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2-Verma modules

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Abstract. We construct a categorification of parabolic Verma modules for symmetrizable Kac–Moody algebras using KLR-like diagrammatic algebras. We show that our construction arises naturally from a dg-enhancement of the cyclotomic quotients of the KLR-algebras. As a consequence, we are able to recover the usual categorification of integrable modules. We also introduce a notion of dg-2-representation for quantum Kac–Moody algebras, and in particular of parabolic 2-Verma modules.

Contents

1. Introduction

The study of categorical actions of (quantum enveloping algebras of) Kac–Moody algebras leads to many interesting results. An impressive example is due to Chuang and Rouquier, who introduced in the work [12] categorical actions of $\mathfrak{sl}_2$ to prove the Broué abelian defect

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group conjecture for symmetric groups. Another interesting result is Webster's construction of homological versions of quantum invariants of links obtained by the Reshetikhin–Turaev machinery [45].

We note that, until recently, only categorifications of integrable representations of quantum Kac–Moody algebras were known. These are given by additive (or abelian) categories, on which the quantum group acts by (exact) endofunctors respecting certain direct sum decompositions, corresponding to the defining relations of the algebra (see for example [16, 20, 26, 27, 39]). In [34], the authors followed a slightly different approach to construct a categorification of the universal Verma module $M(\lambda)$ for quantum $\mathfrak{sl}_2$. The construction of [34] is given in the form of an abelian, bigraded (super)category, where the commutator relation takes the form of a (non-split) natural short exact sequence

$$0 \to \mathsf{FE} \to \mathsf{EF} \to \mathsf{QK} \oplus \Pi\mathsf{QK}^{-1} \to 0,$$

where Π is the parity shift functor, and Q a categorification of $1/(q-q^{-1})$ in the form of an infinite direct sum. This category is obtained as a certain category of modules over cohomology rings of infinite Grassmannians and their Koszul duals. Categorification of Verma modules appeared independently in the literature with a strongly different flavor in [13] and in [5].

Studying the endomorphism ring of $\mathsf{F}^k := \mathsf{F} \circ \cdots \circ \mathsf{F}$ yields a (super)algebra A_k that extends the ubiquitous nilHecke algebra NH_k. This superalgebra was studied by the authors in the follow up [35], where it was used to construct an equivalent categorification of Verma modules for quantum $\mathfrak{sl}_2$. The supercenter of A_k was also studied in [4]. The definition of the superalgebra A_k and is supercenter were extended in [38] to the case of a Weyl group of type B.

The superalgebra A_k comes equipped with a family of differentials d_n for $n \geq 0$. The corresponding dg-algebras are formal, with homology being isomorphic to the n-cyclotomic quotients of the nilHecke algebra. These quotients are known to categorify the irreducible integrable $U_q(\mathfrak{sl}_2)$-representations $V(n)$ of highest weight n. We interpret this as a categorification of the universal property of the Verma module $M(\lambda)$, that is there is a surjection $M(\lambda) \twoheadrightarrow V(n)$ for all n. This also means the dg-algebra (A_k, d_n) can be seen as a dg-enhancement of the cyclotomic nilHecke algebra NH_k^n, and in particular, of categorified $V(n)$.

In [23, 25] and [39], Khovanov–Lauda and Rouquier introduced generalizations of the nilHecke algebra for any Cartan datum. These algebras are presented in the form of braid-like diagrams in [23, 25], with strands labeled by simple roots and decorated with dots. It is proven in [23, 25, 39] that KLR algebras categorify the half quantum group associated with the input Cartan datum. Khovanov and Lauda conjectured that certain quotients of these algebras categorify irreducible, integrable representations of the quantum group. Due to the isomorphism between these quotient algebras and cyclotomic Hecke algebras in type A (see [8, 39]), these quotients have become known as cyclotomic KLR algebras. The corresponding cyclotomic conjecture was first proven in [9, 10, 28] for some special cases, and then for all symmetrizable Kac–Moody algebras by Kang–Kashiwara in [20], and independently by Webster in [45].

In this paper, we introduce a version of KLR algebra associated to a pair $(\mathfrak{p}, \mathfrak{g})$, where $\mathfrak{p}$ is a (standard) parabolic subalgebra of a quantum Kac–Moody algebra $\mathfrak{g}$. This construction generalizes the algebra A_k from [34], which we view as associated to the (standard) Borel subalgebra of $\mathfrak{sl}_2$. The usual KLR algebra is recovered by taking $\mathfrak{p} = \mathfrak{g}$. We prove that certain "cyclotomic quotients" of these $\mathfrak{p}$-KLR algebras categorify parabolic Verma modules induced over the parabolic subalgebra $\mathfrak{p}$, with the cyclotomic quotient depending on the highest weight. The proof goes by showing first that if $\mathfrak{p} = \mathfrak{b}$ is the (standard) Borel subalgebra of $\mathfrak{g}$, then the

$\mathfrak{b}$-KLR algebra is equipped with a categorical $\mathfrak{g}$-action similar to the one constructed in [35]. In particular, it categorifies the universal Verma module of $\mathfrak{g}$. Next, we show that the $\mathfrak{b}$-KLR algebra can be equipped with a family of differentials, turning it into a dg-enhancement of the cyclotomic $\mathfrak{p}$-KLR algebras. This induces a categorical $\mathfrak{g}$-action on the cyclotomic $\mathfrak{p}$-KLR algebra. In particular, we recover the usual categorical action on cyclotomic KLR algebras, and we can reinterpret Kang–Kashiwara's proof of Khovanov–Lauda's cyclotomic conjecture in terms of dg-enhanced KLR algebras. The world of dg-categories also allows to reinterpret the usual categorical $\mathfrak{sl}_2$-commutator relation in terms of mapping cones. More precisely, the derived category of dg-modules over the dg-enhanced KLR algebra comes equipped with functors $\mathsf{E}_i, \mathsf{F}_i$ and an autoequivalence K_i for all simple root α_i, that categorifies the action of the Chevalley generators E_i, F_i and of the Cartan element $K_i = q_i^{H_i}$. Then the $\mathfrak{sl}_2$-commutator relation of the categorical action takes the form of a quasi-isomorphism of mapping cones

$$\operatorname{Cone}(\mathsf{F}_i\mathsf{E}_i \to \mathsf{E}_i\mathsf{F}_i) \xrightarrow{\simeq} \operatorname{Cone}(\mathsf{Q}_i\mathsf{K}_i \to \mathsf{Q}_i\mathsf{K}_i^{-1}),$$

where Q_i is a direct sum of grading shift copies of the identity functor that categories the fraction $1/(q_i^{-1}-q_i)$. Whenever F_i is locally nilpotent, $\operatorname{Cone}(\mathsf{Q}_i\mathsf{K}_i \to \mathsf{Q}_i\mathsf{K}_i^{-1})$ is quasi-isomorphic to a finite direct sum of shifted copies of the identity functor, corresponding to the usual notion of an integrable categorical $\mathfrak{g}$-action (as in [20] for example).

Categorification of parabolic Verma modules have found connections with topology in the work of the authors in [36]. In particular, they have constructed Khovanov–Rozansky's triply graded link homology using parabolic 2-Verma modules of $\mathfrak{gl}_{2k}$. On the decategorified level, the connection between the HOMFPY-PT link polynomial and Verma modules was not known before. We expect to find in the future more connections between categorified Verma modules and low-dimensional topology.

Outline of the paper. In Section 2, we recall the basics about quantum groups and their parabolic Verma modules.

In Section 3, we introduce the $\mathfrak{b}$-KLR algebra $R_\mathfrak{b}$ (Definition 3.3) as a diagrammatic algebra over a unital commutative ring $\Bbbk$, in the same spirit as Khovanov–Lauda's [23]. We construct a faithful action on a polynomial ring and exhibit a basis, proving $R_\mathfrak{b}$ is a free $\Bbbk$-module.

In Section 4, we introduce the $\mathfrak{p}$-KLR algebra $R_\mathfrak{p}$ for any (standard) parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. We also introduce the corresponding N-cyclotomic quotient $R_\mathfrak{p}^N$. We introduce a differential d_N on $R_\mathfrak{b}$, turning it into a dg-enhancement of $R_\mathfrak{p}^N$. In particular, we prove the following theorem:

Theorem 4.4. *The dg-algebra $(R_\mathfrak{b}(m), d_N)$ is formal with homology*

$$H(R_\mathfrak{b}(m), d_N) \cong R_\mathfrak{p}^N(m).$$

In Section 5, we construct a categorical action of $U_q(\mathfrak{g})$ on $R_\mathfrak{b}$, where the action of the Chevalley generators F_i and E_i is given by functors F_i and E_i which are defined in terms of induction and restriction functors for the map that adds a strand labeled i. The $\mathfrak{sl}_2$-commutator relation takes the form of a non-split natural short exact sequence. Let $\bigoplus_{[\beta_i-\alpha_i^\vee(\nu)]_{q_i}} \mathrm{Id}_\nu$ be an infinite direct sum of degree shifts of the identity functor that categorifies the power series $(\lambda_i q_i^{-\alpha_i^\vee(\nu)} - \lambda_i^{-1} q_i^{\alpha_i^\vee(\nu)})/(q_i - q_i^{-1})$ (see equation (5.1) in the beginning of Section 5).

Corollary 5.2. *There is a natural short exact sequence*

$$0 \to \mathsf{F}_i \mathsf{E}_i \,\mathrm{Id}_\nu \to \mathsf{E}_i \mathsf{F}_i \,\mathrm{Id}_\nu \to \bigoplus_{[\beta_i - \alpha_i^\vee(\nu)]_{q_i}} \mathrm{Id}_\nu \to 0$$

for all $i \in I$, and there is a natural isomorphism

$$\mathsf{F}_i \mathsf{E}_j \cong \mathsf{E}_j \mathsf{F}_i$$

for all $i \neq j \in I$.

Fix $\mathfrak{p} \subset \mathfrak{g}$, and let I_f be the set of simple roots for which $F_i \in \mathfrak{p}$. Let $\bigoplus_{[n]_{q_i}} \mathrm{Id}_\nu$ be a finite direct sum of degree shifts of the identity functor that categorifies the quantum integer $[n]_{q_i}$. The categorical $\mathfrak{g}$-action on $R_\mathfrak{b}$ lifts to the dg-algebra $(R_\mathfrak{b}, d_N)$, and thus to $R_\mathfrak{p}^N$ by Theorem 4.4. The short exact sequence of Corollary 5.2 lifts to a short of exact sequence of complexes, inducing a long exact sequence in homology. This allows us to compute the action of the functors of induction F_i^N and restriction E_i^N on $R_\mathfrak{p}^N$:

Theorem 5.17. *For $i \notin I_f$ there is a natural short exact sequence*

$$0 \to \mathsf{F}_i^N \mathsf{E}_i^N \,\mathrm{Id}_\nu \to \mathsf{E}_i^N \mathsf{F}_i^N \,\mathrm{Id}_\nu \to \bigoplus_{[\beta_i - \alpha_i^\vee(\nu)]_{q_i}} \mathrm{Id}_\nu \to 0,$$

and for $i \in I_f$ there are natural isomorphisms

$$\mathsf{E}_i^N \mathsf{F}_i^N \,\mathrm{Id}_\nu \cong \mathsf{F}_i^N \mathsf{E}_i^N \,\mathrm{Id}_\nu \bigoplus_{[n_i - \alpha_i^\vee(\nu)]_{q_i}} \mathrm{Id}_\nu \quad \text{if } n_i - \alpha_i^\vee(\nu) \geq 0,$$

$$\mathsf{F}_i^N \mathsf{E}_i^N \,\mathrm{Id}_\nu \cong \mathsf{E}_i^N \mathsf{F}_i^N \,\mathrm{Id}_\nu \bigoplus_{[\alpha_i^\vee(\nu) - n_i]_{q_i}} \mathrm{Id}_\nu \quad \text{if } n_i - \alpha_i^\vee(\nu) \leq 0.$$

Moreover, there is a natural isomorphism

$$\mathsf{F}_i^N \mathsf{E}_j^N \cong \mathsf{E}_j^N \mathsf{F}_i^N$$

for $i \neq j \in I$.

In Section 6, we compute the asymptotic Grothendieck group of $(R_\mathfrak{b}, d_N)$. The asymptotic Grothendieck group is a refined version of Grothendieck group, that was introduced by the first author in [33]. It allows taking in consideration infinite iterated extensions of objects, such as infinite projective resolutions and infinite composition series (see Definition 6.3). Let $M^\mathfrak{p}(\Lambda, N)$ be the parabolic Verma module of highest weight (Λ, N), and $\mathcal{M}^\mathfrak{p}(\Lambda, N)$ be the c.b.l.f. derived category of $(R_\mathfrak{b}, d_N)$ (see Section 6.1).

Theorem 6.14. *The asymptotic Grothendieck group*

$${}_\mathbb{Q}\boldsymbol{K}_0^\Delta(\mathcal{M}^\mathfrak{p}(\Lambda, N))$$

is a $U_q(\mathfrak{g})$-weight module, with action of E_i, F_i given by $[\mathsf{E}_i], [\mathsf{F}_i]$. Moreover, there is an isomorphism of $U_q(\mathfrak{g})$-modules

$${}_\mathbb{Q}\boldsymbol{K}_0^\Delta(\mathcal{M}^\mathfrak{p}(\Lambda, N)) \cong M^\mathfrak{p}(\Lambda, N).$$

In Section 7, we introduce a notion of categorical dg-action of $\mathfrak{g}$ on a pretriangulated dg-category (Definition 7.2), and of (parabolic) 2-Verma module (Definition 7.6). In particular, we show that $\mathcal{M}^{\mathfrak{p}}(\Lambda, N)$ admits a dg-enhancement $\mathcal{M}^{\mathfrak{p}}_{\mathrm{dg}}(\Lambda, N)$ in the form of a dg-category. It yields an example of parabolic 2-Verma module, for which Theorem 6.14 takes the following form:

Corollary 7.8. *For all $i \in I$ there is a quasi-isomorphism of cones*

$$\mathrm{Cone}\big(\mathsf{F}_i^N \mathsf{E}_i^N \,\mathrm{Id}_\nu \to \mathsf{E}_i^N \mathsf{F}_i^N \,\mathrm{Id}_\nu\big) \xrightarrow{\simeq} \mathrm{Cone}\big(\mathsf{Q}_i \lambda_i q_i^{-\alpha_i^\vee(\nu)} \,\mathrm{Id}_\nu \to \mathsf{Q}_i \lambda_i^{-1} q^{\alpha_i^\vee(\nu)} \,\mathrm{Id}_\nu\big),$$

in $\mathcal{E}nd_{\mathrm{Hqe}}(\mathcal{D}_{\mathrm{dg}}(R_{\mathfrak{b}}, d_N))$.

Finally, in Section A we recall the construction of the homotopy category of dg-categories up to quasi-equivalence, based on Toen [42]. We also recall how to compute the (derived) dg-hom-spaces between pretriangulated dg-categories.

2. Quantum groups and Verma modules

We recall the basics about quantum groups and their (parabolic) Verma modules. Our presentation is close to [19] and [30], where the proofs can be found. References for classical results about Verma modules are [31] and [18] (and [2] for the quantum case).

2.1. Quantum groups. A *generalized Cartan matrix* is a finite-dimensional square matrix $A = \{a_{ij}\}_{i,j \in I} \in \mathbb{Z}^{|I| \times |I|}$ such that

- $a_{ii} = 2$ and $a_{ij} \leq 0$ for all $i \neq j \in I$,
- $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

One says that A is symmetrizable if there exists a diagonal matrix D with positive entries $d_i \in \mathbb{Z}_{>0}$ for all $i \in I$, such that DA is symmetric. A *Cartan datum* consists of

- a symmetrizable generalized Cartan matrix A,
- a free abelian group Y called the *weight lattice*,
- a set of linearly independent elements $\Pi = \{\alpha_i\}_{i \in I} \subset Y$ called *simple roots*,
- a *dual weight lattice* $Y^\vee := \mathrm{Hom}(Y, \mathbb{Z})$,
- a set of *simple coroots* $\Pi^\vee = \{\alpha_i^\vee\}_{i \in I} \subset Y^\vee$,

such that

- $\alpha_i^\vee(\alpha_j) = a_{ij}$,
- for each $i \in I$ there is a *fundamental weight* $\Lambda_i \in Y$ such that $\alpha_j^\vee(\Lambda_i) = \delta_{ij}$ for all $j \in I$.

The abelian subgroup $X := \bigoplus_i \mathbb{Z}\alpha_i \subset Y$ is called the *root lattice*. We also write

$$X^+ := \bigoplus_i \mathbb{N}\alpha_i \subset X$$

for the *positive roots*. Given a Cartan datum, since A is symmetrizable with $d_i a_{ij} = d_j a_{ji}$, one can construct a symmetric bilinear form

$$(-|-) : Y \times Y \to \mathbb{Z},$$

respecting

- $(\alpha_i|\alpha_i) = 2d_i \in \{2, 4, \dots\}$,
- $(\alpha_i|\alpha_j) = d_i a_{ij} \in \{0, -1, -2, \dots\}$ for all $i \neq j$,
- $\alpha_i^\vee(y) = 2\frac{(\alpha_i|y)}{(\alpha_i|\alpha_i)}$ for all $y \in Y$.

In the end, a Cartan datum is completely determined by $(I, X, Y, (-|-))$.

Definition 2.1. The *quantum Kac–Moody algebra* $U_q(\mathfrak{g})$ associated to a Cartan datum $(I, X, Y, (-|-))$ is the associative, unital $\mathbb{Q}(q)$-algebra generated by the set of elements E_i, F_i and K_γ for all $i \in I$ and $\gamma \in Y^\vee$, with relations for all $i \in I$ and $\gamma, \gamma' \in Y^\vee$:

$$K_0 = 1, \qquad K_\gamma K_{\gamma'} = K_{\gamma+\gamma'},$$
$$K_\gamma E_i = q^{\gamma(\alpha_i)} E_i K_\gamma, \qquad K_\gamma F_i = q^{-\gamma(\alpha_i)} F_i K_\gamma.$$

One also imposes the $\mathfrak{sl}_2$-*commutator relation* for all $i, j \in I$:

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

where $q_i := q^{d_i}$ and $K_i := K_{\alpha_i^\vee}$. Finally, there are the *Serre relations* for $i \neq j \in I$:

$$\sum_{r+s=1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} E_i^r E_j E_i^s = 0, \qquad \sum_{r+s=1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} F_i^r F_j F_i^s = 0.$$

This ends the definition of $U_q(\mathfrak{g})$.

Given a sequence $\boldsymbol{i} = i_1 \cdots i_m$ of elements in I, we write

$$F_{\boldsymbol{i}} := F_{i_1} \cdots F_{i_m} \quad \text{and} \quad E_{\boldsymbol{i}} := E_{i_1} \cdots E_{i_m}.$$

We write $\mathrm{Seq}(I)$ for the set of such sequences. Any element of $U_q(\mathfrak{g})$ decomposes as a sum of elements $F_{\boldsymbol{i}} K_\gamma E_{\boldsymbol{j}}$ with $\boldsymbol{i}, \boldsymbol{j} \in \mathrm{Seq}(I)$.

