Reputation systems and nonnegativity

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We present a voting system that is based on an iterative method that assigns a reputation to $n+m$ items, n objects and m raters, applying some filter to the votes. Each rater evaluates a subset of objects leading to an $n \times m$ rating matrix with a given sparsity pattern. From this rating matrix a formula is defined for the reputation of raters and objects. We propose a natural and intuitive nonlinear formula and also provide an iterative algorithm that linearly converges to the unique vector of reputations and this for any rating matrix. In contrast to classical outliers detection, no evaluation is discarded in this method but each one is taken into account with different weights for the reputations of the objects. The complexity of one iteration step is linear in the number of evaluations, making our algorithm efficient for large data set.

1 Introduction

Many measures of reputation have been proposed under the names of reputation, voting, ranking or trust systems and they deal with various contexts ranging from the classification of football teams to the reliability of each individual in peer to peer systems. Surprisingly enough, the most used method for reputation on the Web amounts simply to average the votes. In that case, the reputation is, for instance, the average of scores represented by 5 stars in YouTube, or the percentage of positive transactions in eBay. Therefore such a method trusts evenly each rater of the system. Besides this method, many other algorithms exploit the structure of networks generated by the votes: raters and evaluated items are nodes connected by votes. A great part of these methods use efficient eigenvector based techniques or trust propagation over the network to obtain the reputation of every node [15, 13, 9, 16, 7, 14, 17]. They can be interpreted as a distribution of some reputation flow over the network where reputations satisfy some transitivity: you have a high reputation if you have several incoming links coming from nodes with a high reputation. The average method, the eigenvector based techniques and trust propagation may suffer from noise in the data and bias from dishonest raters. For this reason, they are sometimes accompanied by statistical methods for spam detection [19, 10], like in the context of web pages trying to boost their PageRank scores by adding artificial incoming links [8, 2]. Detected spam is then simply removed from the data. This describes the three main strategies for voting systems: simple methods averaging votes where raters are evenly trusted, eigenvector based techniques and trust propagation where reputations directly depend on reputations of the neighbours, and finally statistical measures to classify and possibly remove some of the items.

Concerning the Iterative Filtering (IF) systems which we introduce here, we will make the following assumption: *Raters diverging often from other raters' opinion are less taken into account.* We label this the *IF*-property and will formally define it later on. This property is at the heart of the filtering process and implies that all votes are taken into account, but with a continuous validation scale, in contrast with the direct deletion of outliers. Moreover, the weight of each rater depends on the distance between his votes and the reputation of the objects he evaluates: typically weights of random raters and outliers decrease during the iterative filtering. The main criticism one can have about the *IF*-property is that it discriminates "marginal" evaluators, i.e., raters who vote differently from the average opinion for many objects. However, IF systems may have different basin of attraction, each corresponding to a group of people with a coherent opinion.

Votes, raters and objects can appear, disappear or change making the system dynamical. This is for example the case when we consider a stream of news like in [5]: news sources and articles are ranked according to their publications over time. Nowadays, most sites driven by raters involve dynamical opinions. For instance, the blogs, the site Digg and the site Flickr are good places to exchange and discuss ideas, remarks and votes about various topics ranging from political election to photos and videos. We will see that IF systems allow to consider evolving voting matrices and then provide time varying reputations.

2 Iterative Filtering Systems

We first consider the case where the votes are fixed, i.e., the voting matrix does not change over time, and all objects are evaluated by all raters, i.e., the voting matrix is full. With these assumptions, we present the main properties of IF systems and then we restrict ourselves to the natural case of quadratic IF systems where the reputations are given by a linear combination of the votes and the weights of the raters are based on the Euclidean distance between the reputations and the votes.

Let $\boldsymbol{X} \in \mathbb{R}^{n\times m}$ be the voting matrix, $\boldsymbol{r} \in \mathbb{R}^n$ be the reputation vector of the objects and $\boldsymbol{w} \in \mathbb{R}^m$ be the weight vector of the raters. The entry $\boldsymbol{X}_{ij}$ is the vote to object i given by rater j and the vector $\boldsymbol{x}_j$, the j^{th} column of $\boldsymbol{X}$, represents the votes of rater j:

$$\boldsymbol{X} = [\boldsymbol{x}_1 \ \dots \ \boldsymbol{x}_m] \,.$$

The bipartite graph formed by the objects, the raters and their votes is represented by the $n \times m$ adjacency matrix $\boldsymbol{A}$, i.e., $\boldsymbol{A}_{ij} = 1$ if object i is evaluated by rater j, and 0 otherwise. For the sake of simplicity, we assume in this section that every object has been evaluated by all raters

$$\boldsymbol{A}_{ij} = 1 \quad \text{for all } i, j. \tag{1}$$

The general case, where the bipartite graph is not complete, will be handled later.

