_d & 0 \\ 0 & A_d - LC_d \end{array} \right], \\ \Delta A_{cd}(t) &= \left[\begin{array}{cc} \Delta A_d(t) & 0 \\ \Delta A_d(t) - L\Delta C_d(t) & 0 \end{array} \right], \quad G_c = \left[\begin{array}{cc} G & 0 \\ 0 & \bar{G} \end{array} \right], \quad B_c = \left[\begin{array}{c} B \\ \bar{B} \end{array} \right], \\ E_c &= \left[\begin{array}{cc} 0 & E \end{array} \right], \ \Delta E_c(t) = \left[\begin{array}{cc} \Delta E(t) & 0 \end{array} \right], \ E_{cd} = \left[\begin{array}{cc} 0 & E_d \end{array} \right], \ \Delta E_{cd}(t) = \left[\begin{array}{cc} \Delta E_d(t) & 0 \end{array} \right], \end{aligned}$$

$$\xi_c(x(t), x(t-\tau), \hat{x}(t), \hat{x}(t-\tau)) = \left[\begin{array}{cc} g(x(t), x(t-\tau))^T & \xi(x(t), x(t-\tau), \hat{x}(t), \hat{x}(t-\tau))^T \end{array} \right]^T.$$

By Assumption 1, it is easy to show that

$$\|\xi_c(x(t), x(t-\tau), \hat{x}(t), \hat{x}(t-\tau))\|^2 \le 2\left\|\tilde{S}_1 \eta(t)\right\|^2 + 2\left\|\tilde{S}_2 \eta(t-\tau)\right\|^2 \quad (39)$$

where

$$\tilde{S}_1 = \left[\begin{array}{cc} S_{1g} & 0 \\ 0 & S_1 \end{array} \right], \ \tilde{S}_2 = \left[\begin{array}{cc} S_{2g} & 0 \\ 0 & S_2 \end{array} \right].$$

Now we write

$$\left[\begin{array}{cc} \Delta A_c(t) & \Delta A_{cd}(t) \\ \Delta E_c(t) & \Delta E_{cd}(t) \end{array} \right] = \left[\begin{array}{c} M_{1c} \\ M_3 \end{array} \right] F(t) \left[\begin{array}{cc} N_{1c} & N_{2c} \end{array} \right]$$

where

$$M_{1c} = \left[\begin{array}{c} M_1 \\ M_1 - LM_2 \end{array} \right], \ N_{1c} = \left[\begin{array}{cc} N_1 & 0 \end{array} \right], \ N_{2c} = \left[\begin{array}{cc} N_2 & 0 \end{array} \right].$$

Let

$$\begin{aligned} P_c &= diag(P_1, P_2), \qquad Q_c = diag(Q_1, Q_2) \\ \Theta_1(\epsilon_1, \epsilon_2, \epsilon_3, P_c, Q_c) &= A_c^T P_c + P_c A_c + (\epsilon_1 + \epsilon_3) N_{1c}^T N_{1c} + E_c^T E_c + 2\epsilon_2 \tilde{S}_1^T \tilde{S}_1 + Q_c \\ \Theta_2(\epsilon_1, \epsilon_2, \epsilon_3, Q_c) &= E_{cd}^T E_{cd} + (\epsilon_1 + \epsilon_3) N_{2c}^T N_{2c} + 2\epsilon_2 \tilde{S}_2^T \tilde{S}_2 - Q_c \\ \Theta_3(\epsilon_1, \epsilon_2) &= diag(-\gamma I, -\epsilon_1 I, -\epsilon_2 I, -\epsilon_2 I, -\epsilon_2 I) \\ \Lambda_c &= \left[\begin{array}{ccc} B_c & M_{1c} & G_c \end{array} \right] \end{aligned}$$