The *half quantum group* $U_q^-(\mathfrak{g})$ of $U_q(\mathfrak{g})$ is the subalgebra generated by the elements $\{F_i\}_{i\in I}$. As a $\mathbb{Q}(q)$-vector space, it admits a basis given by a subset of $\{F_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathrm{Seq}(I)}$.

2.2. Weight modules. Let M be an $U_q(\mathfrak{g})$-module with ground ring $R \supset \mathbb{Q}(q)$. Consider a $\mathbb{Z}$-linear functional

$$\lambda : Y^\vee \to R^\times,$$

where the group structure on $R^\times$ is the product. For each such functional λ and $y \in Y$, we call (λ, y)-*weight space* the set

$$M_{\lambda,y} := \{v \in M \mid K_\gamma v = \lambda(\gamma) q^{\gamma(y)} v \text{ for all } \gamma \in Y^\vee\}.$$

Note that $E_i M_{\lambda,y} \subset M_{\lambda,y+\alpha_i}$ and $F_i M_{\lambda,y} \subset M_{\lambda,y-\alpha_i}$. A weight module is a module that decomposes as a direct sum of weight spaces. A highest weight module is a module M such that $M = U_q(\mathfrak{g})v_\lambda$ for some $v_\lambda \in M_{\lambda,0}$ with $E_i v_\lambda = 0$ for all $i \in I$. In that case, we call λ the *highest weight* and we have

$$M \cong \bigoplus_{y \in X^+} M_{\lambda,-y},$$

as R-module.

One says that a $U_q(\mathfrak{g})$-module M is *integrable* if for each $v \in M$ there exists $k \gg 0$ such that $E_i^k v = 0$ and $F_i^k v = 0$ for all $i \in I$. Any finite-dimensional module is integrable, and any integrable module is a weight module with $\lambda(\Pi^\vee) \subset \mathbb{Z}[q]$. We consider only type 1 modules, that is $\lambda(\Pi^\vee) \subset \mathbb{N}[q]$.

Let M be a highest weight module with highest weight vector $v_\lambda \in M_{\lambda,0}$. Then we set $\lambda_i := \lambda(\alpha_i^\vee)$ for each $i \in I$. We are interested in λ such that each λ_i is either $\lambda_i = q^{n_i}$ for some $n_i \in \mathbb{Z}$ or λ_i is formal. In that case, we write it $\lambda_i = q^{\beta_i}$, where we interpret β_i as a formal parameter.

2.2.1. Parabolic Verma modules. The (standard) *Borel subalgebra* $U_q(\mathfrak{b})$ of $U_q(\mathfrak{g})$ is generated by K_γ and E_i for all $\gamma \in Y^\vee$ and $i \in I$. A (standard) *parabolic subalgebra* of $U_q(\mathfrak{g})$ is a subalgebra containing $U_q(\mathfrak{b})$. It is generated by K_γ, E_i and F_j for all $\gamma \in Y^\vee, i \in I$ and $j \in I_f$ for some fixed subset $I_f \subset I$. The part given by K_γ, E_j and F_j for $j \in I_f$ is called the *Levi factor* and written $U_q(\mathfrak{l})$. The *nilpotent radical* $U_q(\mathfrak{n})$ is generated by E_i for all $i \in I_r := I \setminus I_f$. Note that parabolic subalgebras are in bijection with partitions $I = I_f \sqcup I_r$.

Let $U_q(\mathfrak{p})$ be a parabolic subalgebra determined by $I = I_f \sqcup I_r$. For each $i \in I_f$, we choose a weight $n_i \in \mathbb{N}$. For each $j \in I_r$ we choose a weight $\lambda_j \in \{q^{\beta_j}, q^{n_j}\}$. We write $N = \{n_i\}_{i \in I_f}$ and $\Lambda = \{\lambda_j\}_{j \in I_r}$. Let $V(\Lambda, N)$ be the unique (type 1) integrable, irreducible representation of $U_q(\mathfrak{l})$ on the ground ring $R = \mathbb{Q}(q, \Lambda)$, and with highest weight λ determined by

$$\lambda(\alpha_k^\vee) = \begin{cases} q^{n_i} & \text{if } k = i \in I_f, \\ \lambda_j & \text{if } k = j \in I_r. \end{cases}$$

We extend it to a representation of $U_q(\mathfrak{p})$ by setting $U_q(\mathfrak{n})V(\Lambda, N) = 0$.

Definition 2.2. The *parabolic Verma module* of highest weight (Λ, N) associated to $U_q(\mathfrak{p}) \subset U_q(\mathfrak{g})$ is the induced module

$$M^{\mathfrak{p}}(\Lambda, N) := U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p})} V(\Lambda, N).$$

Whenever $U_q(\mathfrak{p}) \subsetneq U_q(\mathfrak{g})$, we have that $M^{\mathfrak{p}}(\Lambda, N)$ is an infinite-dimensional module. Moreover, for all parabolic Verma modules, there is a $\mathbb{Q}(q)$-linear surjection

$$U_q^-(\mathfrak{g}) \otimes_{\mathbb{Q}(q)} R \twoheadrightarrow M^{\mathfrak{p}}(\Lambda, N).$$

Example 2.3. If $U_q(\mathfrak{p}) = U_q(\mathfrak{b})$, then $N = \emptyset$, and $V(\Lambda, N) \cong \mathbb{Q}(q, \Lambda)v_\Lambda$ is 1-dimensional, and such that

$$E_i v_\Lambda = 0, \quad K_\gamma v_\Lambda = \prod_{j \in I} \lambda_j^{\gamma(\Lambda_j)} v_\Lambda.$$

In this case, we simply call it *Verma module*, and denote it $M^{\mathfrak{b}}(\Lambda)$. If $\lambda_j = q^{\beta}$ is formal for all $j \in I_r$, then we call it the *universal Verma module*.

Example 2.4. If $U_q(\mathfrak{p}) = U_q(\mathfrak{g})$, then $\Lambda = \emptyset$ and $M^{\mathfrak{p}}(\Lambda, N) \cong V(N)$ is an integrable, irreducible $U_q(\mathfrak{g})$ representation.

Since q is a generic parameter, we can apply Jantzen's criterion [18, Theorem 9.12], thanks to the results in [2]. We obtain that $M^{\mathfrak{p}}(\Lambda, N)$ is irreducible whenever $\lambda_j \notin \{q^n \mid n \in \mathbb{N}\}$ for all $j \in I_r$. If $\lambda_j = q^{n_j}$ for $n_j \in \mathbb{N}$, then $M^{\mathfrak{p}}(\Lambda, N)$ contains a non-trivial, proper submodule, which is isomorphic to $M^{\mathfrak{p}}(\Lambda^{n_j}_{-n_j-2}, N)$ for $\Lambda^{n_j}_{-n_j-2}$ given by exchanging q^{n_j} with q^{-n_j-2} in Λ. Moreover, the quotient

$$\frac{M^{\mathfrak{p}}(\Lambda, N)}{M^{\mathfrak{p}}(\Lambda^{n_j}_{-n_j-2}, N)} \cong M^{\mathfrak{p}+j}(\Lambda \setminus \{q^{n_j}\}, N \sqcup \{n_j\})$$

is isomorphic to the parabolic Verma module associated to the parabolic subalgebra $\mathfrak{p} + j$ given by adding j to I_f, that is generated by $\mathfrak{p}$ and F_j.

Furthermore, whenever $\lambda_j = q^{\beta_j}$ is formal, there is a surjective map

$$\mathrm{ev}_{n_j} : M^{\mathfrak{p}}(\Lambda, N) \twoheadrightarrow M^{\mathfrak{p}}(\Lambda^{n_j}_{\beta_j}, N)$$

for all $n_j \in \mathbb{Z}$, given by evaluating $\beta_j = n_j$.

These two facts together allow us to define a partial order on parabolic Verma modules. For this, we say that there is an arrow from $M^{\mathfrak{p}}(\Lambda, N)$ to $M^{\mathfrak{p}'}(\Lambda', N')$ if we have an evaluation map ev_{n_j} such that

$$\mathrm{ev}_{n_j}(M^{\mathfrak{p}}(\Lambda, N)) \cong M^{\mathfrak{p}'}(\Lambda', N'),$$

or if there is a short exact sequence

$$0 \to M^{\mathfrak{p}}(\Lambda^{n_j}_{-n_j-2}, N) \to M^{\mathfrak{p}}(\Lambda, N) \to M^{\mathfrak{p}'}(\Lambda', N') \to 0.$$

For parabolic Verma modules M and M' we say that M is bigger than M' if there is a chain of arrows from M to M'. In that case, there is an M'', which is either trivial or a parabolic Verma module, and a short exact sequence

$$0 \to M'' \to \mathrm{ev}(M) \to M' \to 0,$$

where ev is a composition of evaluation maps ev_{n_j}. With this partial order, the universal Verma module is a maximal element and each integrable, irreducible module is a minimum. This also means that we can recover any parabolic Verma module from the universal one.

2.2.2. The Shapovalov form. Let $\rho : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})^{\mathrm{op}}$ be the $\mathbb{Q}(q)$-linear algebra anti-involution given by

$$\rho(E_i) := q_i^{-1} K_i^{-1} F_i, \quad \rho(F_i) := q_i^{-1} K_i E_i, \quad \rho(K_\gamma) := K_\gamma \tag{2.1}$$

for all $i \in I$ and $\gamma \in Y^\vee$.

Definition 2.5. The *Shapovalov form*

$$(-,-) : M^{\mathfrak{p}}(\Lambda, N) \times M^{\mathfrak{p}}(\Lambda, N) \to \mathbb{Q}(q, \Lambda)$$

is the unique bilinear form respecting

- $(v_{\Lambda,N}, v_{\Lambda,N}) = 1$, for $v_{\Lambda,N}$ the highest weight vector,
- $(uv, v') = (v, \rho(u)v')$, where ρ is defined in (2.1),
- $f(v, v') = (fv, v') = (v, fv')$

for all $v, v' \in M^{\mathfrak{p}}(\Lambda, N), u \in U_q(\mathfrak{g})$ and $f \in \mathbb{Q}(q, \Lambda)$.

2.2.3. Basis. Since parabolic Verma modules are highest weight modules, they admit at least one basis given in terms of elements of the form $F_{\boldsymbol{i}} v_{\Lambda,N}$ for $\boldsymbol{i} \in \mathrm{Seq}(I)$, where $v_{\Lambda,N}$ is a highest weight vector. In particular, as R-modules they are all submodules of $U_q^-(\mathfrak{g}) \otimes_{\mathbb{Q}(q)} R$, meaning that these basis lives in a subset of $\{F_{\boldsymbol{i}} v_{\Lambda,N} \mid \boldsymbol{i} \in \mathrm{Seq}(I)\}$ modded out by the Serre relations. We call such a basis an *induced basis* and write it $\{v_{\Lambda,N} = m_0, m_1, \dots\}$. Any element in such basis takes the form

$$F_{\boldsymbol{i}} = F_{i_r}^{b_r} \cdots F_{i_1}^{b_1}$$

for some $i_1, \dots, i_r \in I$ and $b_1, \dots, b_r \in \mathbb{N}$, with $i_\ell \neq i_{\ell+1}$. Replacing each F_i^b by the *divided power*

$$F_i^{(b)} := \frac{F_i^b}{[b]_{q_i}!}$$

yields another basis $\{v_{\Lambda,N} = m'_0, m'_1, \dots\}$. Lusztig's *canonical basis* [30] is given by a certain choice of such a divided power basis characterized by

$$(m'_i, m'_i) - 1 \in \mathbb{Z}^+_{\prec}[\![q, \Lambda]\!]$$

for any order such that $0 \prec q \prec \lambda_i$ (see Section 6 for a definition of $\mathbb{Z}^+_{\prec}[\![q, \Lambda]\!]$). Whenever $M^{\mathfrak{p}}(\Lambda, N)$ is irreducible, the Shapovalov form is non-degenerate. Therefore, in this case, there is a *dual canonical basis* uniquely determined by

$$(m'_i, m^j) = \delta_{ij}.$$

3. The $\mathfrak{b}$-KLR algebras

Fix once and for all a Cartan datum $(I, X, Y, (-|-))$, and let

$$d_{ij} := -\alpha_i^\vee(\alpha_j) \in \mathbb{N}.$$

For $\nu \in X^+$ we write

$$\nu = \sum_{i \in I} \nu_i \cdot \alpha_i, \quad \nu_i \in \mathbb{N},$$

and we set $|\nu| := \sum_i \nu_i$, and $\mathrm{Supp}(\nu) := \{i \mid \nu_i \neq 0\}$.

We also fix a choice of scalars in a commutative, unital ring $\Bbbk$ as introduced in [40]. Following the conventions in [11], it consists of

- $t_{ij} \in \Bbbk^\times$ for all $i, j \in I$,
- $s_{ij}^{tv} \in \Bbbk$ for $i \neq j$, $0 \leq t < d_{ij}$ and $0 \leq v < d_{ji}$,
- $r_i \in \Bbbk^\times$ for all $i \in I$,

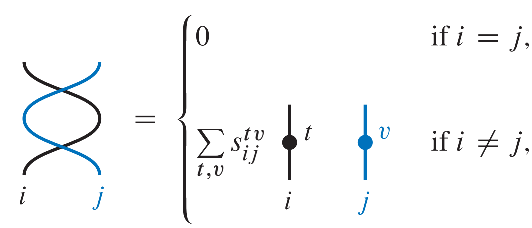
respecting

- $t_{ii} = 1$,
- $t_{ij} = t_{ji}$ whenever $d_{ij} = 0$,
- $s_{ij}^{tv} = s_{ji}^{vt}$,
- $s_{ij}^{tv} = 0$ whenever $t(\alpha_i|\alpha_i) + v(\alpha_j|\alpha_j) \neq -2(\alpha_i|\alpha_j)$.

In addition, whenever $t < 0$ or $v < 0$, we put $s_{ij}^{tv} := 0$. Thus we have $s_{ij}^{pq} = 0$ for $p > d_{ij}$ or $q > d_{ji}$. We will also write $s_{ij}^{d_{ij}0} := t_{ij}$ and $s_{ij}^{0d_{ji}} := t_{ji}$. Hence if $(\alpha_i|\alpha_j) = 0$, we get $s_{ij}^{00} = s_{ji}^{00} = t_{ij} = t_{ji}$.

Definition 3.1 ([23, 39]). For $m \in \mathbb{N}$, the *Khovanov–Lauda–Rouquier (KLR) algebra* $R(m)$ is the $\mathbb{k}$-algebra generated by braid-like diagrams on m strands, read from bottom to top, such that

- two strands can intersect transversally, but no triple intersections are allowed,
- strands can be decorated by dots (we use a dot with a label k to denote k consecutive dots on a strand),
- each strand is labeled by a simple root, written $i \in I$, that we (usually) write at the bottom,
- multiplication is given by concatenation of diagrams, which preserves the labeling (i.e. connecting two strands with different labels gives zero),
- diagrams are taken modulo planar isotopies and the following local relations:

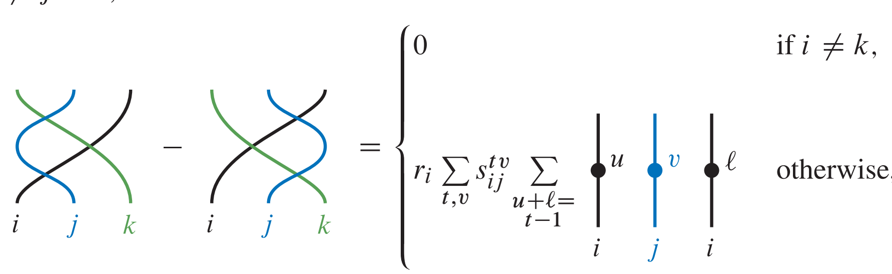
(3.1)


for all $i, j \in I$,

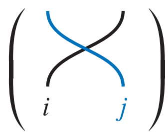
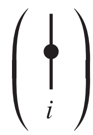
(3.2)


(3.3)


for all $i \neq j \in I$,

(3.4)


for all $i, j, k \in I$. In addition, $R(m)$ is $\mathbb{Z}$-graded by setting

$\deg_q$  $:= -(\alpha_i|\alpha_j)$, $\deg_q$  $:= (\alpha_i|\alpha_i)$.

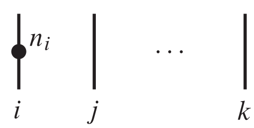
Remark 3.2. Note that in equation (3.1) and equation (3.4), the sum $\sum_{t,v} s_{ij}^{tv}$ can be restricted to the finite number of pairs $t, v \in \mathbb{N}$ such that

$$t(\alpha_i|\alpha_i) + v(\alpha_j|\alpha_j) = -2(\alpha_i|\alpha_j).$$

Moreover, it contains at least two non-zero elements with invertible coefficients, given by $t = d_{ij}, v = 0$ and $t = 0, v = d_{ji}$.

As proven in [23, 25] (see also [39]), these algebras categorify the half quantum group $U_q^-(\mathfrak{g})$ associated to $(I, X, Y, (-|-))$, as a (twisted) bialgebra. The multiplication and comultiplication are categorified using respectively induction and restriction functors, obtained by putting diagrams side by side.

For each non-negative integral highest weight $N := \{n_i \in \mathbb{N} \mid i \in I\}$, there is an N-cyclotomic quotient $R^N(m)$ of $R(m)$ given by modding out the two-sided ideal generated by all diagrams of the form

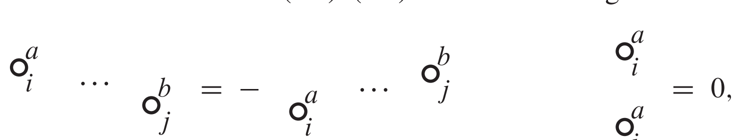
 $= 0.$

As first conjectured in [23] and proven in [20] and independently in [45], these cyclotomic quotients categorify the irreducible integrable $U_q(\mathfrak{g})$-module of highest weight N, where the action of F_i (resp. E_i) is given by induction (resp. restriction) along the map $R(m) \hookrightarrow R(m+1)$ that adds a vertical strand with label i, at the right.