The belief divergence d_j of rater j is the normalized distance between his votes and the reputation vector $\boldsymbol{r}$ (for a particular choice of norm)

$$d_j = \frac{1}{n}\|\boldsymbol{x}_j - \boldsymbol{r}\|^2. \tag{2}$$

Let us already remark that when the bipartite graph is not complete, i.e., Eq.(1) is not satisfied, then the number of votes varies from one rater to another. Therefore the normalization of the belief divergence d_j in Eq.(2) will change depending on this number.

Before introducing IF systems, we define the two basic functions of these systems:

(1) the reputation function $\quad F : \mathbb{R}^n \to \mathbb{R}^m : F(\boldsymbol{w}) = \boldsymbol{r}$,

that gives the reputation vector depending on the weights of the raters and implicitly on the voting matrix $\boldsymbol{X}$;

(2) The filtering function $\quad G : \mathbb{R}^m \to \mathbb{R}^n : G(\boldsymbol{d}) = \boldsymbol{w}$,

that gives the weight vector for the raters depending on the belief divergence $\boldsymbol{d}$ of each rater defined in Eq.(2).
We formalize the so-called **IF-property** described in the introduction that claims that raters diverging often from the opinion of other raters are less taken into account. We will make the reasonable assumption that raters with identical belief divergence receive equal weights. Hence, we can write

$$G(\boldsymbol{d}) = \begin{bmatrix} g(d_1) \\ \vdots \\ g(d_m) \end{bmatrix}. \tag{3}$$

We call the scalar function g the discriminant function associated with G.

A filtering function G satisfies the *IF*-property if its associated discriminant function g is positive and decreases with d. Therefore, the *IF*-property merely implies that a decrease in belief divergence d_j for any rater j corresponds to a larger weight w_j. Eq.(3) indicates that every rater has the same discriminant function g, but we could also consider personalized functions g_j penalizing differently the raters. In [4] three choices of function g are shown to have interesting properties

$$g(d) = d^{-k}, \tag{4}$$
$$g(d) = e^{-k\,d}, \tag{5}$$
$$g(d) = 1 - k\,d. \tag{6}$$

All discriminant function g are positive and decrease with d for positive k and therefore satisfy the IF-property. However k must be small enough to keep g positive in Eq.(6) and hence to avoid negative weights.

Definition 1. *IF systems are systems of equations in the reputations $\boldsymbol{r}^t$ of the objects and the weights $\boldsymbol{w}^t$ of the raters that evolve over discrete time t according to the voting matrix $\boldsymbol{X}$*

$$\boldsymbol{r}^{t+1} = F(\boldsymbol{w}^t), \tag{7}$$
$$\boldsymbol{w}^{t+1} = G(\boldsymbol{d}^{t+1}) \qquad \text{with } d_j^{t+1} = \frac{1}{n}\|\boldsymbol{x}_j - \boldsymbol{r}^{t+1}\|^2 \tag{8}$$

for $j = 1, \ldots, m$ and some initial vector of weights $\boldsymbol{w}^0$.

Definition 1 does not imply any convergence properties, nor robustness to initial conditions. The system (7-8) can have several converging solutions and it allows the existence of cycles in the iterative processes.

Fixed points and quadratic systems. The fixed points of (7-8) satisfy

$$\boldsymbol{r}^* = F(\boldsymbol{w}^*), \tag{9}$$

$$\boldsymbol{w}^* = G(\boldsymbol{d}^*) \qquad \text{with } d_j^* = \frac{1}{n}\|\boldsymbol{x}_j - \boldsymbol{r}^*\|^2 \tag{10}$$

for $j = 1, \ldots, m$. Let us remark that IF systems can be interpreted as a particular iterative search method to find the stable fixed points of Eq.(9-10). IF systems are a simple iterative scheme for this system with the advantage to be easily extended to take into account dynamical voting matrices $\boldsymbol{X}^t$ with $t \geq 0$.

In this paper, we focus on IF systems where we fix the reputation function F appearing in Eq.(7,9) and the norm $\|.\|$ given in the definition of the belief divergence in Eq.(2).