then it follows from (22) that

$$\begin{aligned} &\left[\begin{array}{cccc} \Theta_1(\epsilon_1, \epsilon_2, \epsilon_3, P_c, Q_c) & P_c A_{dc} + E_c^T E_{cd} + (\epsilon_1 + \epsilon_3) N_{1c}^T N_{2c} & P_c \Lambda_c & E_c^T M_3 \\ A_{dc}^T P_c + E_{cd}^T E_c + (\epsilon_1 + \epsilon_3) N_{2c}^T N_{1c} & \Theta_2(\epsilon_1, \epsilon_2, \epsilon_3, Q_c) & 0 & E_{cd}^T M_3 \\ \Lambda_c^T P_c & 0 & \Theta_3(\epsilon_1, \epsilon_2) & 0 \\ M_3^T E_c & M_3^T E_{cd} & 0 & M_3^T M_3 - \epsilon_3 I \end{array} \right] \\ = &\left[\begin{array}{cccc} \Phi_1(\epsilon_1, \epsilon_2, \epsilon_3, P_1, P_2, Q_1, Q_2, Z) & \Upsilon_1(\epsilon_1, \epsilon_3, P_1, P_2, Z) & \Upsilon_2(P_1, P_2, Z) & \Xi_1 \\ \Upsilon_1(\epsilon_1, \epsilon_3, P_1, P_2, Z)^T & \Phi_2(\epsilon_1, \epsilon_2, \epsilon_3, Q_1, Q_2) & 0 & \Xi_2 \\ \Upsilon_2(P_1, P_2, Z)^T & 0 & \Phi_3(\epsilon_1, \epsilon_2) & 0 \\ \Xi_1^T & \Xi_2^T & 0 & M_3^T M_3 - \epsilon_3 I \end{array} \right] < 0. \end{aligned}$$

Therefore, by Lemma 1 the desired result follows immediately. This completes the proof. □

Remark 1. Theorem 3.1 provides a method for designing robust H_∞ filters for the uncertain nonlinear system (Σ), and the desired H_∞ filter can be constructed by solving an LMI. It is worth pointing out that the LMI in (22) can be solved efficiently, and no tuning of parameters is required [3], although there are several parameters and matrices to be determined.

Remark 2. In the context of Theorem 3.1, we can determine the lowest γ by solving the following optimization problem:

$$minimize\ \gamma$$

subject to $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 > 0$, $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$, $Q_2 > 0$ and (22).

In the case when there are no nonlinearities in system (Σ), that is, the system (Σ) reduces to the following uncertain time-delay system:

$$\begin{aligned}(\Sigma_N): \quad \dot{x}(t) &= (A+\Delta A(t))\,x(t) + (A_d+\Delta A_d(t))\,x(t-\tau) + B\omega(t) &(40)\\ y(t) &= (C+\Delta C(t))\,x(t) + (C_d+\Delta C_d(t))\,x(t-\tau) + D\omega(t) &(41)\\ z(t) &= (E+\Delta E(t))\,x(t) + (E_d+\Delta E_d(t))\,x(t-\tau) &(42)\\ x(t) &= \phi(t),\ \forall t\in[-\tau,0], &(43)\end{aligned}$$

where $\Delta A(t)$, $\Delta A_d(t)$, $\Delta C(t)$ and $\Delta C_d(t)$, $\Delta E(t)$, $\Delta E_d(t)$ are unknown matrices satisfying (5) and (6). Then, from Theorem 3.1, we have the following robust H_∞ filter design result for the above system.

Corollary. *Consider the uncertain time-delay system (Σ_N). If there exist scalars $\alpha > 0$, $\beta > 0$, matrices $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$, $Q_2 > 0$ and Z such that the following LMI holds:*

$$\begin{bmatrix} \Phi_1(\alpha,0,\beta,P_1,P_2,Q_1,Q_2,Z) & \Upsilon_1(\alpha,\beta,P_1,P_2,Z) & \hat{\Upsilon}_2(P_1,P_2,Z) & \Xi_1 \\ \Upsilon_1(\alpha,\beta,P_1,P_2,Z)^T & \Phi_2(\alpha,0,\beta,Q_1,Q_2) & 0 & \Xi_2 \\ \hat{\Upsilon}_2(P_1,P_2,Z)^T & 0 & \hat{\Phi}_3(\alpha) & 0 \\ \Xi_1^T & \Xi_2^T & 0 & M_3^TM_3-\beta I \end{bmatrix} < 0 \qquad (44)$$

where

$$\begin{aligned}\hat{\Upsilon}_2(P_1,P_2,Z) &= \begin{bmatrix} P_1B & P_1M_1 \\ P_2B-ZD & P_2M_1-ZM_2 \end{bmatrix}\\ \hat{\Phi}_3(\epsilon_1) &= diag(-\gamma I, -\epsilon_1 I)\end{aligned}$$

and $\Phi_1(\alpha,\gamma,\beta,P_1,P_2,Q_1,Q_2,Z)$, $\Phi_2(\alpha,\gamma,\beta,Q_1,Q_2)$, $\Upsilon_1(\alpha,\beta,P_1,P_2,Z)$, Ξ_1, and Ξ_2 are given in (23)–(27) and (30), respectively, then the robust H_∞ filtering problem for the uncertain time-delay system (Σ_N) is solvable. Furthermore, when (44) is satisfied, a suitable robust H_∞ filter is given as follows:

$$\begin{aligned}(\hat{\Sigma}_f): \quad \dot{\hat{x}} &= A\hat{x}(t) + A_d\hat{x}(t-\tau) + L_o\left[y(t) - C\hat{x}(t) - C_d\hat{x}(t-\tau)\right] &(45)\\ \hat{z}(t) &= Ex(t) + E_dx(t-\tau) &(46)\end{aligned}$$

where $L_o = P_2^{-1}Z$.