3.1. $\mathfrak{b}$-KLR algebra. Our first goal is to construct a dg-enhancement of the cyclotomic KLR algebras $R^N(m)$, in the same spirit as in [35]. We introduce the following algebra:

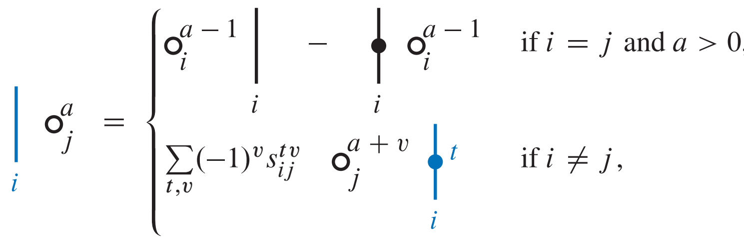
Definition 3.3. For $m \in \mathbb{N}$, the $\mathfrak{b}$-*KLR algebra* $R_{\mathfrak{b}}(m)$ is the $\Bbbk$-algebra generated by braid-like diagrams on m strands, read from bottom to top, such that

- two strands can intersect transversally, but no triple intersections are allowed,
- strands can be decorated by dots,
- regions in-between strands can be decorated by *floating dots*, which are labeled by a subscript in I and a superscript in $\mathbb{N}$,
- each strand is labeled by a simple root, written $i \in I$,
- multiplication is given by concatenation of diagrams, which preserves the labeling,
- diagrams are taken modulo planar isotopies that preserve the relative height of the floating dots, and modulo the KLR relations (3.1)–(3.4) and the following local relations:

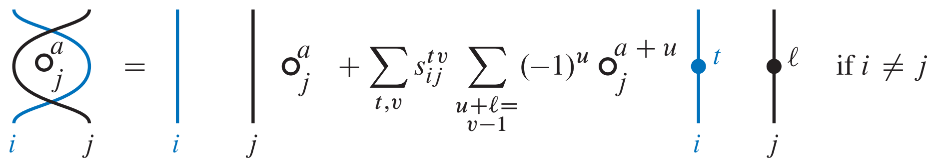
(3.5) 

meaning floating dots anti-commute with each other for all $i, j \in I$ and $a, b \in \mathbb{N}$,

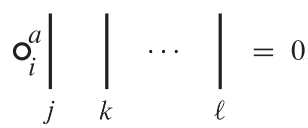
(3.6)



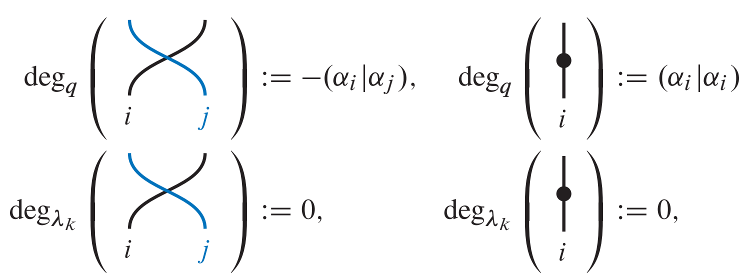
(3.7)



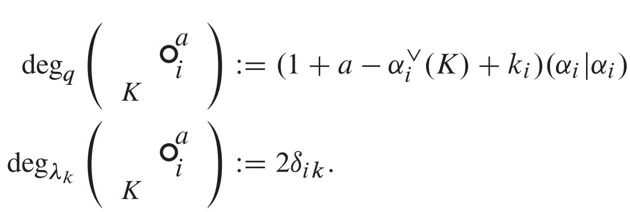
Moreover, a floating dot in the left-most region is zero



Given a diagram, it is sometimes useful to decorate some of its regions with an element $K := \sum_{i \in I} k_i \cdot \alpha_i \in X^+$, where k_i denotes the number of strands with label i to the left of the region. The algebra $R_{\mathfrak{b}}$ is $\mathbb{Z}^{1+|I|}$-graded (a q-grading and a λ_k-grading for each $k \in I$) with



and

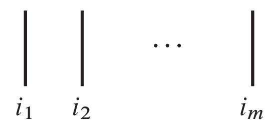


This ends the definition of $R_{\mathfrak{b}}(m)$.

3.2. Tightened basis. Before going any further, let us introduce some useful notations borrowed from [23]. First, let $R_{\mathfrak{b}}(\nu)$ be the subalgebra of $R_{\mathfrak{b}}(m)$ given by diagrams where there are exactly ν_i strands labeled i, for each $i \in I$. We also denote $\mathrm{Seq}(\nu)$ the set of all ordered sequences $\boldsymbol{i} = i_1 i_2 \cdots i_m$ with $i_k \in I$ and i appearing ν_i times in the sequence.

The symmetric group S_m acts on $\mathrm{Seq}(\nu)$ with the simple transposition $\sigma_k \in S_m$ acting on $\boldsymbol{i} = i_1 i_2 \cdots i_m \in \mathrm{Seq}(\nu)$ by permuting i_k and i_{k+1}. Sometimes, for $K = \sum_{i \in I} k_i \cdot \alpha_i \in X^+$, we abuse notation by writing σ_K instead of $\sigma_{|K|}$.

For $\boldsymbol{i} = i_1 i_2 \cdots i_m \in \mathrm{Seq}(\nu)$, let $1_{\boldsymbol{i}} \in R_{\mathfrak{b}}(\nu)$ be the idempotent given by m vertical strands with labels $i_1, i_2, \ldots, i_m$, that is

$1_{\boldsymbol{i}} :=$  .

We have $1_{\boldsymbol{i}} 1_{\boldsymbol{j}} = \delta_{\boldsymbol{i}\boldsymbol{j}}$ for all $\boldsymbol{i}, \boldsymbol{j} \in \mathrm{Seq}(\nu)$, and so there is a decomposition of $\Bbbk$-modules

$$R_{\mathfrak{b}}(\nu) \cong \bigoplus_{\boldsymbol{i}, \boldsymbol{j} \in \mathrm{Seq}(\nu)} 1_{\boldsymbol{j}} R_{\mathfrak{b}}(\nu) 1_{\boldsymbol{i}}.$$

Our goal is to construct a basis of $1_{\boldsymbol{j}} R_{\mathfrak{b}}(\nu) 1_{\boldsymbol{i}}$ as $\Bbbk$-module.

3.2.1. An action of $R_{\mathfrak{b}}(\nu)$ on a polynomial space. We construct a polynomial representation of $R_{\mathfrak{b}}(\nu)$ with a similar flavor as in [23, Section 2.3]. We fix $\nu \in X^+$ with $|\nu| = m$. For each $i \in I$ we define

$$Q_i := \Bbbk[x_{1,i}, \ldots, x_{\nu_i,i}] \otimes \textstyle\bigwedge^{\bullet}\langle \omega_{1,i}, \ldots, \omega_{\nu_i,i} \rangle.$$

We write $Q_I := \bigotimes_{i \in I} Q_i$, where $\otimes$ means the supertensor product in the sense that

$$\omega_{\ell,i}\omega_{\ell',j} = -\omega_{\ell',j}\omega_{\ell,i}$$

for all $i, j \in I$ and $x_{i,\ell}$ commutes with everything. Thus, Q_I is a supercommutative superring. Then we construct the ring

$$Q_\nu := \bigoplus_{\boldsymbol{i} \in \mathrm{Seq}(\nu)} Q_I 1_{\boldsymbol{i}},$$

where the elements $1_{\boldsymbol{i}}$ are central idempotents. It is $\mathbb{Z}^{1+|I|}$-graded by setting

$$\begin{aligned} \deg_q(x_{\ell,i}) &= (\alpha_i|\alpha_i), & \deg_q(\omega_{\ell,i}) &= (1-\ell)(\alpha_i|\alpha_i), \\ \deg_{\lambda_j}(x_{\ell,i}) &= 0, & \deg_{\lambda_j}(\omega_{\ell,i}) &= 2\delta_{ij}. \end{aligned}$$

We first construct an action of the symmetric group S_m on Q_ν by letting the simple transposition

$$\sigma_k : Q_I 1_{\boldsymbol{i}} \to Q_I 1_{\sigma_k \boldsymbol{i}}$$

to act by sending

$$x_{p,i} 1_{\boldsymbol{i}} \mapsto \begin{cases} x_{p+1,i} 1_{\sigma_k \boldsymbol{i}} & \text{if } i_k = i_{k+1} = i,\ p = \#\{s \le k \mid i_s = i\}, \\ x_{p-1,i} 1_{\sigma_k \boldsymbol{i}} & \text{if } i_k = i_{k+1} = i,\ p = 1 + \#\{s \le k \mid i_s = i\}, \\ x_{p,i} 1_{\sigma_k \boldsymbol{i}} & \text{otherwise,} \end{cases}$$

for $i \in I$, $p \in \{1, \ldots, \nu_i\}$ and $\boldsymbol{i} = i_1 \ldots i_m$, and by sending

$$\omega_{p,i} 1_{\boldsymbol{i}} \mapsto \begin{cases} (\omega_{p,i} + (x_{p,i} - x_{p+1,i})\omega_{p+1,i}) 1_{\sigma_k \boldsymbol{i}} & \text{if } i_k = i_{k+1} = i,\ p = \#\{s \le k \mid i_s = i\}, \\ \omega_{p,i} 1_{\sigma_k \boldsymbol{i}} & \text{otherwise,} \end{cases}$$

which we extend to Q_ν by setting $\sigma_k(fg) := \sigma_k(f)\sigma_k(g)$ for all $f, g \in Q_\nu$.

Proposition 3.4. *The procedure described above yields a well-defined action of S_m on Q_ν.*

Proof. The proof is a straightforward computation. We leave the details to the reader. □

Then we define inductively the element $\omega_{p,j}^a \in Q_I$ for $a \in \mathbb{N}$ as

$$\omega_{p,j}^0 := \omega_{p,j}, \quad \omega_{p,j}^{a+1} := \omega_{p-1,j}^a - x_{p,j}\omega_{p,j}^a.$$

For $K = \sum_{i\in I} k_i \cdot i \in X^+$ such that $k_i \leq \nu_i$, we define $\omega_j^a(K) \in Q_I$ inductively as

$$\omega_j^a(K) := \begin{cases} 0 & \text{if } k_j = 0, \\ \omega_{k_j,j}^a & \text{if } k_i = 0 \text{ for all } i \neq j, \\ \sum\limits_{t,v} (-1)^t s_{ij}^{tv} x_{k_i,i}^t \omega_j^{a+v}(K-i) & \text{otherwise,} \end{cases}$$

where $K - i$ is a shorthand for $K - 1 \cdot \alpha_i$.

Lemma 3.5. *The element $\omega_j^a(K)$ is well-defined.*

Proof. Take $i \neq i' \neq j \in I$ such that $k_i > 0$ and $k_{i'} > 0$. We can suppose by induction that $\omega_j^b(K - i - i')$ is well-defined for all $b \geq 0$. Then we have

$$\begin{aligned} &\sum_{t,v} (-1)^t s_{ij}^{tv} x_{k_i,i}^t \sum_{t',v'} (-1)^{t'} s_{i'j}^{t'v'} x_{k_{i'},i'}^{t'} \omega_j^{a+v}(K - i - i') \\ &= \sum_{t',v'} (-1)^{t'} s_{i'j}^{t'v'} x_{k_{i'},i'}^{t'} \sum_{t,v} (-1)^t s_{ij}^{tv} x_{k_i,i} \omega_j^{a+v}(K - i' - i) \end{aligned}$$

for all $i \neq i' \neq j \in I$. □

It will be useful to give $\omega_j^a(K)$ a non-inductive expression. We write

$$K^{\setminus j} := \sum_{i \neq j} k_i \cdot \alpha_i.$$

For a given non-negative integer $n_i \in \mathbb{N}$ we define

$$\varepsilon_{n_i,i}^j(\underline{x}_{k_i,i}) := \sum_{|V_i| = n_i} \left(\prod_{\ell=1}^{k_i} s_{ji}^{v_\ell t_\ell} x_{\ell,i}^{t_\ell} \right) \in P_i, \tag{3.8}$$

with the sum being over all partitions $V_i : v_1 + \cdots + v_{k_i} = v_i$ such that $(\alpha_i|\alpha_i)|v_\ell(\alpha_j|\alpha_j)$ for each $\ell \in \{1, \dots, k_i\}$, and with

$$t_\ell := \frac{-2(\alpha_i|\alpha_j) - v_\ell(\alpha_j|\alpha_j)}{(\alpha_i|\alpha_i)}.$$

This is a symmetric polynomial of q-degree $-2k_i(\alpha_i|\alpha_j) - v_i(\alpha_j|\alpha_j)$ whenever it is non-zero. Clearly, we can suppose $v_\ell \leq d_{ji}$, and therefore we can also suppose that $n_i \leq d_{ji}k_i$. For $n \in \mathbb{N}$ we define

$$\varepsilon_v^j(\underline{x}_K) := \sum_{|V| = n} \left(\prod_{i \neq j} \varepsilon_{n_i,i}^j(\underline{x}_{k_i,i}) \right) \in P_I, \tag{3.9}$$

with the sum being over all partitions $V : \sum_{i \neq j} n_i = n$. Notice that $\varepsilon_v^j(\underline{x}_K)$ is a polynomial of q-degree $(-\alpha_j^\vee(K^{\setminus j}) - n)(\alpha_j|\alpha_j)$.

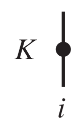
Lemma 3.6. *We have*

$$\omega_j^a(K) = \sum_{n=0}^{-\alpha_j^\vee(K^{\setminus j})} (-1)^n \omega_{k_j,j}^{a+n} \varepsilon_n^j(\underline{x}_K) \in P_I. \tag{3.10}$$

Proof. A straightforward computation shows that the right-hand side of equation (3.10) respects the recursive definition of $\omega_j^a(K)$, which proves the equality. □

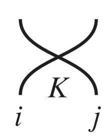
We now have all the tools we need to define an action of $R_{\mathfrak{p}}(\nu)$ on P_ν. First, we choose an arbitrary orientation $i \leftarrow j$ or $i \rightarrow j$ for each pair of distinct $i, j \in I$. Then we let $a \in R_{\mathfrak{p}}(\nu)1_{\boldsymbol{j}}$ act as zero on $P_I 1_{\boldsymbol{i}}$ whenever $\boldsymbol{j} \neq \boldsymbol{i}$. Otherwise, we declare that

- the dot



  acts as multiplication by $x_{k_i+1,i} 1_{\boldsymbol{i}}$,
- the floating dot

  acts as multiplication by $\omega_j^a(K)1_{\boldsymbol{i}}$,
- the crossing



  acts as

$$\begin{aligned} f1_{\boldsymbol{i}} &\mapsto r_i \frac{f1_{\boldsymbol{i}} - \sigma_K(f1_{\boldsymbol{i}})}{x_{k_i,i} - x_{k_i+1,i}} && \text{if } i = j, \\ f1_{\boldsymbol{i}} &\mapsto \Big(\sum_{t,v} s_{ij}^{tv} x_{k_i,i}^t x_{k_j+1,j}^v\Big)\sigma_K(f1_{\boldsymbol{i}}) && \text{if } i \rightarrow j, \\ f1_{\boldsymbol{i}} &\mapsto \sigma_K(f1_{\boldsymbol{i}}) && \text{if } i \leftarrow j. \end{aligned}$$

Proposition 3.7. *The rules above define an action of $R_{\mathfrak{b}}(\nu)$ on Q_ν.*

Proof. We have to check the validity of the KLR relations (3.1)–(3.4) and of the relations involving floating dots (3.5)–(3.7), as well as the relations coming from regular isotopies.

We start by proving the KLR relations. Clearly equations (3.1), (3.2) and (3.3) are satisfied. The case $i \neq k$ of equation (3.4) is also straightforward. For $i \leftarrow j$ and $k = i$ we compute the action of the left-hand side of equation (3.4) on $f \in Q_\nu$ as

$$\begin{aligned} f \mapsto & \Big(\sum_{t,v} s_{ji}^{tv} y^t x_1^v\Big)\sigma_1\partial_2\sigma_1(f) - \sigma_2\partial_1\Big(\sum_{t,v} s_{ji}^{tv} y^t x_2^v \sigma_2(f)\Big) \\ &= \Big(\sum_{t,v} s_{ji}^{tv} y^t x_1^v\Big)\frac{f - \sigma_1\sigma_2\sigma_1(f)}{x_1 - x_2} - \frac{\big(\sum_{t,v} s_{ji}^{tv} y^t x_2^v\big) f - \big(\sum_{t,v} s_{ji}^{tv} y^t x_1^v\big)\sigma_2\sigma_1\sigma_2(f)}{x_1 - x_2} \\ &= \sum_{t,v} s_{ji}^{tv} y^t \frac{x_1^v f - x_2^v}{x_1 - x_2} = \sum_{t,v} s_{ji}^{tv} y^t \sum_{r+s=v-1} x_1^r x_2^s, \end{aligned}$$

where x_1, x_2 correspond with the $x_{k_i,i}, x_{k_i+1,i}$ and y with $x_{k_j,j}$. What remains coincides with the right-hand side of equation (3.4). A similar computation applies for the case $i \to j$.

For the relations involving floating dots, we remark that (3.5) follows from the supercommutativity of Q_ν, and $\omega_j^a(K)$ respects (3.6) by construction. For relation (3.7), we apply the action of the left-hand side on some $f \in Q_\nu$ and we obtain

$$f \mapsto \Big(\sum_{t,v} s_{ji}^{tv} y^t x^v\Big)\omega_j^a(K + j) f,$$

and for the right-hand side we obtain

$$\begin{aligned} f \mapsto & \big(\omega_j^a(K + i + j) + \sum_{t,v} s_{ij}^{tv} \sum_{u+\ell=v-1} (-1)^u \omega_j^{a+u}(K) x^t y^\ell\big) f \\ &= \Big(\sum_{t,v} (-1)^v s_{ij}^{tv} x^t \omega_j^{a+v}(K + j) + \sum_{t,v} s_{ij}^{tv} \sum_{u+\ell=v-1} (-1)^u \omega_j^{a+u}(K) x^t y^\ell\Big) f. \end{aligned}$$

Then we compute

$$\omega_j^{a+v}(K + j) = \Big(\sum_{u+\ell=v-1} (-1)^{v-1-u} y^\ell \omega_j^{a+u}(K)\Big) + (-1)^v y^v \omega_j^a(K + j),$$

which implies that the action of the right-hand side of equation (3.7) coincides with the one of the left-hand side.

The only non-trivial relation coming from regular isotopies we need to verify is given by the commutation of a floating dot and a crossing at its left. This is a consequence of the fact that equation (3.8) is a symmetric polynomial, which commutes with divided difference operators. □

3.2.2. Left-adjusted expressions. We recall from [35, Section 2.2.1] that a reduced expression $\sigma_{i_r} \cdots \sigma_{i_1}$ of $w \in S_n$ is *left-adjusted* if $i_r + \cdots + i_1$ is minimal. Equivalently, it is left-adjusted if and only if

$$\min_{t \in \{0,\dots,r\}} \sigma_{i_t} \cdots \sigma_{i_1}(k) \leq \min_{t \in \{0,\dots,r\}} \sigma_{j_t} \cdots \sigma_{j_1}(k)$$

for all $k \in \{0, \dots, n\}$ and all other reduced expression $\sigma_{j_r} \cdots \sigma_{j_1} = w$. In this condition, we write

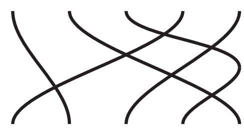
$$\min_w(k) := \min_{t \in \{0,\dots,r\}} \sigma_{i_t} \cdots \sigma_{i_1}(k).$$

Note that a left adjusted expression always exists and is unique up to distant permutation ($\sigma_i\sigma_j \leftrightarrow \sigma_j\sigma_i$ for $|i-j|>1$), so that $\min_w(k)$ is well-defined. In particular, one can obtain a left-adjusted reduced expression for any permutation by taking its representative in the coset decomposition

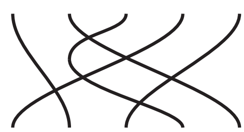
$$S_n = \bigsqcup_{a=1}^{n} S_{n-1}\sigma_{n-1}\cdots\sigma_a, \tag{3.11}$$

applied recursively. If we think of a reduced expression in terms of string diagrams, then it is left-adjusted if all strings are pulled as far as possible to the left.