Definition 2. *Quadratic IF systems are IF systems where the reputation function F and the belief divergence are respectively given by*

$$F(\boldsymbol{w}) = \boldsymbol{X}\frac{\boldsymbol{w}}{\mathbf{1}^T\boldsymbol{w}}, \tag{11}$$

$$\boldsymbol{d} = \frac{1}{n}(\boldsymbol{X}^T - \mathbf{1}\boldsymbol{r}^T)^{\circ 2}\mathbf{1}, \tag{12}$$

where $\mathbf{1}$ is the vector of all 1's and $(\boldsymbol{X}^T - \mathbf{1}\boldsymbol{r}^T)^{\circ 2}$ is the componentwise product $(\boldsymbol{X}^T - \mathbf{1}\boldsymbol{r}^T) \circ (\boldsymbol{X}^T - \mathbf{1}\boldsymbol{r}^T)$.

In that definition, the reputation function $F(\boldsymbol{w})$ is naturally given by taking the weighted average of the votes and the belief divergence $\boldsymbol{d}$ (given in the matrix form) is defined using the Euclidian norm. Therefore Eq.(12) are quadratic equations in $\boldsymbol{r}$ and amount to consider an estimate of the variances of the votes for every rater according to a given reputation vector $\boldsymbol{r}$.
For any positive vector $\boldsymbol{w}$, the reputation vector $\boldsymbol{r}$ then belongs to the polytope

$$\mathcal{P} = \{\boldsymbol{r} \in \mathbb{R}^n \,|\, \boldsymbol{r} = \sum_{j=1}^{m} \lambda_j \boldsymbol{x}_j \text{ with } \sum_{j=1}^{m} \lambda_j = 1 \text{ and } \lambda_j \geq 0\}. \tag{13}$$

From Eq.(11), the iterations and the fixed point in Eq.(7,9) are given by quadratic equations in $\boldsymbol{r}$ and $\boldsymbol{w}$

$$\boldsymbol{r}^{t+1}(\mathbf{1}^T\boldsymbol{w}^t) = \boldsymbol{X}\boldsymbol{w}^t, \tag{14}$$

$$\boldsymbol{r}^*(\mathbf{1}^T\boldsymbol{w}^*) = \boldsymbol{X}\boldsymbol{w}^*. \tag{15}$$

The next theorem establishes the correspondence between the iterations of quadratic IF systems and some steepest descent methods minimizing some energy function. The fixed points in Eq.(14,15) are then the stationary points of that energy function.

Theorem 1. *(see [4]) The fixed points of quadratic IF systems with integrable discriminant function g, are the stationary points of the energy function*

$$E(\boldsymbol{r}) = \sum_{j=1}^{m} \int_{0}^{d_j(\boldsymbol{r})} g(u)\, du, \tag{16}$$

where d_j is the belief divergence of rater j that depends on $\boldsymbol{r}$. Moreover one iteration step in quadratic IF systems corresponds to a steepest descent direction with a particular step size

$$\boldsymbol{r}^{t+1} = \boldsymbol{r}^t - \alpha^t \nabla_{\boldsymbol{r}} E(\boldsymbol{r}^t), \tag{17}$$

with $\alpha^t = \frac{n}{2(\mathbf{1}^T \boldsymbol{w}^t)}$.

3 Iterative Filtering with affine discriminant function

We look at the quadratic IF system with the discriminant function g defined in Eq.(6) where the iterations are given by

$$\boldsymbol{r}^{t+1} = \; F(\boldsymbol{w}^t) \; = \boldsymbol{X} \, \frac{\boldsymbol{w}^t}{\mathbf{1}^T \boldsymbol{w}^t}, \tag{18}$$

$$\boldsymbol{w}^{t+1} = G(\boldsymbol{d}^{t+1}) = \mathbf{1} - k\, \boldsymbol{d}^{t+1}, \tag{19}$$

starting with equal weights $\boldsymbol{w}^0 = \mathbf{1}$. By substituting $\boldsymbol{w}$, the fixed point of the system is given by a system of cubic equations in $\boldsymbol{r}^*$

$$(\boldsymbol{X} - \boldsymbol{r}^* \mathbf{1}^T)(\mathbf{1} - \frac{k}{n}(\boldsymbol{X}^T - \mathbf{1}(\boldsymbol{r}^*)^T)^{\circ 2} \mathbf{1}) = 0, \tag{20}$$

with $\boldsymbol{r}^*$ in the polytope $\mathcal{P}$ defined in Eq.(13). Theorem 2 claims that $\boldsymbol{r}^*$ is unique in $\mathcal{P}$ if k is such that the weights are strictly positive for all vectors of reputations $\boldsymbol{r} \in \mathcal{P}$. This result uses the associated energy function that we define for affine IF systems.