Remark 3. It is worth noting that the robust H_∞ filtering problem for time-delay systems without nonlinearities was studied in [16], where the considered uncertain system involved no delays in the measurement and the estimated variables. Under the assumption that the matrix M_2 is of full row rank it was shown in [16] that the solution to the problem involves solving a pair of indefinite algebraic Riccati equations. Corollary 3 in this paper, however, shows that for a more general uncertain time-delay system, the robust H_∞ filtering problem can be solved by an LMI without any assumptions on the matrix M_2. It is worth pointing out that solving Riccati equations requires tuning of a symmetric positive definite matrix, while solving an LMI involves no parameter tuning. Thus using Corollary 3 will make the H_∞ filter design relatively direct and simple.

4 Numerical Example

In this section, we give an example to demonstrate the effectiveness of the proposed method.

Consider the uncertain nonlinear delay system (Σ) with the following parameters:

$$
\begin{aligned}
A &= \begin{bmatrix} -1.5 & 0.5 \\ 1 & -3 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0.2 \\ 0.5 & -0.5 \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}, G = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.1 \end{bmatrix}, M_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} \\
C &= \begin{bmatrix} 1 & 0.5 \\ 0.8 & 0.2 \end{bmatrix}, C_d = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}, H = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.2 \end{bmatrix} \\
E &= \begin{bmatrix} 0.1 & 0.3 \\ 0.5 & 0.1 \end{bmatrix}, E_d = \begin{bmatrix} 0 & -0.1 \\ 0.6 & 1 \end{bmatrix}, M_3 = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, N_1 = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.2 \end{bmatrix}, N_2 = \begin{bmatrix} 0.1 & 0.3 \\ 0 & 0.2 \end{bmatrix}.
\end{aligned}
$$

The nonlinearities $g(x(t), x(t-\tau))$ and $h(x(t), x(t-\tau))$ are assumed to satisfy Assumption 1 with

$$
S_{1g} = \begin{bmatrix} 0.3 & 0 \\ 0.1 & 0.2 \end{bmatrix}, S_{2g} = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.1 \end{bmatrix}, S_{1h} = \begin{bmatrix} 0.6 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, S_{2h} = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix}.
$$

In this example we assume that $\tau = 1.5$ and we require a noise attenuation level $\gamma = 1.5$.

Using the Matlab LMI Control Toolbox to solve the LMI (22), we obtain the solution as follows:

$$
\begin{aligned}
P_1 &= \begin{bmatrix} 1.1184 & -1.2419 \\ -1.2419 & 4.4571 \end{bmatrix}, P_2 = \begin{bmatrix} 22.6096 & -30.6174 \\ -30.6174 & 56.7138 \end{bmatrix}, Q_1 = \begin{bmatrix} 1.6058 & -2.5660 \\ -2.5660 & 8.0843 \end{bmatrix}, \\
Q_2 &= \begin{bmatrix} 51.4539 & -79.8173 \\ -79.8173 & 147.5590 \end{bmatrix}, Z = \begin{bmatrix} 7.4073 & -11.0126 \\ -2.6541 & 18.6921 \end{bmatrix} \\
\epsilon_1 &= 1.1154, \quad \epsilon_2 = 0.4265, \quad \epsilon_3 = 0.3666.
\end{aligned}
$$

Therefore, by Theorem 3.1 the robust H_∞ filtering problem is solvable, and a desired nonlinear filter is given by (32) and (33) with

$$
L = \begin{bmatrix} 0.9826 & -0.1516 \\ 0.4836 & 0.2478 \end{bmatrix}.
$$

5 Conclusions

In this paper, we have studied the problem of robust H_∞ filter design for a class of state-delayed nonlinear systems with time-varying norm-bounded parameter uncertainty in all the matrices of the linear part of the system model. A sufficient condition for the solvability of this problem has been presented. The desired filter which ensures not only the robust stability but also a prescribed H_∞ performance of the filtering error dynamics for all admissible uncertainties, has been constructed by solving a certain LMI. A numerical example has shown the effectiveness of the proposed approach.

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