Example 3.8. The permutation $(1\ 3\ 2\ 4) \in S_4$ admits as left-adjusted reduced expression the word $\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2$ which comes from the summand $S_2\sigma_3\sigma_2$ in the first step of the recursive decomposition (3.11). Note that $\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2$ is also left-adjusted while $\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2$ and $\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3$ are not. In terms of string diagrams, we consider as example the following reduced expression of the permutation $w = (1\ 4\ 3\ 5\ 2) \in S_5$:



It is not left-adjusted since the 4th strand (read at the bottom) can be pulled to the left. Hence we obtain the following left-adjusted minimal presentation:



Suppose $\sigma_{i_r}\cdots\sigma_{i_1}$ is a left-adjusted reduced expression of w. Then we can choose for each $k \in \{1,\dots,n\}$ an index $t_k \in \{1,\dots,r\}$ such that

$$\sigma_{i_{t_k}}\cdots\sigma_{i_1}(k) = \min_w(k).$$

Clearly this choice is not necessarily unique and we can have $t_k = t_{k'}$ for $k \neq k'$. However, it defines a partial order $\prec$ on the set $\{1,\dots,n\}$ where $k \prec k'$ whenever $t_k \leq t_{k'}$. We extend this order arbitrarily and we write $<_t$ for it. There is a bijective map $\zeta : \{1,\dots,n\} \to \{1,\dots,n\}$ which sends $k < k'$ to $\zeta(k) <_t \zeta(k')$, so that $t_{\zeta(k)} \leq t_{\zeta(k')}$. In terms of string diagrams, the map ζ tells us in which order the strands attain their (chosen) leftmost position while reading from bottom to top. In particular, $\zeta(k)$ gives the starting point of the strand that attains its leftmost position in kth position.

Example 3.9. Consider again the following left-adjusted string diagram:

Both the first and third strand attain their leftmost position at the bottom of the diagram, thus we can choose $\zeta(1) = 1$ and $\zeta(2) = 3$. Then both the second and fourth strand attain their leftmost position, thus we can take $\zeta(3) = 4$ and $\zeta(4) = 2$. Finally, the fifth strand attains its leftmost position and we put $\zeta(5) = 5$.

For $k \in \{1, \dots, n+1\}$, we put

$$w^k := \sigma_{i_{t_{\zeta(k)}}} \cdots \sigma_{i_{t_{\zeta(k-1)}}},$$

where it is understood that $t_{\zeta(0)} := 0$ and $t_{\zeta(n+1)} := r$. It defines a partition of the reduced expression of $\sigma_{i_r} \cdots \sigma_{i_1} = w$. Moreover, it is constructed so that

$$w^k \cdots w^1(\zeta(k)) = \min_w(\zeta(k))$$

for all $1 \le k \le n$.

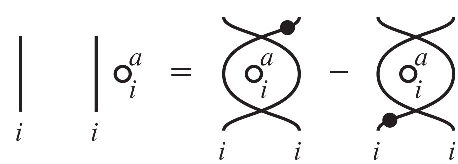
Example 3.10. Consider again $w = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2$ with $i_1 = 2, i_2 = 3, i_3 = 1, i_4 = 2, i_5 = 1$. We can choose for example $t_1 = 0, t_2 = 0, t_3 = 3, t_4 = 5$. Then we can put $\zeta(1) = 1$ (or 2), $\zeta(2) = 2$ (or 1), $\zeta(3) = 3$ and $\zeta(4) = 4$, with $w^1 = 1, w^2 = 1$, $w^3 = \sigma_1\sigma_3\sigma_2$ and $w^4 = \sigma_1\sigma_2$.

3.2.3. A generating set. We say that a floating dot is *tight* if it is placed immediately to the right of the left-most strand, and has superscript 0. We can also suppose it has the same subscript as the label of the strand at its left (otherwise it would slide to the left and be zero).

Lemma 3.11. *The algebra $R_{\mathfrak{b}}(\nu)$ is generated by KLR elements (i.e. dots and crossings) and tight floating dots.*

Proof. We first compute

(3.12)



for all $a \ge 0, i \in I$. Equation (3.12), together with equations (3.7) and (3.6) allows to bring all floating dots to the left. Then applying equation (3.6) recursively allows to transform all floating dots with superscript bigger than zero into dots and tight floating dots. □

We write ω for a tight floating dot, τ_a for a crossing between the ath and $(a+1)$st strands (counting from left), and x_a for a dot on the ath strand, where we suppose the label of the strands given by the context, in the form of an idempotent $1_{\boldsymbol{i}}$. We also define the *tightened floating dot* in $R_{\mathfrak{b}}(\nu)$ as $\theta_a := \tau_{a-1} \cdots \tau_1 \omega \tau_1 \cdots \tau_{a-1}$, or diagrammatically

$\theta_a :=$

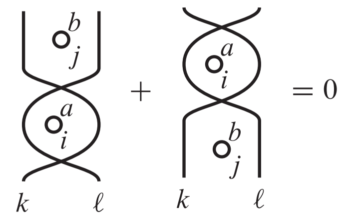
We also write $\theta_a^0 := \theta_a$ and $\theta_a^{-1} := 1$.

Lemma 3.12. *Tightened floating dots anticommute with each others, up to adding terms with a smaller number of crossings, that is*

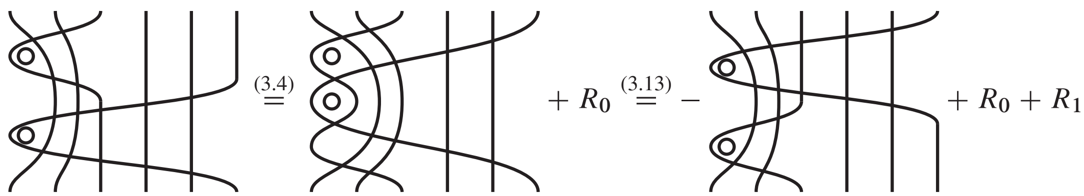
$$\theta_a\theta_b = -\theta_b\theta_a + R, \quad (\theta_a)^2 = 0 + R',$$

where R (resp. R') possesses strictly less crossings than $\theta_a\theta_b$ (resp. $(\theta_a)^2$), for all $1 \le a, b \le m$.

Proof. We first compute that

(3.13) 

for all $i, j, k, \ell \in I$ and $a, b \in \mathbb{N}$. Then we obtain



where both R_0 and R_1 have less crossings. □

Fix $\boldsymbol{i}, \boldsymbol{j} \in \mathrm{Seq}(\nu)$. Since they are both sequences of the same elements, there is a subset ${}_{\boldsymbol{j}}S_{\boldsymbol{i}} \subset S_m$ of permutations $w \in S_m$ such that $i_k = j_{w(k)}$ for all $k \in \{1, \dots, m\}$. Given such a permutation $w \in {}_{\boldsymbol{j}}S_{\boldsymbol{i}}$, we can choose a left-adjusted reduced expression. It comes with a partition $w^{m+1} \cdots w^2 w^1 = w$ and a bijection $\zeta : \{1, \dots, m\} \to \{1, \dots, m\}$ such that

$$w^k \cdots w^1(\zeta(k)) = \min_w(\zeta(k))$$

for all $1 \leq k \leq m$. Then consider the collection of elements

$$(3.14) \quad {}_{\boldsymbol{j}}B_{\boldsymbol{i}} := \{x_m^{a_m} \cdots x_1^{a_1} \tau_{w^{m+1}} \theta^{\ell_m}_{\min_w(\zeta(m))} \tau_{w^m} \cdots \theta^{\ell_2}_{\min_w(\zeta(2))} \tau_{w^2} \theta^{\ell_1}_{\min_w(\zeta(1))} \tau_{w^1} \mid a_i \in \mathbb{N},\ \ell_i \in \{0, -1\},\ w \in {}_{\boldsymbol{j}}S_{\boldsymbol{i}}\}$$

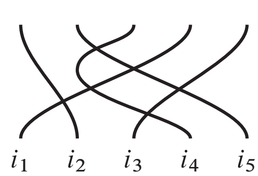
in $1_{\boldsymbol{j}} R_{\mathfrak{b}}(\nu) 1_{\boldsymbol{i}}$. Diagrammatically, elements in ${}_{\boldsymbol{j}}B_{\boldsymbol{i}}$ can be constructed using the following algorithm:

(i) Choose a permutation $w \in {}_{\boldsymbol{j}}S_{\boldsymbol{i}}$, consider its corresponding string diagram and make it left-adjusted by bringing all strands to the left.

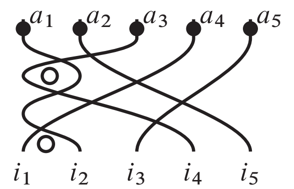
(ii) For each strand, choose whether we want to add a floating dot. If so, add a tightened floating dot where the strand attains its left-most position by pulling the strand to the far left and adding the floating dot immediately at its right.

(iii) For each strand, choose a number of dots to add at the top of the diagram.

Example 3.13. Take $\boldsymbol{i} = i_1 i_2 i_3 i_4 i_5$ and $\boldsymbol{j} = i_2 i_5 i_4 i_1 i_3$, and consider the following left-adjusted permutation $w \in {}_{\boldsymbol{j}}S_{\boldsymbol{i}}$:

$w =$  .

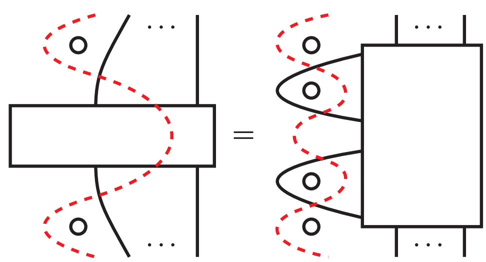
Take $\ell_1 = 1, \ell_2 = 0, \ell_3 = 1, \ell_4 = 0$ and $\ell_5 = 0$ (for the same ζ as in Example 3.9). Then we obtain the following element in ${}_{\boldsymbol{j}}B_{\boldsymbol{i}}$:



Proposition 3.14. *Elements in* ${}_{\boldsymbol{j}}B_{\boldsymbol{i}}$ *generate* $1_{\boldsymbol{j}} R_{\mathfrak{b}}(\nu) 1_{\boldsymbol{i}}$ *as a* $\Bbbk$*-vector space.*

Proof. The proof is an induction on the number of crossings. By Lemma 3.11, we can assume that all floating dots are tight. By equations (3.2) and (3.3) we can bring all the dots to the top of any diagram, at the cost of adding diagrams with fewer crossings. Moreover, all braid isotopies hold up to adding terms with a lower amount of crossings thanks to equations (3.1) and (3.4).

We claim that we can also assume that there is at most one floating dot at the immediate right of each strand. Indeed, suppose there are two tight floating dots at the right of the same strand. Then we can apply a braid-isotopy to bring it as most as possible to the left, until it is possibly blocked by other tight floating dots. We depict it by the following picture:



where the dashed strand in red represents the one we want to pull, and the boxes represent other elements in $R_{\mathfrak{b}}(\nu)$. If there is no floating dot in-between, then it is zero by equation (3.5). Otherwise, we apply equation (3.13) to jump the bottom floating dot over all the floating dots in-between, until we have two floating dots in the same region at the top, which is zero by equation (3.5). This proves the claim.

Finally, we observe that given a strand with a single tight floating dot, we can tighten it by braid isotopy, until we end up with a tightened floating dot. Since by Lemma 3.12 tightened floating dots anticommute, this concludes the proof. □

3.2.4. The basis theorem.

Proposition 3.15. *The action in Proposition* 3.7 *is faithful.*

Proof. The proof is inspired by [40, Proposition 3.8] (see also [23, Theorem 2.5] for a different approach). We claim that elements of ${}_{\boldsymbol{j}}B_{\boldsymbol{i}}$ act as linearly independent endomorphisms on P_ν. The action yields morphisms

$$P_I 1_{\boldsymbol{i}} \to P_I 1_{\boldsymbol{j}},$$

that we will consider as endomorphisms of P_I.

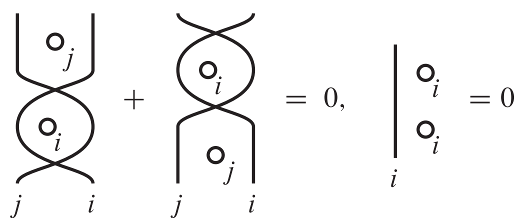
First we extend the scalars to $\Bbbk(x_{1,i}, \dots, x_{\nu_i,i})$ in P_i for all $i \in I$. We claim that different choices of $w \in {}_{\boldsymbol{j}}S_{\boldsymbol{i}}$ and $\ell_i \in \{-1, 0\}$ give linearly independent operators. Notice that since $\boldsymbol{i}, \boldsymbol{j}$ is fixed, w is given by choices of permutations between strands of the same label. Since crossings between strands with different labels act as multiplication by a polynomial, we can ignore them as being multiplication by a scalar. By [35, Corollary 3.9], we know that different choices of permutations and tightened floating dots for strands with label i act as linearly independent operators on P_i, hence proving our claim. Finally, taking into account the multiplication by the polynomial given by the choice of the $a_i \in \mathbb{N}$ as in equation (3.14) concludes the proof. □

Theorem 3.16. *The $\Bbbk$-module $1_{\boldsymbol{j}} R_{\mathfrak{b}}(\nu) 1_{\boldsymbol{i}}$ is free with basis ${}_{\boldsymbol{j}}B_{\boldsymbol{i}}$.*

Proof. It follows from Propositions 3.14 and 3.15. □

From this, we also deduce the following:

Corollary 3.17. *The $\mathfrak{b}$-KLR algebra admits a presentation given by the KLR-generators and tight floating dots, subjected to the KLR-relations* (3.1)–(3.4) *together with*

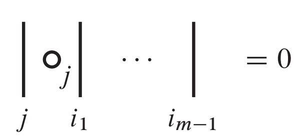


for all $i, j \in I_r$.

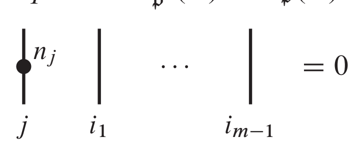
4. Dg-enhancement

We fix a subset $I_f \subset I$ and consider the associated parabolic subalgebra $U_q(\mathfrak{p}) \subset U_q(\mathfrak{g})$ as defined in Section 2.2. For each index $j \in I_f$, we also choose a weight $n_j \in \mathbb{N}$, and write $N := \{n_j\}_{j \in I_f}$.

Definition 4.1. The $\mathfrak{p}$-KLR algebra $R_{\mathfrak{p}}(m)$ is given by forgetting the λ_j-degree for each $j \in I_f$ in $R_{\mathfrak{b}}(m)$ and modding out by



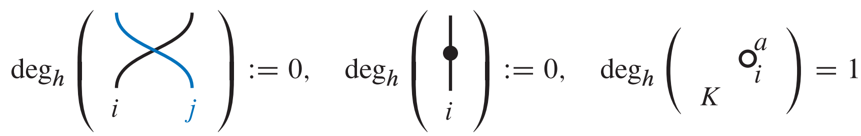
for all $j \in I_f$. The *N-cyclotomic quotient* $R_{\mathfrak{p}}^N(m)$ of $R_{\mathfrak{p}}(m)$ is given by modding out by



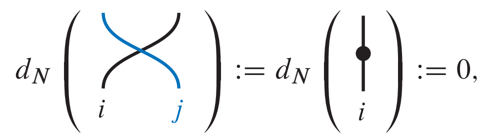
for all $j \in I_f$.

In particular, $R_{\mathfrak{g}}(m)$ is the usual KLR algebra $R(m)$ (see Definition 3.1). Its N-cyclotomic quotient $R^N_{\mathfrak{g}}(m)$ is also the usual cyclotomic quotient of the KLR algebra. Taking $I_f = \emptyset$ gives $\mathfrak{p} = \mathfrak{b}$ and we recover Definition 3.3.

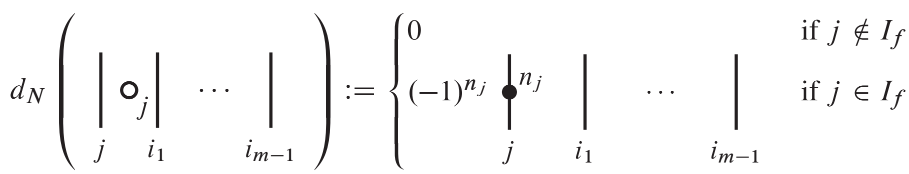
We equip $R_{\mathfrak{b}}(m)$ with a homological $\mathbb{Z}$-grading, denoted h, by setting



for all $i, j \in I$. Then we equip $R_{\mathfrak{b}}(m)$ with a differential d_N by setting



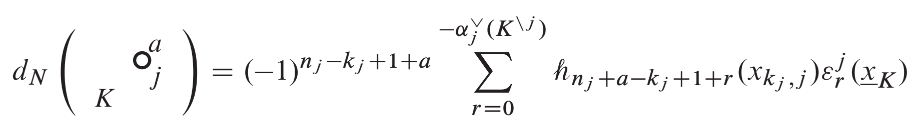
and



We extend the definition of d_N to the whole algebra using the graded Leibniz rule

$$d_N(xy) = d_N(x)y + (-1)^{\deg_h(x)} x d_N(y)$$

and Lemma 3.11. Checking that d_N is well-defined is straightforward using Corollary 3.17. From this, we derive that for $j \in I_r$ we have

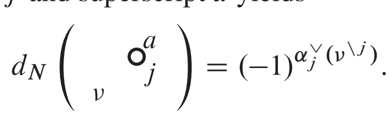


where $x_{\ell,i}$ is a dot on the ℓth strand with label i, $\hbar_n$ is the nth complete homogeneous polynomial, and $\varepsilon_r^j(\underline{x}_K)$ is defined in equation (3.9).

Definition 4.2. We will refer to the dg-algebra $(R_{\mathfrak{b}}(m), d_N)$ as the *dg-enhanced KLR algebra*.