3.1 The energy function

The energy function in Eq.(16) associated with system (18,19) is given by

$$E(\boldsymbol{r}) = -\frac{1}{2\,k}\, \boldsymbol{w}^T \boldsymbol{w}, \tag{21}$$

where $\boldsymbol{w}$ depends on $\boldsymbol{r}$ according the function $G(\boldsymbol{r})$. We will see later that this energy function decreases with the iterations, i.e., $(E(\boldsymbol{r}^t))_{t \geq 0}$ decreases, and under some assumption on k, it converges to the unique minimum.

The iterations in system (18,19) can be written as a particular minimization step on the function E,

$$\boldsymbol{r}^{t+1} = \arg\min_{r} \left[-\frac{1}{2\,k}\, G(\boldsymbol{r})^T G(\boldsymbol{r}^t) \right].$$

Therefore, we have for all t that $(\boldsymbol{w}^{t+1})^T (\boldsymbol{w}^t) \geq (\boldsymbol{w}^t)^T (\boldsymbol{w}^t)$.

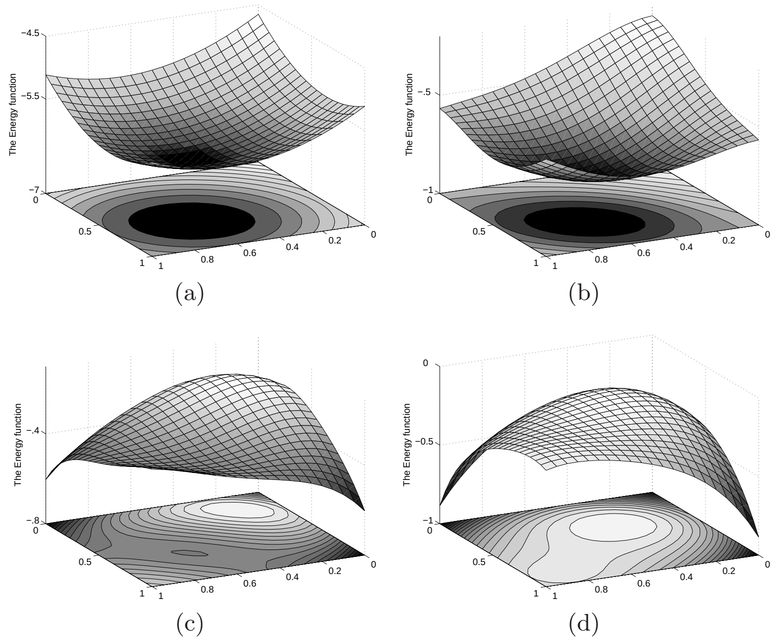

Fig. 1. Four energy functions with two objects and increasing values of k. We have in the unit square: (a) a unique minimum; (b) a unique minimum but other stationary points are close to the border; (c) a unique minimum and other stationary points; (d) a unique maximum.

3.2 Uniqueness

The following theorem proves that the stable point of quadratic IF systems with g defined in Eq.(6) is unique, under some condition on parameter k. This result follows directly from the energy function E that is a fourth-order polynomial equation.

Theorem 2. *(see [4]) The system (18,19) has a unique fixed point* $\boldsymbol{r}$ *in* $\mathcal{P}$ *if*

$$k < \min_{\boldsymbol{r} \in \mathcal{P}} \|\boldsymbol{d}\|_\infty^{-1}.$$

3.3 Convergence of the method

We analyze the convergence of system (18,19) that reaches the minimum of the energy function E in $\mathcal{P}$. Let $\boldsymbol{r}^t$ and $\boldsymbol{r}^{t+1}$ be two subsequent points of the iterations given by some search method. Then the next point $\boldsymbol{r}^{t+1}$ is obtained by choosing a direction vector $\boldsymbol{v}$ and a scalar γ such that

$$\boldsymbol{r}^{t+1} = \boldsymbol{r}^t + \gamma \boldsymbol{v}. \tag{22}$$

We will see that the direction $\boldsymbol{v}$ corresponds to a steepest descent in the first method and a coordinate descent for the second one. This corresponds to some line search on the scalar energy function

$$e(y) = E(\boldsymbol{r}^t + y\,\boldsymbol{v}) \tag{23}$$

that is a polynomial of degree 4. We have that $e(0)$ is the energy at $\boldsymbol{r}^t$ and $e(\gamma)$ is the energy at $\boldsymbol{r}^{t+1}$. Finally it is useful for the sequel to define the scalar that minimizes e given by

$$\beta = \arg \min_{y \text{ with } \boldsymbol{r}^t + y\,\boldsymbol{v} \in \mathcal{P}} E(\boldsymbol{r}^t + y\,\boldsymbol{v}). \tag{24}$$