Proposition 4.3. *If* $n_j - \nu_j - \alpha_j^\vee(\nu^{\setminus j}) < 0$, *then* $(R_{\mathfrak{b}}(\nu), d_N)$ *is acyclic.*

Proof. Taking $a := -(n_j - \nu_j - \alpha_j^\vee(\nu^{\setminus j}) + 1)$ and considering the floating dot placed on the far right with subscript j and superscript a yields



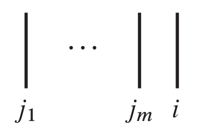
Thus, $H(R_{\mathfrak{b}}(\nu), d_N) \cong 0$. □

Our goal for the rest of the section will be to prove the following:

Theorem 4.4. *The dg-algebra* $(R_{\mathfrak{b}}(m), d_N)$ *is formal with homology*

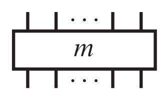
$$H(R_{\mathfrak{b}}(m), d_N) \cong R^N_{\mathfrak{p}}(m).$$

4.1. Proof of Theorem 4.4. Denote $1_{(m,i)} := \sum_{\boldsymbol{j} \in I^m} 1_{\boldsymbol{j}i}$, or diagrammatically

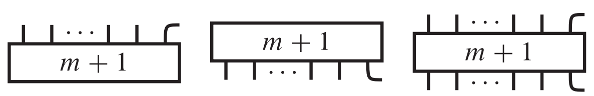
$$1_{(m,i)} := \sum_{(j_1,\dots,j_m)\in I^m}$$ 

It is an idempotent of $R_{\mathfrak{b}}(m+1)$. We also define $1_{(\nu,i)} := \sum_{\boldsymbol{j}\in \mathrm{Seq}(\nu)} 1_{\boldsymbol{j}i}$ for $\nu \in X^+$. The algebra $R_{\mathfrak{b}}(m)$ acts on $1_{(m,i)}R_{\mathfrak{b}}(m+1)$ by first adding a vertical strand labeled i at the right of $D \in R_{\mathfrak{b}}(m)$ and then multiplying on the left in $R_{\mathfrak{b}}(m+1)$.

We now introduce some other diagrammatic notations as in [35, Section 3.1]. We draw $R_{\mathfrak{b}}(m)$ (viewed as $R_{\mathfrak{b}}(m)$-$R_{\mathfrak{b}}(m)$-bimodule) as a box labeled by m

$$R_{\mathfrak{b}}(m) =$$ 

and $\otimes_m := \otimes_{R_{\mathfrak{b}}(m)}$ becomes stacking boxes on top of each other. Moreover, when $R_{\mathfrak{b}}(m+1)$ is viewed as a left $R_{\mathfrak{b}}(m)$-module, as a right $R_{\mathfrak{b}}(m)$-module or as an $R_{\mathfrak{b}}(m)$-$R_{\mathfrak{b}}(m)$-bimodule, we draw respectively

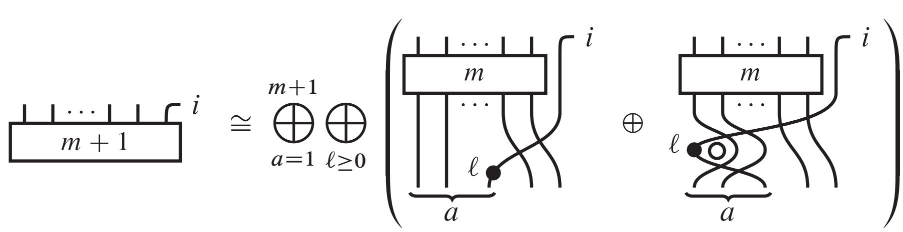


Lemma 4.5. *As a left $R_{\mathfrak{b}}(m)$-module, $1_{(m,i)}R_{\mathfrak{b}}(m+1)$ is free with decomposition*

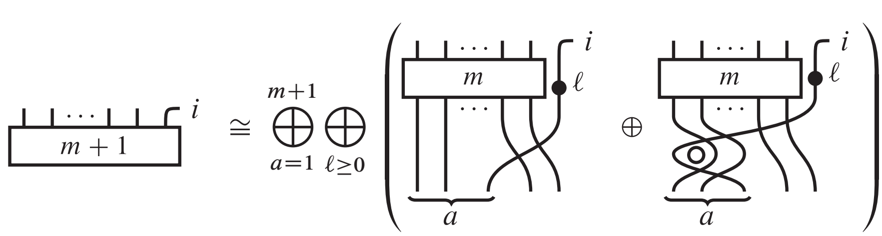
$$\bigoplus_{a=1}^{m+1}\bigoplus_{\ell\geq 0}\left(R_{\mathfrak{b}}(m)1_{(m,i)}\tau_m\cdots\tau_a x_a^{\ell} \oplus R_{\mathfrak{b}}(m)1_{(m,i)}\tau_m\cdots\tau_a\theta_a^{\ell}\right),$$

where $\theta_a^{\ell} := \tau_{a-1}\cdots\tau_1\omega x_1^{\ell}\tau_1\cdots\tau_{a-1}$.

We draw this as

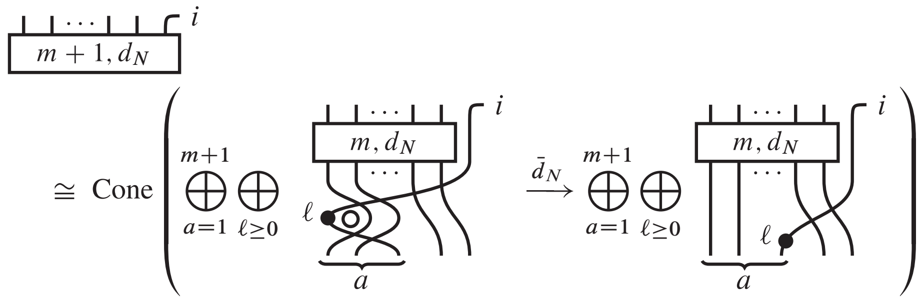


Proof. By Theorem 3.16 we obtain



We then apply equations (3.2) and (3.3) to bring all the dots to the desired position. It is a triangular change of basis, concluding the proof. □

From now on, we draw boxes with label "m, d_N" to denote the dg-algebra $(R_{\mathfrak{b}}(m), d_N)$. Similarly, a box with label $H(m)$ denotes its homology $H(R_{\mathfrak{b}}(m), d_N)$. Then the decomposition in Lemma 4.5 lifts directly to a direct sum decomposition of dg-modules whenever $i \notin I_f$. Otherwise, for $i \in I_f$, it lifts to the mapping cone

(4.1) 

where the map $\bar{d}_N$ is induced by the differential of $(R_{\mathfrak{b}}(m+1), d_N)$.

We will prove Theorem 4.4 using induction on the number of strands m. Therefore, we can assume $(R_{\mathfrak{b}}(m), d_N)$ to be formal.

Following [21], recall that for a dg-algebra (A, d_A), we say that a dg-module is a *relatively projective module* if it is a direct summand of a free module in (A, d_A)-mod. Moreover, an (A, d_A)-module Y *satisfies property* (P) if there is an exhaustive filtration of (A, d_A)-modules

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_r \subset F_{r+1} \subset \cdots \subset Y$$

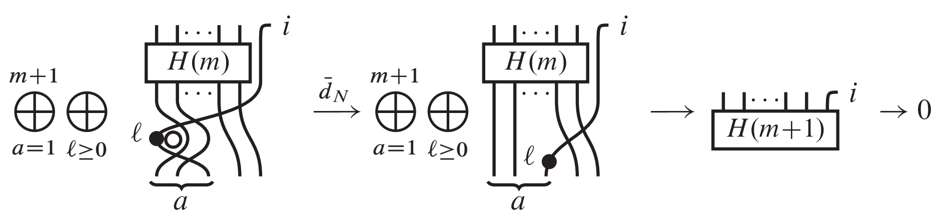
such that each F_{r+1}/F_r is isomorphic in (A, d_A)-mod to a relatively projective module. An (A, d_A)-direct summand of a property (P) module is called *cofibrant*. Also recall the following result of homological algebra:

Lemma 4.6. *Let (A, d_A) be a dg-algebra, let (M, d_M) be a right (A, d_A)-module, and let (N, d_N) be a left one. If (M, d_M) is formal and (N, d_N) is cofibrant, then we have*

$$H\big((M, d_M) \otimes_{(A,d_A)} (N, d_N)\big) \cong H\big(H(M, d_M) \otimes_{(A,d_A)} (N, d_N)\big).$$

Proof. Tensoring with a cofibrant dg-module preserves quasi-isomorphisms. □

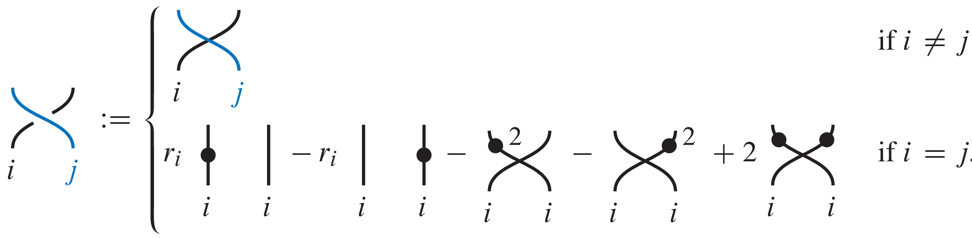
Therefore we obtain an exact sequence

(4.2) 

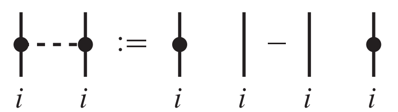
thanks to Proposition 4.6 and equation (4.1).

Proposition 4.7. *The exact sequence equation* (4.2) *is a short exact sequence, with $\bar{d}_N$ being injective.*

Theorem 4.4 above is a direct consequence of Proposition 4.7. Therefore, we now focus on proving this proposition. This is in fact similar to Kang–Kashiwara's [20, equation (4.13)], with basically only a change of basis, and thus we will follow the same ideas. We introduce the equivalent of "g_a" from the reference and draw it as an *undercrossing*:

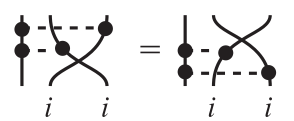


In order to shorten out our diagrams, we introduce the convenient notation



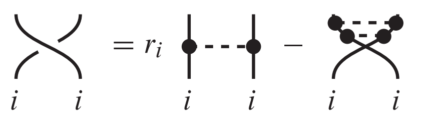
It respects the relation

(4.3)

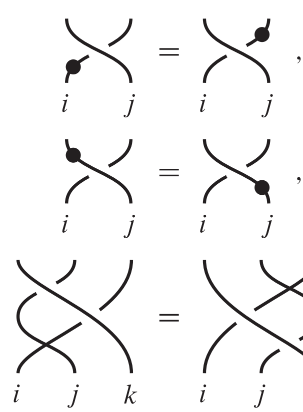


We also have that

(4.4)

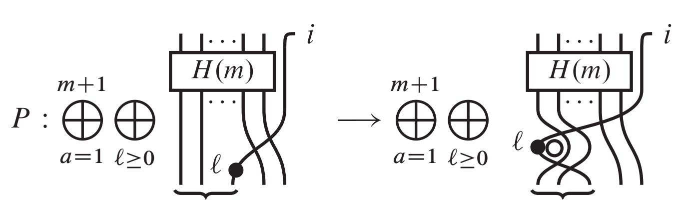


Lemma 4.8 ([20, Lemma 4.12]). *Undercrossings respect the following relations:*



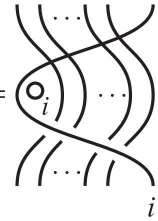
for all $i, j, k \in I$.

Still as in [20], in order to construct a "nearly inverse" for $\bar{d}_N$, we define the map



as multiplication on the left (or diagrammatically stacking above) with the element

$$\widetilde{\theta}_{m+1} :=$$



.

Lemma 4.9. *The map P defined above is a map of $H(R_{\mathfrak{b}}(m), d_N)$-modules.*

Proof. We need to verify that $\widetilde{\theta}_{m+1}$ commutes with the elements in $H(R_{\mathfrak{b}}(m), d_N)$. Crossings and dots slide over the upper part of the $(m+1)$st strand in $\widetilde{\theta}_{m+1}$ at the cost of adding diagrams with fewer crossings. Because there are fewer crossings, we can slide the floating dot coming from $\widetilde{\theta}_{m+1}$ to the part $H(R_{\mathfrak{b}}(m), d_N)$ of the diagram, which gives zero. The crossings and dots in the remaining terms then slide over the lower part thanks to Lemma 4.8. Tight floating dots with subscript $j \notin I_f$ also slide over $\widetilde{\theta}_{m+1}$ thanks to equation (3.7). □

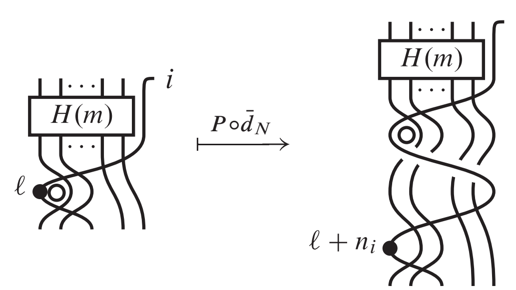
Lemma 4.10. *The composition $P \circ \bar{d}_N$ is given on $H(R_{\mathfrak{b}}(m), d_N) \otimes_m 1_{(\nu,i)} R_{\mathfrak{b}}(m+1)$ by multiplication by*

$$r_i^{2\nu_i} \sum_{p=0}^{2\nu_i - \alpha_i^\vee(\nu)} x_{m+1}^{n_i+p} \varepsilon_p^i(\underline{x}_\nu), \tag{4.5}$$

where $\varepsilon_p^i(\underline{x}_\nu)$ is as in equation (3.9).

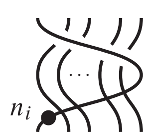
Proof. The proof is similar to [20, Theorem 4.15]. We have

(4.6)



.

We prove by induction on the number of strands m that



$$\equiv r_i^{2\nu_i} \sum_{p=0}^{2\nu_i - \alpha_i^\vee(\nu)} x_{m+1}^{n_i+p} \varepsilon_p^i(\underline{x}_\nu),$$

where $\equiv$ means equality up to adding elements killed in the quotient

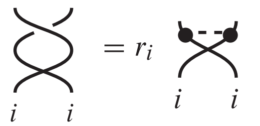
$$H(R_{\mathfrak{b}}(m), d_N) \cong R_{\mathfrak{p}}^N(m).$$

If $m = 0$, then it is trivial. Thus we suppose by induction that it holds for $m-1$. We fix the label of the strands on the diagram above as $i\,\boldsymbol{j}$ with $\boldsymbol{j} = j_1 \cdots j_m \in \mathrm{Seq}(\nu)$, and we consider the different possible cases.

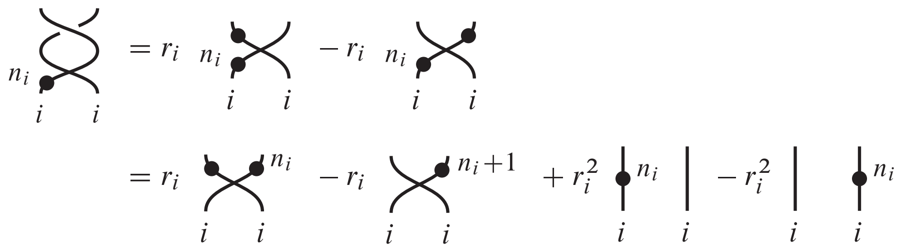
If $j_m \neq i$, then the result follows by applying equation (3.1) with Lemma 4.8, and using the induction hypothesis.

If $j_m = i$, we first observe that

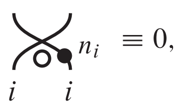
(4.7)



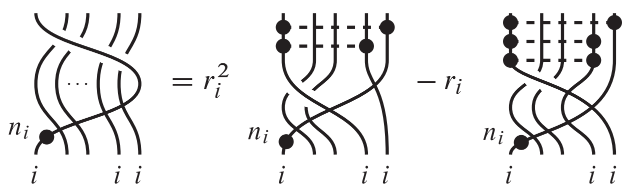
Then we need to consider j_{m-1}. If $m = 1$, we have that



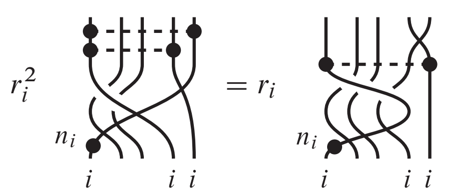
Moreover, we observe that



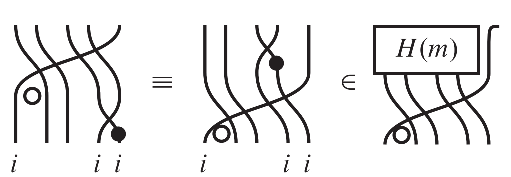
which finishes the case $m = 1$. For $j_{m-1} = i$, we have



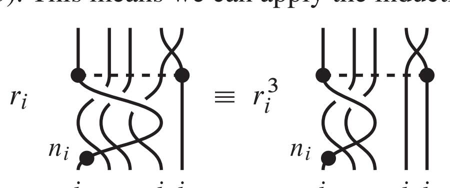
using equations (4.7), (4.4) and Lemma 4.8. Using equation (4.3) followed by Lemma 4.8 and equation (4.7) we obtain



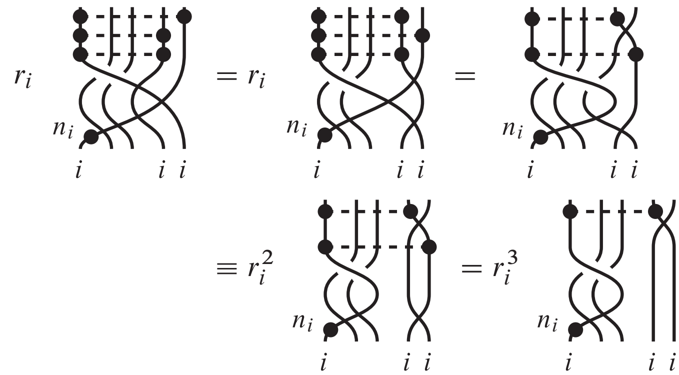
Keeping in mind equation (4.6), we have



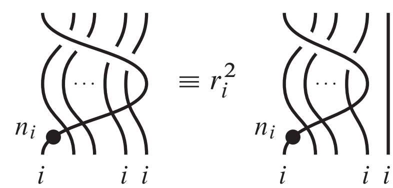
by equations (3.4) and (3.3). This means we can apply the induction hypothesis to get



Similarly, we have

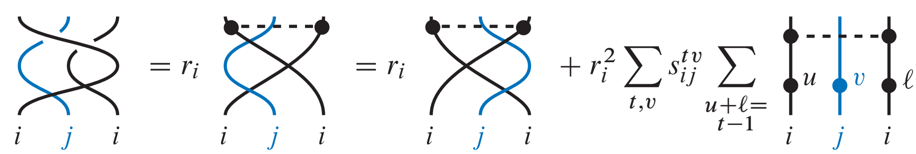


Putting these two results together and using equation (3.3), we obtain

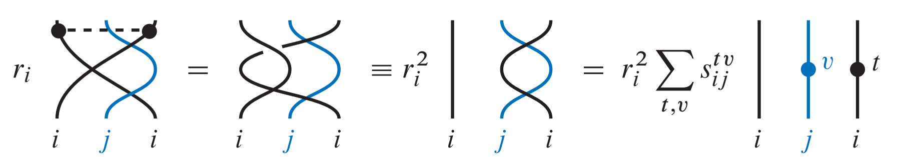


which concludes this case.

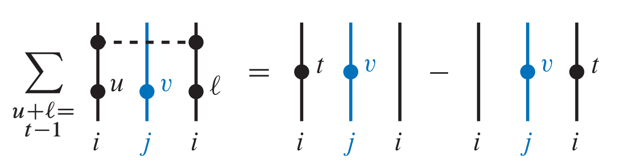
For the final case $j_{m-1} = j \neq i$, we compute



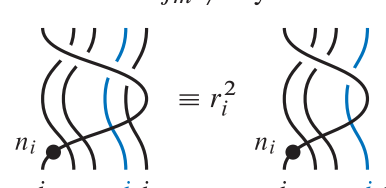
using equation (3.4). Then we obtain for the first term on the right-hand side of the second equality, using the induction hypothesis together with equation (3.1)



On the other hand, we have for all t, v that



Putting these results together with the case $j_m \neq i$ yields



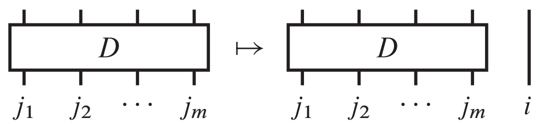
which concludes the proof. □

Proof of Proposition 4.7. The polynomial (4.5) is monic (up to invertible scalar) with leading terms $x_{m+1}^{n_i+2\nu_i-\alpha_i^\vee(\nu)}$. Therefore, multiplication by equation (4.5) yields an injective map. Thus, Lemma 4.10 tells us that $P \circ \bar{d}_N$ is injective, and so is $\bar{d}_N$. □

As a consequence, this also ends the proof of Theorem 4.4.

5. Categorical action

For each $i \in I$ there is a (non-unital) inclusion $R_{\mathfrak{b}}(m) \hookrightarrow R_{\mathfrak{b}}(m+1)1_{(m,i)}$, given by adding a vertical strand with label i to the right of a diagram $D \in R_{\mathfrak{b}}(m)$:



This gives rise to induction and restriction functors

$$\mathrm{Ind}_m^{m+i} : R_{\mathfrak{b}}(m)\text{-mod} \to R_{\mathfrak{b}}(m+1)\text{-mod}, \quad \mathrm{Ind}_m^{m+i}(-) \cong R_{\mathfrak{b}}(m+1)1_{(m,i)} \otimes_m -$$

and

$$\mathrm{Res}_m^{m+i} : R_{\mathfrak{b}}(m+1)\text{-mod} \to R_{\mathfrak{b}}(m)\text{-mod}, \quad \mathrm{Res}_m^{m+i}(-) \cong 1_{(m,i)} R_{\mathfrak{b}}(m+1) \otimes_{m+1} -.$$

which are adjoint.

We write

$$R_{\mathfrak{b}}^{\xi_i}(\nu) := R_{\mathfrak{b}}(\nu) \otimes \Bbbk[\xi_i] \cong \bigoplus_{\ell \geq 0} q_i^{2\ell} R_{\mathfrak{b}}(\nu),$$

with $\deg_q(\xi_i) = (\alpha_i|\alpha_i)$. We will prove the following theorem in the next subsection:

Theorem 5.1. *There is a short exact sequence*

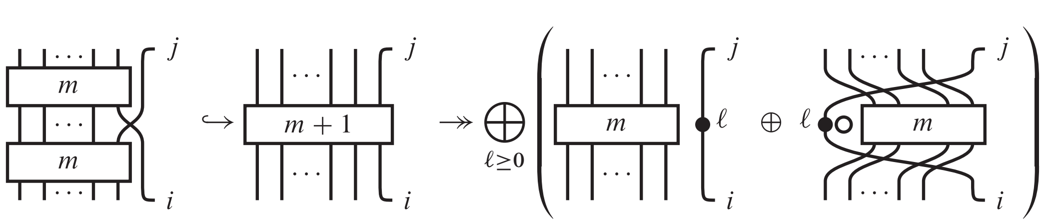
$$\begin{aligned} 0 \to q_i^{-2} R_{\mathfrak{b}}(\nu)1_{(\nu-i,i)} \otimes_{m-1} 1_{(\nu-i,i)} R_{\mathfrak{b}}(\nu) \to 1_{(\nu,i)} R_{\mathfrak{b}}(\nu+i)1_{(\nu,i)} \\ \to R_{\mathfrak{b}}^{\xi_i}(\nu) \oplus \lambda_i^2 q_i^{-2\alpha_i^\vee(\nu)} R_{\mathfrak{b}}^{\xi_i}(\nu)[1] \to 0 \end{aligned}$$

of $R_{\mathfrak{b}}(\nu)$-$R_{\mathfrak{b}}(\nu)$-bimodules for all $i \in I$. Moreover, there is an isomorphism

$$q^{-(\alpha_i|\alpha_j)} R_{\mathfrak{b}}(\nu)1_{(\nu-i,i)} \otimes_{m-1} 1_{(\nu'-j,j)} R_{\mathfrak{b}}(\nu') \cong 1_{(\nu',j)} R_{\mathfrak{b}}(\nu+i)1_{(\nu,i)}$$

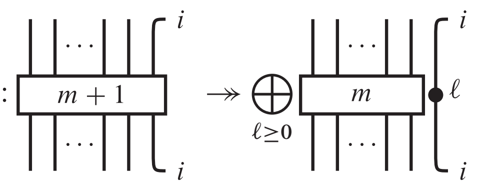
for all $i \neq j \in I$ and $\nu' + j = \nu + i$.

As we will see in the proof of Theorem 5.1, we can picture these facts as a short exact sequence of diagrams



,

where the cokernel vanishes whenever $i \neq j$. We write π for the projection

$\pi :$ 

We write $\mathrm{Id}_\nu := R_{\mathfrak{b}}(\nu) \otimes_m (-)$ and we define

$$\mathsf{F}_i := \bigoplus_{m \geq 0} \mathrm{Ind}_m^{m+i}, \quad \mathsf{E}_i := \bigoplus_{m \geq 0} \bigoplus_{|\nu| = m} \lambda_i^{-1} q_i^{1+\alpha_i^\vee(\nu)} \mathrm{Res}_m^{m+i} \mathrm{Id}_{\nu+i}.$$

These are exact functors thanks to Lemma 4.5. Define

$$\bigoplus_{[\beta_i - \alpha_i^\vee(\nu)]_{q_i}} \mathrm{Id}_\nu := \bigoplus_{\ell \geq 0} q_i^{1+2\ell} \big(\lambda_i^{-1} q_i^{\alpha_i^\vee(\nu)} \mathrm{Id}_\nu \oplus \lambda_i q_i^{-\alpha_i^\vee(\nu)} \mathrm{Id}_\nu[1] \big). \tag{5.1}$$

It is a categorification of the fraction $\frac{\lambda_i q_i^{-\alpha_i^\vee(\nu)} - \lambda_i^{-1} q_i^{\alpha_i^\vee(\nu)}}{q_i - q_i^{-1}}$. We obtain:

Corollary 5.2. *There is a natural short exact sequence*

$$0 \to \mathsf{F}_i \mathsf{E}_i \,\mathrm{Id}_\nu \to \mathsf{E}_i \mathsf{F}_i \,\mathrm{Id}_\nu \to \bigoplus_{[\beta_i - \alpha_i^\vee(\nu)]_{q_i}} \mathrm{Id}_\nu \to 0$$

for all $i \in I$, and there is a natural isomorphism

$$\mathsf{F}_i \mathsf{E}_j \cong \mathsf{E}_j \mathsf{F}_i$$

for all $i \neq j \in I$.

Proposition 5.3. *For each $i, j \in I$ there is a natural isomorphism*

$$\bigoplus_{a=0}^{\lfloor \frac{d_{ij}+1}{2} \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a \end{bmatrix}_{q_i} \mathsf{F}_i^{2a} \mathsf{F}_j \mathsf{F}_i^{d_{ij}+1-2a} \cong \bigoplus_{a=0}^{\lfloor \frac{d_{ij}}{2} \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a+1 \end{bmatrix}_{q_i} \mathsf{F}_i^{2a+1} \mathsf{F}_j \mathsf{F}_i^{d_{ij}-2a},$$

and in particular for $(\alpha_i|\alpha_j) = 0$ we have $\mathsf{F}_i \mathsf{F}_j 1_\nu \cong \mathsf{F}_j \mathsf{F}_i 1_\nu$. By adjunction, the same isomorphism exists for the $\mathsf{E}_i, \mathsf{E}_j$.

Proof. Similarly as in the case of the usual KLR algebras, it follows from equations (3.3) and (3.4) (the proof of [25, Proposition 6] can be applied directly). □

5.1. Proof of Theorem 5.1. By symmetry along the horizontal axis, we obtain a decomposition of $R_{\mathfrak{b}}(m+1)$ as a right $R_{\mathfrak{b}}(m)$-module similar to the one of Lemma 4.5. Note that the left and right decompositions are not compatible, and therefore we do not have a decomposition as a $R_{\mathfrak{b}}(m)$-$R_{\mathfrak{b}}(m)$-bimodule. However, the surjection $R_{\mathfrak{b}}(m+1) \twoheadrightarrow q^{2\ell} R_{\mathfrak{b}}(m)$ that projects on the summand $R_{\mathfrak{b}}(m) x_{m+1}^{\ell}$, given by taking $a = m+1$ in Lemma 4.5, is a (left-invertible) map of bimodules.

We define the map

$$\pi_L^\ell : 1_{(\nu,i)} R_{\mathfrak{b}}(m+1) 1_{(\nu,i)} \twoheadrightarrow \lambda_i^2 q_i^{2\ell - 2\alpha_i^\vee(\nu)} R_{\mathfrak{b}}(\nu)[1],$$

as the projection map on the summand $R_{\mathfrak{b}}(m)\theta^{\ell}_{n+1}$ in the left decomposition of $R_{\mathfrak{b}}(m+1)$ as $R_{\mathfrak{b}}(m)$-module in Lemma 4.5. Similarly, let

$$\pi^{\ell}_{R} : 1_{(\nu,i)}R_{\mathfrak{b}}(m+1)1_{(\nu,i)} \twoheadrightarrow \lambda_i^2 q_i^{2\ell-2\alpha_i^{\vee}(\nu)} R_{\mathfrak{b}}(\nu)[1]$$

be the projection map on $\theta^{\ell}_{n+1}R_{\mathfrak{b}}(m)$ in the right decomposition.

Lemma 5.4. *We have*

$$\pi^{\ell}_{L}(y) = (-1)^{\deg_h(y)}\pi^{\ell}_{R}(y)$$

for all $y \in R_{\mathfrak{b}}(m+1)$.

Proof. We can suppose $y = \theta^{\ell}_{m+1}y'$ with $y' \in R_{\mathfrak{b}}(m)$. We want to prove that

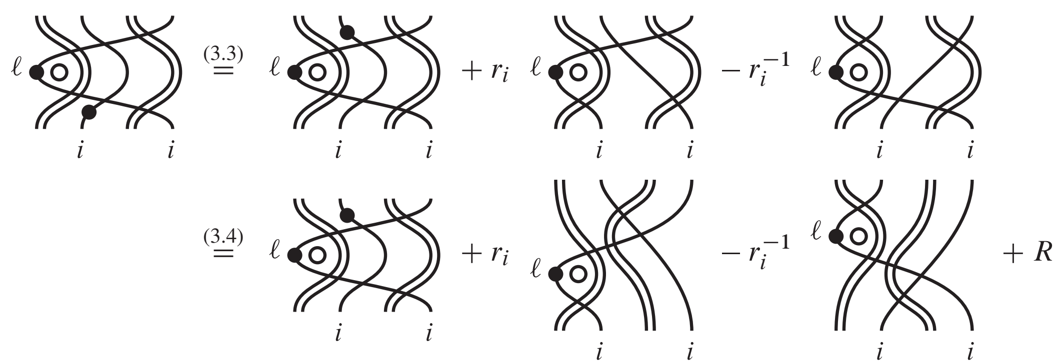
$$y = (-1)^{\deg_h(y)}y'\theta^{\ell}_{m+1} + y_0$$

for some $y_0 \notin R_{\mathfrak{b}}(m)\theta^{\ell}_{m+1}$. For this, it is enough to show that

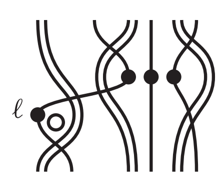
$$y_1\theta_{m+1}zy_2 = (-1)^{\deg_h(z)}y_1z\theta_{n+1}y_2 + z_0,$$

where $y_1, y_2 \in R_{\mathfrak{b}}(m)$, $z_0 \notin R_{\mathfrak{b}}(m)\theta^{\ell}_{m+1}$ and z is any generator of $R_{\mathfrak{b}}(m)$ (i.e. crossing, dot or tight floating dot).

If $z = x_a$ and is on a strand labeled $j \neq i$, then it slides freely over θ_{m+1} thanks to equation (3.2). If the strand is labeled i, then we compute



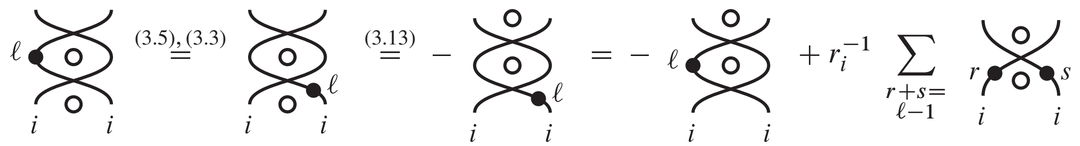
where the double strands represent multiple parallel strands (the number depending on m and a), and R is a sum of terms of the form



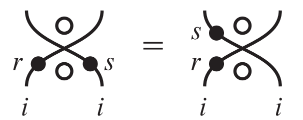
and its mirror along the horizontal axis. Note that it is implicitly assumed that each of these diagrams have the element y_1 at the top and y_2 at the bottom. Using Lemma 4.5, we can rewrite the composition of the last three terms in the equation above with y_2 as elements in $\bigoplus_{a=1}^{n}\bigoplus_{p\geq 0} R_{\mathfrak{b}}(m-1)\tau_m\tau_{m-1}\cdots\tau_a x_a^p \not\subset R_{\mathfrak{b}}(m)\theta^{\ell}_{m+1}$. Hence they form the term z_0.

If $z = \tau_i$ is a crossing, then we obtain the desired property by equation (3.4), and applying a similar reasoning as above.

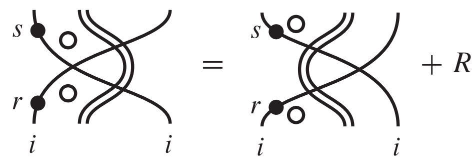
Finally, if $z = \omega$ and is at right of a strand labeled $j \neq i$, it follows directly from equation (3.13). Otherwise, if the strand is labeled i, we compute



Then for all $r, s \geq 0$ we compute using equation (3.3) again



Looking at these elements in the global picture yields



which is an element not contained in $R_{\mathfrak{b}}(m)\theta_{n+1}^{\ell}$ for the same reasons as before. We see that together they form the element z_0, concluding the proof. □

We now have all the ingredients we need to prove Theorem 5.1.

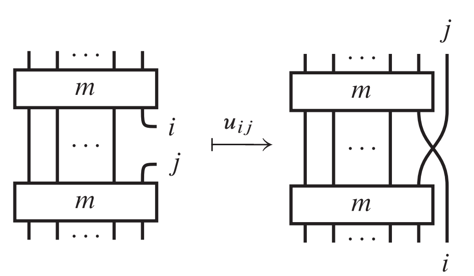
Proof of Theorem 5.1. We first construct an injective map

$$u_{ij} : q^{-(\alpha_i|\alpha_j)} R_{\mathfrak{b}}(\nu)1_{(m-1,i)} \otimes_{m-1} 1_{(m_1,j)} R_{\mathfrak{b}}(\nu) \hookrightarrow 1_{(\nu,j)} R_{\mathfrak{b}}(m+1)1_{(\nu,i)} \tag{5.2}$$

of $R_{\mathfrak{b}}(m)$-$R_{\mathfrak{b}}(m)$-bimodules, by setting (as in [20, Proposition 3.3])

$$u_{ij}(x \otimes_{m-1} y) := x\tau_m y.$$

In terms of diagrams, it consists of adding a crossing at the right



Then we construct a surjective map

$$1_{(\nu,i)} R_{\mathfrak{b}}(m+1)1_{(\nu,i)} \twoheadrightarrow R_{\mathfrak{b}}^{\xi_i}(\nu) \oplus \lambda_i^2 q_i^{-2\alpha_i^{\vee}(\nu)} R_{\mathfrak{b}}^{\xi_i}(\nu)[1],$$

by projecting onto the direct summands $\bigoplus_{\ell \geq 0} x_{m+1}^{\ell} R_{\mathfrak{b}}(m) \oplus \theta_{m+1}^{\ell} R_{\mathfrak{b}}(m)$ of the decomposition of $R_{\mathfrak{b}}(m+1)$ as right $R_{\mathfrak{b}}(m)$-module. By Lemma 5.4 we know that this is a map of

$R_{\mathfrak{b}}(m)$-$R_{\mathfrak{b}}(m)$-bimodules. Finally, exactness follows directly from Lemma 4.5, since

$$\begin{aligned} &R_{\mathfrak{b}}(\nu)1_{(m-1,i)} \otimes_{m-1} 1_{(m_1,j)} R_{\mathfrak{b}}(\nu) \\ &\quad\cong R_{\mathfrak{b}}(\nu)1_{(m-1,i)} \otimes_{m-1} \left(\bigoplus_{a=1}^{m} \bigoplus_{\ell \geq 0} (R_{\mathfrak{b}}(m-1)1_{(m,i)}\tau_{m-1}\cdots\tau_a x_a^{\ell} 1_{\boldsymbol{j}} \right. \\ &\qquad\qquad\qquad \left. \oplus\, R_{\mathfrak{b}}(m-1)1_{(m,i)}\tau_{m-1}\cdots\tau_a \theta_a^{\ell} 1_{\boldsymbol{j}}) \right), \end{aligned}$$

and so

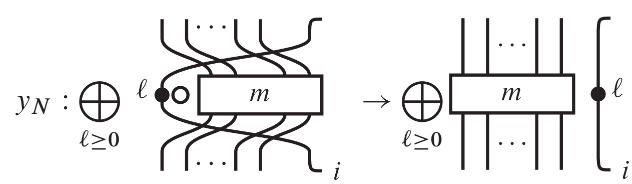
$$\begin{aligned} &u_{ij}(R_{\mathfrak{b}}(\nu)1_{(m-1,i)} \otimes_{m-1} 1_{(m_1,j)} R_{\mathfrak{b}}(\nu)) \\ &\quad\cong \bigoplus_{a=1}^{m} \bigoplus_{\ell \geq 0} (R_{\mathfrak{b}}(m-1)1_{(m,i)}\tau_m\tau_{m-1}\cdots\tau_a x_a^{\ell} 1_{\boldsymbol{j}} \\ &\qquad\qquad\qquad \oplus\, R_{\mathfrak{b}}(m-1)1_{(m,i)}\tau_m\tau_{m-1}\cdots\tau_a \theta_a^{\ell} 1_{\boldsymbol{j}}). \end{aligned}$$