System (18,19) provides a steepest descent method with a particular step size. The direction $\boldsymbol{v}$ and the scalar γ in Eq.(22) are

$$\boldsymbol{v} = -\nabla_{\boldsymbol{r}} E(\boldsymbol{r}^t) \qquad \text{and} \qquad \gamma = \alpha^t = \frac{n}{2\,(\mathbf{1}^T \boldsymbol{w}^t)},$$

so that we recover Eq.(17). This particular step size α^t can be compared to the step size β that minimizes E in the same direction given in Eq.(24). We have that the particular step size α^t is generally smaller than β in numeric simulations, meaning that the step stops before reaching the minimum of the energy function E in the direction $\boldsymbol{v}$. The sequence $(E(\boldsymbol{r}^t))_{t\geq 0}$ can be shown to decrease so that we have the following convergence result.

Theorem 3. *(see [4]) The steepest descent method given by system (18,19) converges to the unique fixed point in $\mathcal{P}$ if*

$$k < \min_{\boldsymbol{r} \in \mathcal{P}} \|\boldsymbol{d}\|_\infty^{-1}.$$

There exist greater values of k such that the minimum of E remains unique and the previous methods converge to this minimum. By increasing k, we allow the maxima of E to appear in the polytope $\mathcal{P}$, see Fig. 1(c). Then, we need to verify during the iterations if $(\boldsymbol{r}^t)$ remains in the basin of attraction of E.

Theorem 4. *(see [4]) If the energy function E in Eq.(21) has a minimum, then the system (18,19) is locally convergent and its asymptotic rate of convergence is linear.*

Let us remark that for a singular matrix $\boldsymbol{X}$, the rate of convergence will be faster. In particular, when $\boldsymbol{X}$ is a rank 1 matrix, we have $\boldsymbol{X} = \boldsymbol{r}^*\mathbf{1}^T$ (every object receives m identical votes from the raters) and the method converges in one step. When we take greater values of k maxima of the function E may appear in $\mathcal{P}$. However if the sequence $(\mathbf{1}^T\boldsymbol{w}^t)$ remains positive, the sequence $(E(\boldsymbol{r}^t))$ remains decreasing and converges to a stationary point of E. In order to avoid saddle points and maxima, we need to avoid to reach the minimum. The idea of increasing k is to make the discriminant function g more penalizing and therefore to have a better separation between honest and dishonest raters. We refer to [4] for more details on this.

4 Sparsity pattern and dynamical votes

This section extends some previous results to the case where the voting matrix has some sparsity pattern, that is when an object is not evaluated by all raters. Moreover we analyze dynamical voting matrices representing votes that evolve over time.

4.1 Sparsity pattern

In general, the structure of real data is sparse. We hardly find a set of raters and objects with a vote for all possible pairs. An absence of vote for object i from rater j will imply that the entry (i,j) of the matrix $\boldsymbol{X}$ is equal to zero, that is, by using the adjacency matrix $\boldsymbol{A}$,

$$\text{if } \boldsymbol{A}_{ij} = 0, \text{ then } \boldsymbol{X}_{ij} = 0.$$

These entries must not be considered as votes but instead as missing values. Therefore the previous equations presented in matrix form require some modifications that will include the adjacency matrix $\boldsymbol{A}$. We write the new equations and their implications using the order of the previous section. Let us already mention that some theorems will be simply stated without proof. Whenever their extensions with an adjacency matrix $\boldsymbol{A}$ are straightforward.

The belief divergence for IF systems in Eq.(2) becomes

$$d_j = \frac{1}{n_j} \|\boldsymbol{x}_j - \boldsymbol{a}_j \circ \boldsymbol{r}\|,$$

where $\boldsymbol{a}_j$ is the j^{th} column of the adjacency matrix $\boldsymbol{A}$ and n_j is the j^{th} entry of the vector $\boldsymbol{n}$ containing the numbers of votes given to each item, i.e.,

$$\boldsymbol{n} = \boldsymbol{A}^T \mathbf{1}.$$

On the other hand, the scalar n remains the total number of objects, i.e., the number of rows in $\boldsymbol{A}$. Therefore, when $\boldsymbol{A}$ is full, then $\boldsymbol{n} = n\mathbf{1}$.