We remark that whenever $i \neq j$, we have

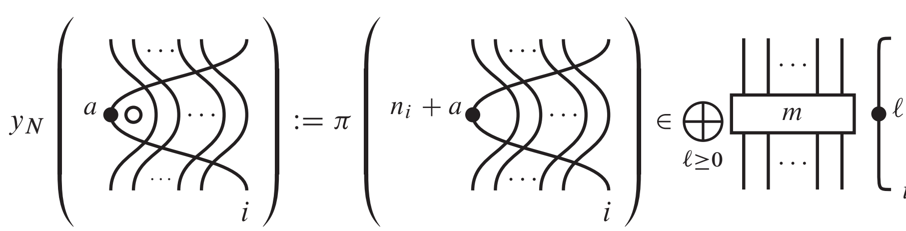
$$\bigoplus_{\ell \geq 0} 1_{(\nu,j)} x_{m+1}^{\ell} R_{\mathfrak{b}}(m) 1_{(\nu,i)} \oplus 1_{(\nu,j)} R_{\mathfrak{b}}(m) \theta_{m+1}^{\ell} 1_{(\nu,i)} = 0,$$

and thus u_{ij} is an isomorphism, concluding the proof. □

5.2. Long exact sequence. In this subsection, we would like to lift Theorem 5.1 to the dg-world of $(R_{\mathfrak{b}}(m), d_N)$, and study the long exact sequence that it induces. Therefore we define



as the $R_{\mathfrak{b}}(m)$-$R_{\mathfrak{b}}(m)$-bimodule map given by



whenever $i \in I_f$, and $y_N = 0$ for $i \notin I_f$. Then we define

$$\begin{aligned} &\big(R_{\mathfrak{b}}^{\xi_i}(\nu) \oplus \lambda_i^2 q_i^{-2\alpha_i^{\vee}(\nu)} R_{\mathfrak{b}}^{\xi_i}(\nu)[1], d_N\big) \\ &\quad := \mathrm{Cone}\Big((\lambda_i^2 q_i^{-2\alpha_i^{\vee}(\nu)} R_{\mathfrak{b}}^{\xi_i}(\nu)[1], d_N) \xrightarrow{y_n} (R_{\mathfrak{b}}^{\xi_i}(\nu), d_N)\Big), \end{aligned}$$

and

$$\begin{aligned} &(R_{\mathfrak{b}}(\nu)1_{(m-1,i)} \otimes_{m-1} 1_{(m-1,i)} R_{\mathfrak{b}}(\nu), d_N) \\ &\quad := (R_{\mathfrak{b}}(\nu)1_{(m-1,i)}, d_N) \otimes_{(R_{\mathfrak{b}}(m-1),d_N)} (1_{(m-1,i)} R_{\mathfrak{b}}(\nu), d_N). \end{aligned}$$

Proposition 5.5. *There is a short exact sequence of dg-bimodules*

$$0 \to q_i^{-2}(R_{\mathfrak{b}}(\nu)1_{(\nu-i,i)} \otimes_{m-1} 1_{(\nu-i,i)}R_{\mathfrak{b}}(\nu), d_N) \to (1_{(\nu,i)}R_{\mathfrak{b}}(\nu+i)1_{(\nu,i)}, d_N)$$
$$\to \big(R_{\mathfrak{b}}^{\xi_i}(\nu) \oplus \lambda_i^2 q_i^{-2\alpha_i^\vee(\nu)} R_{\mathfrak{b}}^{\xi_i}(\nu)[1], d_N\big) \to 0$$

for all $i \in I$. Moreover, there is an isomorphism

$$q^{-(\alpha_i|\alpha_j)}(R_{\mathfrak{b}}(\nu)1_{(\nu-i,i)} \otimes_{m-1} 1_{(\nu'-j,j)}R_{\mathfrak{b}}(\nu'), d_N) \cong (1_{(\nu',j)}R_{\mathfrak{b}}(\nu+i)1_{(\nu,i)}, d_N)$$

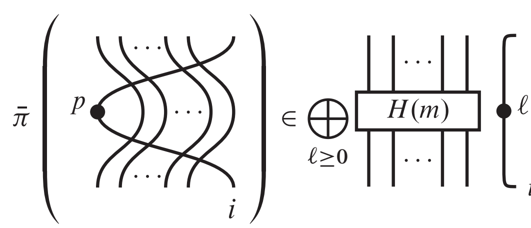
for all $i \neq j \in I$ and $\nu + i = \nu' + j$.

Proof. It is a straightforward consequence of Theorem 5.1. □

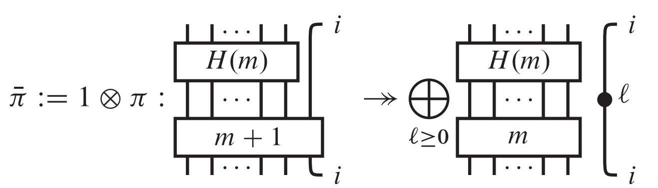
In order to understand the consequences of this short exact sequence in homology, we need to compute the homology

$$H\big(R_{\mathfrak{b}}^{\xi_i}(\nu) \oplus \lambda_i^2 q_i^{-2\alpha_i^\vee(\nu)} R_{\mathfrak{b}}^{\xi_i}(\nu)[1], d_N\big)$$

for all $i \in I_f$. Therefore, we want to compute the projection of the element

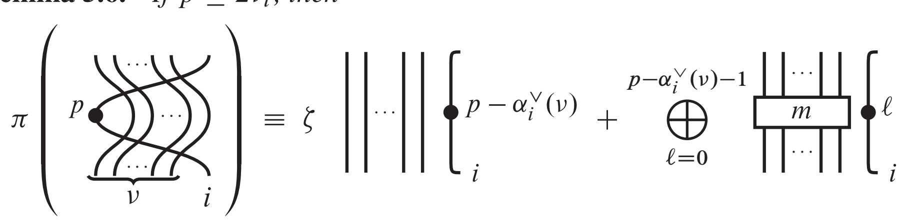


for all $p \geq n_i$. Note that we project on the homology of $(R_{\mathfrak{b}}(m), d_N)$. This will ease some of the computations we need to do. We write $\bar{\pi}$ when we take the composite of π with the projection on the homology of $(R_{\mathfrak{b}}(m), d_N)$. More precisely, $\bar{\pi}$ is given by

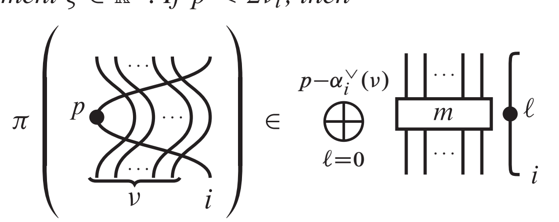


Similarly, we write $\bar{y}_N$.

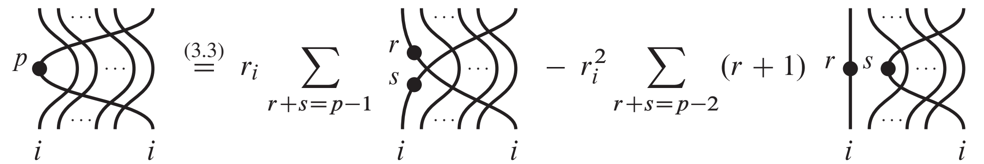
Lemma 5.6. *If $p \geq 2\nu_i$, then*



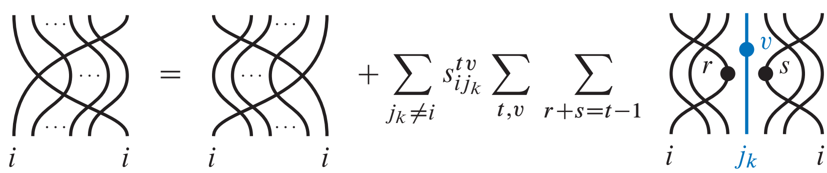
for some invertible element $\zeta \in \Bbbk^\times$. If $p < 2\nu_i$, then



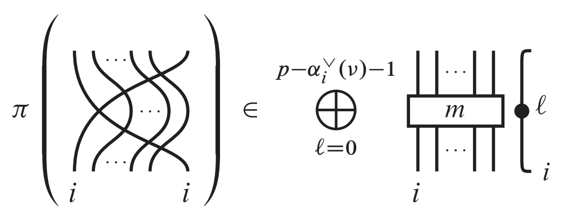
Proof. The proof is an induction on m. If $m = 0$, then it is trivial. Suppose the statement holds for $m-1$. We fix the labels of the strands as the bottom as $\boldsymbol{j} = j_1 \cdots j_m \in \mathrm{Seq}(\nu)$. If $j_1 = i$, then we compute



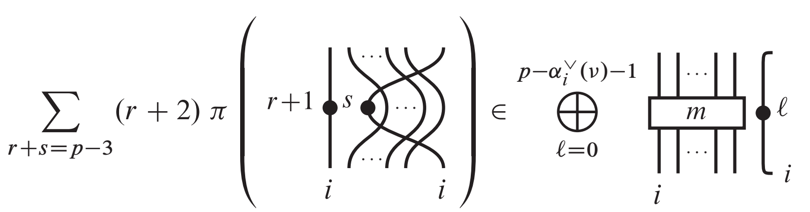
Then, using equation (3.4), we have



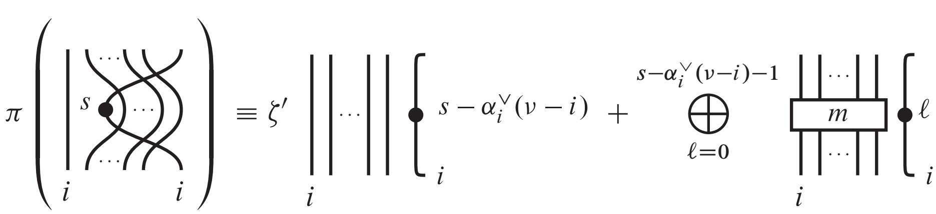
so that, since $s < d_{ij_k}$, we obtain by the induction hypothesis



Moreover, still by the induction hypothesis, we have



Finally, if $p \geq 2\nu_i$, by the induction hypothesis we get for $s = p-2$,



which concludes the case by observing that

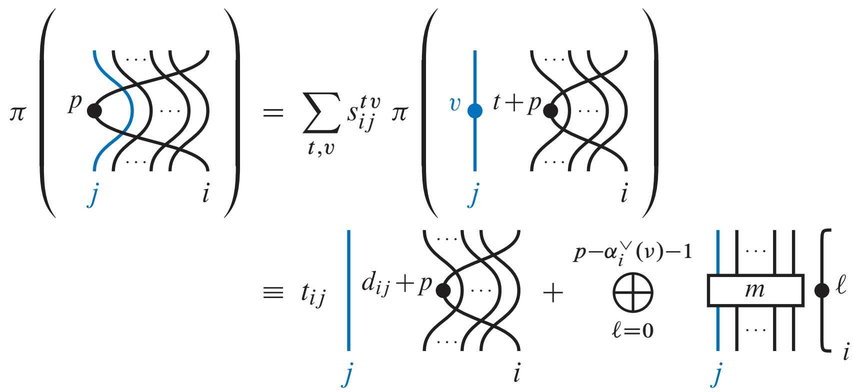
$$s - \alpha_i^\vee(\nu - i) = p - \alpha_i^\vee(\nu),$$

and taking

$$\zeta = r_i^2 \zeta'.$$

If $p < 2\nu_i$, the claim is immediate by the induction hypothesis.

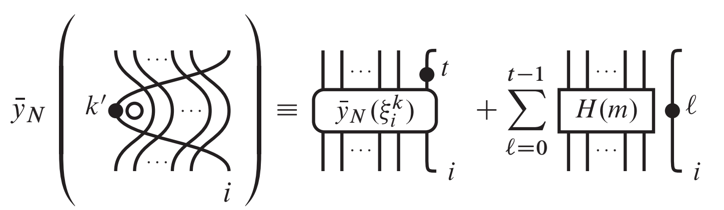
For the case $j_1 = j \neq i$, we use equation (3.1) and then the induction hypothesis to get



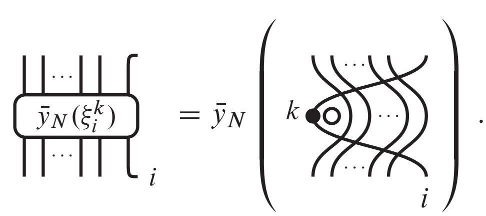
where we recall that $s_{ij}^{d_{ij}0} = t_{ij}$. We conclude by applying the induction hypothesis, observing that $d_{ij} + p - \alpha_i^\vee(\nu - j) = p - \alpha_i^\vee(\nu)$. □

Consider also the following result, which is akin to [20, Lemma 5.4].

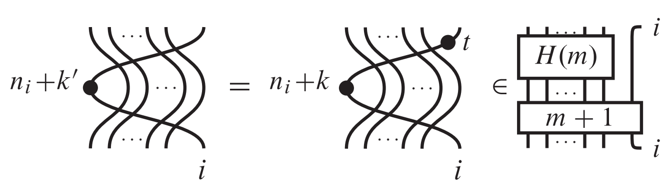
Lemma 5.7. *We have for $k < k'$ and $t = k' - k$,*



where

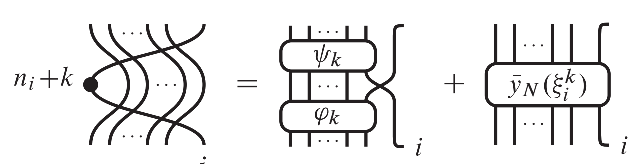


Proof. First we observe that



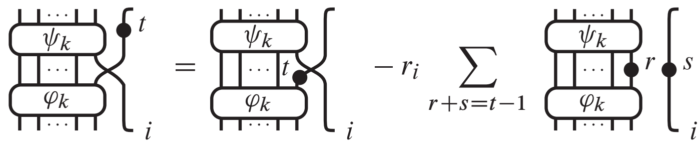
using equations (3.3) and (3.2), and the fact that n_i dots on the left strand is annihilated in $H(R_{\mathfrak{b}}(m), d_N)$. Then, using Lemma 4.5, we obtain

(5.3)



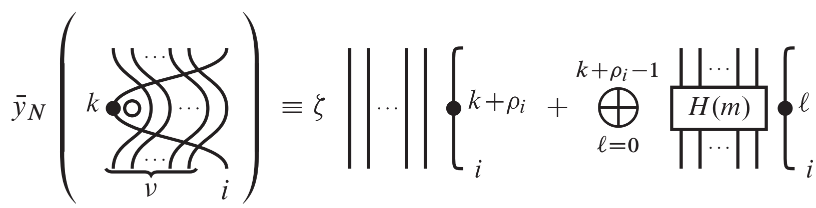
for some $\varphi_k, \psi_k \in R_{\mathfrak{b}}(m)$. We conclude by observing that

(5.4)



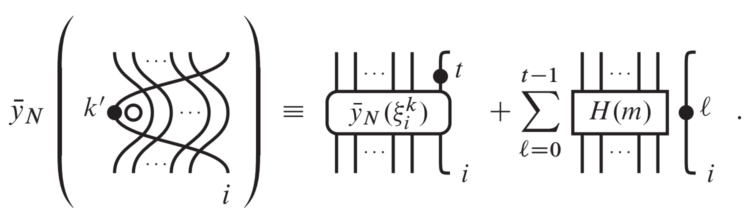
thanks to equation (3.3). □

Proposition 5.8. *Putting $\rho_i := n_i - \alpha_i^\vee(\nu)$, we have*

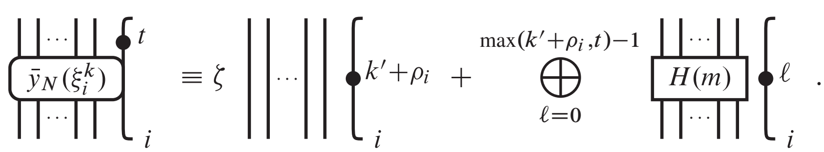


which is 0 *whenever $k + \rho_i < 0$, and where $\zeta \in \Bbbk^\times$.*

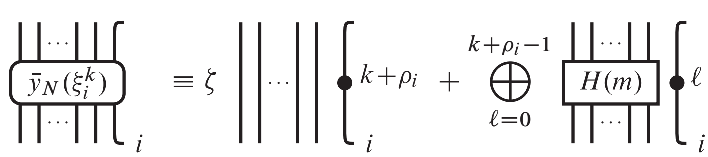
Proof. If $n_i \geq 2\nu_i$, the result follows from Lemma 5.6. Otherwise, we take $k' = 2\nu_i - n_i$ and the result follows from Lemma 5.6 for $k \geq k'$. Suppose $k < k'$ and put $t = k' - k$. Then, by Lemma 5.7 we obtain



Therefore, we have



From this, we deduce



which concludes the proof. □

We now have all the tools we need to compute the homology of the cokernel of the short exact sequence of Proposition 5.5.

Proposition 5.9. *There is an isomorphism of $R_{\mathfrak{p}}^N(\nu)$-$R_{\mathfrak{p}}^N(\nu)$-bimodules*

$$H\big(R_{\mathfrak{b}}^{\xi_i}(\nu) \oplus \lambda_i^2 q_i^{-2\alpha_i^\vee(\nu)} R_{\mathfrak{b}}^{\xi_i}(\nu)[1], d_N\big) \cong \begin{cases} \bigoplus_{\ell=0}^{\rho_i-1} q_i^{2\ell} R_{\mathfrak{p}}^N(\nu) & \text{if } \rho_i \geq 0, \\ \lambda_i^2 q_i^{-2\alpha_i^\vee(\nu)} \bigoplus_{\ell=0}^{-\rho_i-1} q_i^{2\ell} R_{\mathfrak{p}}^N(\nu)[1] & \text{if } \rho_i \leq 0, \end{cases}$$

where $\rho_i = n_i - \alpha_i^\vee(\nu)$.

Proof. First suppose $\rho_i \geq 0$. Then Proposition 5.8 tells us that $\bar{y}_N(\xi_i^k)$ is a monic polynomial (up to invertible scalar) with leading terms $\xi_i^{k+\rho_i}$. This gives us the first case. If $\rho_i \leq 0$, then we have $\bar{y}_N(\xi_i^k) = 0$ for $k < -\rho_i$. Moreover, $\zeta^{-1}\bar{y}_N(\xi_i^{-\rho_i}) = 1$, and in general $\bar{y}_N(\xi_i^k)$ is a monic polynomial with leading term $\xi_i^{k+\rho_i}$ for $k > -\rho_i$. This concludes the proof. □

5.3. Strongly projective dg-modules. The following notions were originally introduced by Moore [32]. We use the presentation given in [6], which is best suited for our notations.

Definition 5.10 ([6, Definition 8.5]). Let $(R, 0)$ be a ring R viewed as a dg-$\mathbb{Z}$-algebra concentrated in degree zero. An $(R, 0)$-module (Q, d_Q) is *strongly projective* if $H(Q, d_Q)$ and $\operatorname{im} d_Q$ are both projective R-modules.