Eq.(11-12) for quadratic IF systems can be replaced by the following ones: the reputation function, that remains the weighted average of the votes, is given in matrix form by

$$F(\boldsymbol{w}) = \frac{[\boldsymbol{X}\boldsymbol{w}]}{[\boldsymbol{A}\boldsymbol{w}]},$$

where $\frac{[\cdot]}{[\cdot]}$ is the componentwise division. Let us remark that every entry of $\boldsymbol{A}\boldsymbol{w}$ must be strictly positive. This means that every object is evaluated by at least one rater with nonzero weight.
Then all possible vectors of reputations $\boldsymbol{r}$ are include in the polytope

$$\bar{\mathcal{P}} = \{\boldsymbol{r} \in \mathbb{R}^n \,|\, r_i = \sum_{j=1}^{m} \lambda_j \boldsymbol{x}_j \text{ with } \sum_{j=1}^{m} \lambda_j \, a_{ij} = 1 \text{ and } \lambda_j \geq 0\}.$$

The third equation (12) for the belief divergence with the Euclidian norm is changed into

$$\boldsymbol{d} = \frac{\left[(\boldsymbol{X}^T - \boldsymbol{A}^T \circ \mathbf{1}\boldsymbol{r}^T)^{\circ 2}\mathbf{1}\right]}{\left[\boldsymbol{A}^T\mathbf{1}\right]}. \tag{25}$$

With these modifications, the iterations and the fixed point in Eq.(7,9) are given by quadratic equations in $\boldsymbol{r}$ and $\boldsymbol{w}$

$$(\boldsymbol{A} \circ \boldsymbol{r}^{t+1}\mathbf{1}^T)\,\boldsymbol{w}^t = \boldsymbol{X}\boldsymbol{w}^t \tag{26}$$
$$(\boldsymbol{A} \circ \boldsymbol{r}^{*}\mathbf{1}^T)\,\boldsymbol{w}^* = \boldsymbol{X}\boldsymbol{w}^*. \tag{27}$$

Hence we expect an energy function to exist and Theorem 1 is generalized by the following theorem.

Theorem 5. *(see [4]) The fixed points of quadratic IF systems with integrable discriminant function g, are the singular points of the energy function*

$$E(\boldsymbol{r}) = \frac{1}{n} \sum_{j=1}^{m} n_j \int_0^{d_j(\boldsymbol{r})} g(u)\, du, \tag{28}$$

where d_j is the belief divergence of rater j that depends on $\boldsymbol{r}$. Moreover one iteration step in quadratic IF systems corresponds to a dilated steepest descent direction with a particular step size

$$\boldsymbol{r}^{t+1} = \boldsymbol{r}^t - \boldsymbol{\alpha}^t \circ \nabla_{\boldsymbol{r}} E(\boldsymbol{r}^t) \tag{29}$$

with $\boldsymbol{\alpha}^t = \frac{n}{2} \frac{[\mathbf{1}]}{[\boldsymbol{A}\boldsymbol{w}^t]}$.

The number of votes n_j gives somehow a weight of importance for the minimization of the surface $\int_0^{d_j} g(u)\, du$. Therefore a rater with more votes receives more attention in the minimization process.

4.2 Affine quadratic IF systems.

The system for the discriminant function $g(d) = 1 - k\, d$ is given by

$$\boldsymbol{r}^{t+1} = F(\boldsymbol{w}^t) = \frac{[\boldsymbol{X}\boldsymbol{w}]}{[\boldsymbol{A}\boldsymbol{w}]}, \tag{30}$$

$$\boldsymbol{w}^{t+1} = G(\boldsymbol{d}^{t+1}) = \mathbf{1} - k\, \boldsymbol{d}^{t+1}, \tag{31}$$

with the belief divergence defined in Eq.(25).
The energy function is given by

$$E(\boldsymbol{r}) = -\frac{1}{2\,kn}\, \boldsymbol{w}^T [\boldsymbol{w} \circ \boldsymbol{n}], \tag{32}$$

where $\boldsymbol{w}$ depends on $\boldsymbol{r}$ according to the function $G(\boldsymbol{r})$.
Theorem 2 remains valid for the system (30-31) and the arguments are similar. The steepest descent method adapted to the system (30-31) converges with the property that the sequence $(E(\boldsymbol{r}^t))$ decreases. The proofs are closely related to the ones presented in Theorems 3.

Theorem 6. *(see [4]) The steepest descent method given by system (30,31) converges to the unique fixed point in $\bar{\mathcal{P}}$ if*

$$k < \min_{\boldsymbol{r} \in \bar{\mathcal{P}}} \|\boldsymbol{d}\|_\infty^{-1}.$$

The choice of k can be made larger to better separate honest from dishonest raters. Theorem 4 remains valid with a few modifications in its proof to take into account the adjacency matrix $\boldsymbol{A}$.

Theorem 7. *(see [4]) If the energy function E in Eq.*(32) *has a minimum, then (30,31) is locally convergent and its asymptotic rate of convergence is linear.*

This section shows that most of the earlier analysis can still be applied when we introduce a sparsity pattern in the voting matrix.

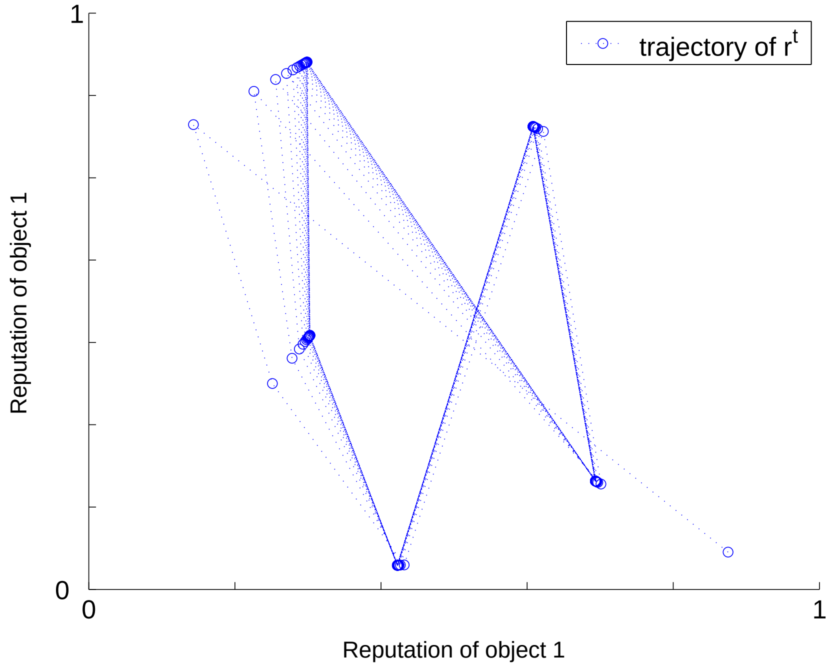

Fig. 2. Trajectory of reputations (circles) for a 5-periodic voting matrix

4.3 Dynamical votes

We consider in this section the case of time-varying votes. Formally, we have discrete sequences

$$(\boldsymbol{X}^t)_{t\geq 0}, \qquad (\boldsymbol{A}^t)_{t\geq 0}$$

of voting matrices and adjacency matrices evolving over time t. Hence the IF system (7,8) takes into account the new voting matrix $\boldsymbol{X}^{t+1}$ in the functions F_{t+1} and G_{t+1} that become time-dependent:

$$\boldsymbol{r}^{t+1} = F_{t+1}(\boldsymbol{w}^t), \tag{33}$$

$$\boldsymbol{w}^{t+1} = G_{t+1}(\boldsymbol{d}^{t+1}). \tag{34}$$

The system (30,31) for dynamical voting matrices is then given by (30,31)

$$\boldsymbol{r}^{t+1} = F_{t+1}(\boldsymbol{w}^t) = \frac{[\boldsymbol{X}^{t+1}\boldsymbol{w}^t]}{[\boldsymbol{A}^{t+1}\boldsymbol{w}^t]}, \tag{35}$$

$$\boldsymbol{w}^{t+1} = G_{t+1}(\boldsymbol{d}^{t+1}) = \mathbf{1} - k\,\boldsymbol{d}^{t+1}, \tag{36}$$

with the belief divergence $\boldsymbol{d}^{t+1}$ defined as in Eq.(25) after replacing $\boldsymbol{X}$ and $\boldsymbol{r}$ by $\boldsymbol{X}^{t+1}$ and $\boldsymbol{r}^{t+1}$. We already now that for subsequent constant matrices $\boldsymbol{X}^t$ with $T_1 \leq t \leq T_2$, the iterations on $\boldsymbol{r}^t$ and $\boldsymbol{w}^t$ of system (35,36) and its variant for coordinate descent tend to fixed vectors $\boldsymbol{r}^*$ and $\boldsymbol{w}^*$ provided that k is not too large. In [4] we give derive stronger results for the case of 2-periodic voting sequences.

5 Concluding remarks

The general definition of Iterative Filtering systems provides a new framework to analyze and evaluate voting systems. We emphasized the need for a differentiation of trusts between the raters unlike what is usually done on the Web. The originality of the approach lies in the continuous validation scale for the votes. Next, we assumed that the set of raters is characterized by various possible behaviors including raters who are clumsy or partly dishonest. However, the outliers being in obvious disagreement with the other votes remain detectable by the system as shown in the simulations in the cases of alliances, random votes and spammers.

Our paper focuses on the subclass of quadratic IF systems and we show the existence of an energy function that allows us to link a steepest descent to each step of the iteration. It then follows that the system minimizes the belief divergence according to some norm defined from the choice of the discriminant function.

This method was illustrated in [4] using two data sets: (i) the votes of 43 countries during the final of the EuroVision 2008 and (ii) the votes of 943 movie lovers in the website of MovieLens. It was shown that the IF method penalizes certain types of votes. In the first set of data, this yielded a difference in the ranking used by Eurovision and the ranking obtained by our method, in the sense that countries trading votes with e.g. neighboring countries, would get a smaller weight. The second set of data was used to verify the desired property mentioned in the introduction: *raters diverging often from other raters' opinion are less taken into account.*

We see two application areas of voting systems: first, the general definition of IF systems offers the possibility to analyze various systems depending on the context and the objectives we aim for; second, the experimental tests and the comparisons are crucial to validate the desired properties (including dynamical properties) and to discuss the choice of the IF systems.

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References

1. G. Akerloff, *The Market for Lemons: Quality Uncertainty and the Market Mechanism*, *Quaterly Journal of Economics, vol. 84 pp. 488-500*, 1970.
2. R. Baeza-Yates and C. Castillo and V. López, *PageRank Increase under Different Collusion Topologies*, *First International Workshop on Adversarial Information Retrieval on the Web,* `http://airweb.cse.lehigh.edu/2005/baeza-yates.pdf`, 2005.
3. C. de Kerchove and P. Van Dooren, *Reputation Systems and Optimization*, *Siam News, March 14 2008.*
4. C. de Kerchove and P. Van Dooren, *Iterative Filtering in Reputation Systems*, submitted, 2009.
5. G. M. Del Corso, A. Gull and F. Romani, *Ranking a stream of news*, *Proceedings of the 14th international conference on World Wide Web*, 2005.
6. V. Ginsburgh and A. Noury, *Cultural Voting. The Eurovision Song Contest*, *mimeo*, 2004.

7. R. Guha, R. Kumar, P. Raghavan, A. Tomkins, *Propagation of Trust and Distrust*, *Proceedings of the 13th International Conference on World Wide Web, pp. 403-412*, 2004.
8. Z. Gyöngyi and H. Garcia-Molina, *Link spam alliances*, *VLDB '05: Proceedings of the 31st international conference on Very large data bases, pp. 517–528*, 2005.
9. S. Kamvar, M. Schlosser and H. Garcia-molina, *The Eigentrust Algorithm for Reputation Management in P2P Networks*, *Proceedings of the 12th International Conference on World Wide Web, pp. 640-651*, 2003.
10. E. Kotsovinos, P. Zerfos, N. M. Piratla, N. Cameron and S. Agarwal, *Jiminy: A Scalable Incentive-Based Architecture for Improving Rating Quality, iTrust06, LNCS 3986, pp. 221-236*, 2006.
11. P. Laureti, L. Moret, Y.-C. Zhang and Y.-K. Yu, *Information Filtering via Iterative Refinement*, *EuroPhysic Letter 75, pp. 1006-1012*, 2006.
12. G. McLachlan and T. Krishnan, *The EM algorithm and extensions*, *John Wiley & Sons, New York, 1996.*
13. L. Mui, M. Mohtashemi and A. Halberstadt, *A Computational Model of Trust and Reputation*, *Proceedings of the 35th Annual Hawaii International Conference, pp. 2431-2439*, 2002.
14. J. O'Donovan and B. Smyth, *Trust in recommender systems*, *Proceedings of the 10th International Conference on Intelligent User Interfaces, pp. 167-174*, 2005.
15. L. Page and S. Brin and R. Motwani and T. Winograd, *The PageRank Citation Ranking: Bringing Order to the Web*, *Stanford Digital Library Technologies Project*, 1998.
16. M. Richardson, R. Agrawal and P. Domingos, *Trust Management for the Semantic Web*, *LNCS 2870, pp. 351-368*, 2003.
17. G. Theodorakopoulos and J. Baras, *On Trust Models and Trust Evaluation Metrics for Ad Hoc Neworks*, *Selected Areas in Communications, IEEE Journal on Vol. 24, Issue 2, pp. 318 - 328*, 2006.
18. Y.-K. Yu, Y.-C. Zhang, P. Laureti and L. Moret, *Decoding information from noisy, redundant, and intentionally distorted sources*, *Physica A, Volume 371, Issue 2, p. 732-744, 2006.*
19. S. Zhang, Y. Ouyang, J. Ford, F. Make, *Analysis of a Lowdimensional Linear Model under Recommendation Attacks*, *Proceedings of the 29th annual International ACM SIGIR conference on Research and development in information retrieval, pp. 517-524*, 2006.