Lemma 5.11 ([44, Theorem 9.3.2]). *Suppose that (P, d_P) is a strongly projective right $(R, 0)$-module and (N, d_N) any left $(R, 0)$-module. Then*

$$H\big((P, d_P) \otimes_{(R,0)} (N, d_N)\big) \cong H(P, d_P) \otimes_R H(N, d_N).$$

Definition 5.12 ([6, Definition 8.17]). Let (A, d_A) be a dg-R-algebra. A left (respectively, right) (A, d_A)-module (P, d_P) is called *strongly projective* if it is a dg-direct summand of $(A, d_A) \otimes_{(R,0)} (Q, d_Q)$ (respectively, $(Q, d_Q) \otimes_{(R,0)} (A, d_A)$) for some strongly projective $(R, 0)$-module (Q, d_Q).

Proposition 5.13 ([6, Lemma 8.23]). *Suppose that (P, d_P) is a strongly projective right (A, d_A)-module and (N, d_N) is any left (A, d_A)-module. Then*

$$H\big((P, d_P) \otimes_{(A,d_A)} (N, d_N)\big) \cong H(P, d_P) \otimes_{H(A,d_A)} H(N, d_N).$$

Note that if (P, d_P) is a strongly projective (A, d_A)-module, then $H(P, d_P)$ is a projective $H(A, d_A)$-module. Indeed, we can assume $(P, d_P) = (A, d_A) \otimes_{(R,0)} (Q, d_Q)$, and we have

$$H(P, d_P) \cong H(A, d_A) \otimes_R H(Q, d_Q).$$

Since $H(Q, d_Q)$ is a projective R-module, it is a direct summand of a free R-module F. Therefore $H(P, d_P)$ is a direct summand of $H(A, d_A) \otimes_R F$, which is a free $H(A, d_A)$-module.

Remark 5.14. This result does not hold in general. As a counterexample we can take $(A, d) = (\mathbb{Q}[x], 0)$ and consider the dg-module $(X, d_X) = \operatorname{Cone}(\mathbb{Q}[x] \xrightarrow{x} \mathbb{Q}[x])$. In this case we have that $H(X, d_X) \cong \mathbb{Q}$ but $H((X, d_X) \otimes_{(A,d)} (X, d_X)) \cong \mathbb{Q} \oplus \mathbb{Q}[1]$.

5.3.1. Strong projectivity of $R_{\mathfrak{b}}(m+1)$. Our next goal is to show the following:

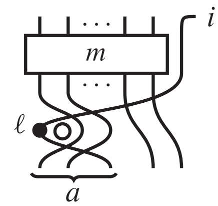
Proposition 5.15. *The $(R_{\mathfrak{b}}(m), d_N)$-module $(1_{(m,i)}R_{\mathfrak{b}}(m+1), d_N)$ is strongly projective.*

It is obvious for $i \notin I_f$ by Lemma 4.5, and thus we can assume $i \in I_f$. We first construct the mapping cone

$$(Q, d_Q) := \operatorname{Cone}\left(\bigoplus_{a=1}^{m+1} \bigoplus_{\ell \geq 0} R_{\mathfrak{g}}(m) 1_{(\nu,i)} \tau_m \cdots \tau_a \theta_a^\ell \xrightarrow{d_Q} \bigoplus_{a=1}^{m+1} \bigoplus_{\ell \geq 0} R_{\mathfrak{g}}(m) 1_{(\nu,i)} \tau_m \cdots \tau_a x_a^\ell \right),$$

where we think of $\tau_m \cdots \tau_a \theta_a^\ell$ as a formal symbol that represents a degree shift corresponding to the degree of the element $1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^\ell$ in $R_{\mathfrak{b}}(m+1)$. The map d_Q is given by first embedding $R_{\mathfrak{g}}(m)$ into $R_{\mathfrak{b}}(m+1)$ through the diagrams

$$R_{\mathfrak{g}}(m) 1_{(\nu,i)} \tau_m \cdots \tau_a \theta_a^\ell \hookrightarrow$$



then applying d_N of $(R_{\mathfrak{b}}(m+1), d_N)$, then decomposing the image in the left-decomposition

$$\bigoplus_{a=1}^{m+1} \bigoplus_{\ell \geq 0} R_{\mathfrak{b}}(m) 1_{(m,i)} \tau_m \cdots \tau_a x_a^\ell,$$

and finally projecting unto the part in homological degree zero of $R_{\mathfrak{b}}(m)$, which is trivially isomorphic to $R_{\mathfrak{g}}(m)$. Moreover, $(R_{\mathfrak{b}}(m), d_N)$ is a (right) module over $(R_{\mathfrak{g}}, 0)$ which acts by gluing KLR diagrams on the bottom. Then we have, as $(R_{\mathfrak{b}}(m), d_N)$-modules

$$(R_{\mathfrak{b}}(m+1), d_N) \cong (R_{\mathfrak{b}}(m), d_N) \otimes_{(R_{\mathfrak{g}}(m),0)} (Q, d_Q).$$

Therefore, we want to show that (Q, d_Q) is strongly projective as $(R_{\mathfrak{g}}(m), 0)$-module. We write

$$Q_1[\xi_i] := \bigoplus_{a=1}^{m+1} \bigoplus_{\ell \geq 0} R_{\mathfrak{g}}(m) 1_{(\nu,i)} \tau_m \cdots \tau_a \theta_a^\ell,$$

$$Q_0[\xi_i] := \bigoplus_{a=1}^{m+1} \bigoplus_{\ell \geq 0} R_{\mathfrak{g}}(m) 1_{(\nu,i)} \tau_m \cdots \tau_a x_a^\ell,$$

where we identify ξ_i with x_a in Q_0, and ξ_i^ℓ with x_1^ℓ in θ_a^ℓ. Note that d_Q is not $\Bbbk[\xi_i]$-linear.

Lemma 5.16. *The map*

$$d_Q : Q_1[\xi_i] \to Q_0[\xi_i]$$

defined above is injective.

Proof. Recall the map P of Lemma 4.9 given by multiplication by $\widetilde{\theta}_{m+1}$. Since floating dots are also annihilated in $R_{\mathfrak{g}}(m)$, multiplication by $\widetilde{\theta}_{m+1}$ also defines a map

$$P' : Q_0[\xi_i] \to Q_1[\xi_i]. \tag{5.5}$$

We reconsider the proof of Lemma 4.10 to show that $P' \circ d_Q$ is injective. First, we introduce an order on the summands of $Q_1[\xi_i] = \bigoplus_{a=1}^{m+1} \bigoplus_{\ell \geq 0} R_{\mathfrak{g}}(m) 1_{(\nu,i)} \tau_m \cdots \tau_a \theta_a^{\ell}$ by declaring that

$$R_{\mathfrak{g}}(m)1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^{\ell} \prec R_{\mathfrak{g}}(m)1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^{\ell'},$$
$$R_{\mathfrak{g}}(m)1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^{\ell} \prec R_{\mathfrak{g}}(m)1_{(\nu,i)}\tau_m \cdots \tau_{a'} \theta_{a'}^{\ell''}$$

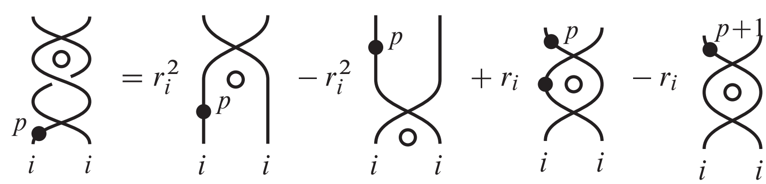
for all $a > a'$, $\ell < \ell'$, and for all ℓ''. In other words, if there are more crossings under the floating dot, then the term is smaller. If there is the same amount of crossings, then we consider the amount of dots at the left of the floating dot, and lesser dots meaning a smaller term.

We claim that if $Z \in R_{\mathfrak{g}}(m)1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^{\ell}$ then

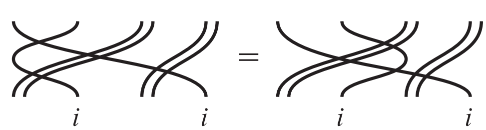
$$P' \circ d_Q(Z) = r_i^{2\nu_i} \sum_{p=0}^{2\nu_i - \alpha_i^{\vee}(\nu)} Z x_{m+1}^{n_i+p} \varepsilon_p^i(\underline{x}_\nu) + H,$$

where $H \prec Z x_{m+1}^{\ell+n_i+2\nu_i-\alpha_i^{\vee}(\nu)}$. This implies that $P' \circ d_Q$ is in echelon form (with pivot being invertible scalars), and thus is injective. By consequence, so is d_Q.

In order to prove our claim, we need to tweak the proof of Lemma 4.10. We need to keep track of the terms that are annihilated when working over the cyclotomic quotient, and show these appear as lower terms in the order defined above. The case $j_m \neq i$ remains the same. The case $j_m = i$ and $m = 1$ becomes



where $p = n_i + \ell$. The first term is the leading term. The second term possesses less dots on the left of the floating dot, and so it is smaller. If $a = 0$, then the last two terms possess one more crossing at the bottom of the floating dot, and therefore they are smaller. If $a = 1$, then they are annihilated by equation (3.1). Finally, the two remaining cases $j_{m-1} \neq i$ and j_{m-1} follow from the same arguments as in the proof of Lemma 4.10, with the lower terms in the induction hypothesis only adding lower terms because:



by (3.4), and,

by equation (3.4) and equation (3.1). This concludes the proof of the claim, and therefore of the proposition. □

Proof of Proposition 5.15. The proof is a revisit of the proof of [20, Lemma 4.18] that applies to our particular case.

Recall the map P' from equation (5.5). We know that $P' \circ d_Q$ is given by multiplying by a monic polynomial with leading term $x_{m+1}^{n_i+2\nu_i-\alpha_i^\vee(\nu)}$ plus some remaining map giving lower terms. In particular, it is injective and we have a short exact sequence

$$0 \to Q_1[\xi_i] \xrightarrow{P'\circ d_Q} Q_1[\xi_i] \to \mathrm{cok}(P' \circ d_Q) \to 0.$$

Since $P' \circ d_Q$ is in echelon form, it means that $\mathrm{cok}(P' \circ d_Q)$ is a projective $R_{\mathfrak{g}}(m)$-module. Thus, the sequence splits as $R_{\mathfrak{g}}(m)$-modules with splitting map $\sigma : Q_1[\xi_i] \to Q_1[\xi_i]$, and we get $\sigma \circ P' \circ d_Q = \mathrm{Id}_{Q_1[\xi_i]}$. Then the short exact sequence

$$0 \longrightarrow Q_1[\xi_i] \underset{\sigma\circ P'}{\overset{d_Q}{\rightleftarrows}} Q_0[\xi_i] \longrightarrow H(Q, d_Q) \longrightarrow 0$$

obtained thanks to Lemma 5.16 splits with splitting map given by $\sigma \circ P'$. Since $Q_0[\xi_i]$ is a projective $R_g(m)$-module, so is $H(Q, d_Q)$. Finally, $d_Q(Q_1[\xi_i])$ is also projective since d_Q is injective and $Q_1[\xi_i]$ is projective. □

5.4. Functors. We define for all $i \in I$ the functors

$$\mathsf{F}_i^N(-) := \bigoplus_{m\geq 0} R_{\mathfrak{p}}^N(m+1)1_{(m,i)} \otimes_{R_{\mathfrak{p}}^N(m)} (-),$$
$$\mathsf{E}_i^N(-) := \bigoplus_{m\geq 0} \bigoplus_{|\nu|=m} \lambda_i^{-1} q_i^{1+\alpha_i^\vee(\nu)} 1_{(\nu,i)} R_{\mathfrak{p}}^N(m+1) \otimes_{R_{\mathfrak{p}}^N(m+1)} (-),$$

where we interpret $\lambda_i = q^{n_i}$ whenever $i \in I_f$. Thanks to Proposition 5.15, these are exact. For $n \in \mathbb{N}$, we write

$$\bigoplus_{[n]_{q_i}} \mathrm{Id}_\nu := \bigoplus_{\ell=0}^{n-1} q_i^{1-n+2\ell}\, \mathrm{Id}_\nu$$

for the finite direct sum that categorifies $[n]_{q_i}$.

Theorem 5.17. *For $i \notin I_f$ there is a natural short exact sequence*

$$0 \to \mathsf{F}_i^N \mathsf{E}_i^N \,\mathrm{Id}_\nu \to \mathsf{E}_i^N \mathsf{F}_i^N \,\mathrm{Id}_\nu \to \bigoplus_{[\beta_i-\alpha_i^\vee(\nu)]_{q_i}} \mathrm{Id}_\nu \to 0 \tag{5.6}$$

and for $i \in I_f$ there are natural isomorphisms

$$\mathsf{E}_i^N \mathsf{F}_i^N \,\mathrm{Id}_\nu \cong \mathsf{F}_i^N \mathsf{E}_i^N \,\mathrm{Id}_\nu \bigoplus_{[n_i-\alpha_i^\vee(\nu)]_{q_i}} \mathrm{Id}_\nu \qquad \text{if } n_i - \alpha_i^\vee(\nu) \geq 0, \tag{5.7}$$
$$\mathsf{F}_i^N \mathsf{E}_i^N \,\mathrm{Id}_\nu \cong \mathsf{E}_i^N \mathsf{F}_i^N \,\mathrm{Id}_\nu \bigoplus_{[\alpha_i^\vee(\nu)-n_i]_{q_i}} \mathrm{Id}_\nu \qquad \text{if } n_i - \alpha_i^\vee(\nu) \leq 0.$$

Moreover, there is a natural isomorphism

$$\mathsf{F}_i^N \mathsf{E}_j^N \cong \mathsf{E}_j^N \mathsf{F}_i^N \tag{5.8}$$

for $i \neq j \in I$.

Proof. The short exact sequence (5.6) and the isomorphism (5.8) are immediate consequences of Propositions 5.5 and 5.15. For the isomorphisms (5.7), Propositions 5.5 and 5.15 give a long exact sequence of $R_{\mathfrak{p}}^N(\nu)$-$R_{\mathfrak{p}}^N(\nu)$-bimodules. By Proposition 5.9 it truncates to a short exact sequence

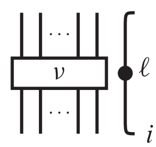
$$0 \to \mathsf{F}_i^N \mathsf{E}_i^N \operatorname{Id}_\nu \to \mathsf{E}_i^N \mathsf{F}_i^N \operatorname{Id}_\nu \to \bigoplus_{[\rho_i]} \operatorname{Id}_\nu \to 0$$

if $\rho_i = n_i - \alpha_i^\vee(\nu) \geq 0$, and a short exact sequence

$$0 \to \bigoplus_{[-\rho_i]} \operatorname{Id}_\nu \to \mathsf{F}_i^N \mathsf{E}_i^N \operatorname{Id}_\nu \to \mathsf{E}_i^N \mathsf{F}_i^N \operatorname{Id}_\nu \to 0$$

if $\rho_i = n_i - \alpha_i^\vee(\nu) \leq 0$. In the first case, we can identify

$$\bigoplus_{[\rho_i]} \operatorname{Id}_\nu \cong q_i^{1-\rho_i} \bigoplus_{\ell=0}^{\rho_i-1}$$



and the map

$$\mathsf{E}_i^N \mathsf{F}_i^N \operatorname{Id}_\nu \to \bigoplus_{[\rho_i]_{q_i}} \operatorname{Id}_\nu$$

is induced by the projection π. Thus the sequence splits with the splitting map

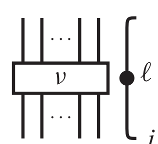
$$\bigoplus_{[\rho_i]_{q_i}} \operatorname{Id}_\nu \to \mathsf{E}_i^N \mathsf{F}_i^N \operatorname{Id}_\nu,$$

given by the sum of maps

$$R_{\mathfrak{p}}^N(\nu)\xi^\ell \to R_{\mathfrak{p}}^N(\nu + i)$$

that add a vertical strand labeled i carrying ℓ dots at the right of a diagram in $R_{\mathfrak{p}}^N(\nu)$. In the second case, we also identify

$$\bigoplus_{[-\rho_i]_{q_i}} \operatorname{Id}_\nu \cong q_i^{1+\rho_i} \bigoplus_{\ell=0}^{-\rho_i-1}$$



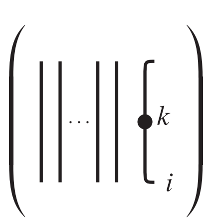
.

Moreover, the map

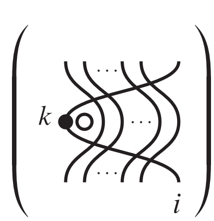
$$\bigoplus_{[-\rho_i]} \operatorname{Id}_\nu \to \mathsf{F}_i^N \mathsf{E}_i^N \operatorname{Id}_\nu$$

is induced by the connecting homomorphism δ. Using the notations of equation (5.3), it takes the form

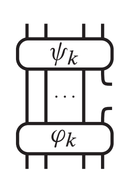
δ



$= u_{ij}^{-1}$



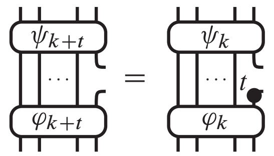
$=$



,

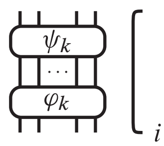
where u_{ij} is the monomorphism defined in equation (5.2), and $0 \leq k < -\rho_i$. We also note that equation (5.4) tells us that

(5.9)



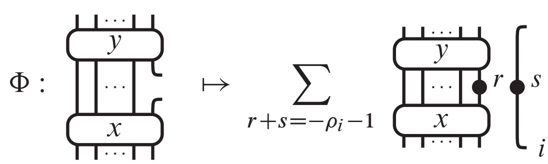
Moreover, since $\bar{y}_N(\xi_i^{-\rho_i}) = \zeta$ and $\bar{y}_N(\xi_i^{\ell}) = 0$ for $\ell < -\rho_i$, we obtain by equation (5.4) again that

(5.10)

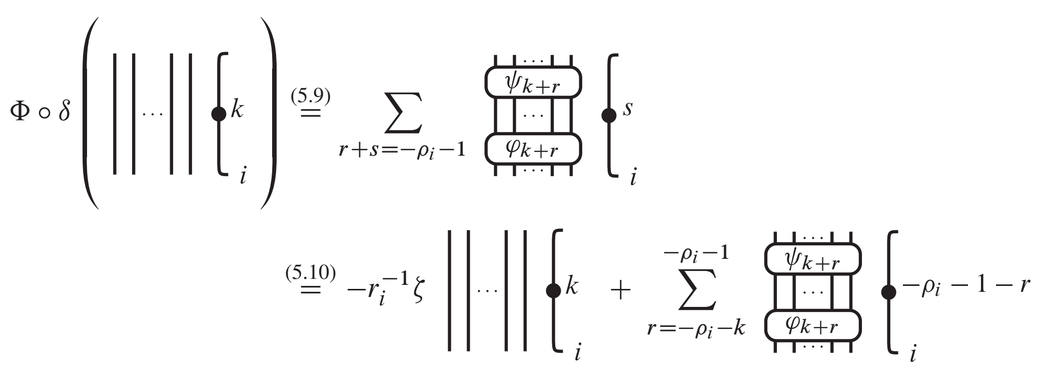


$$= \begin{cases} -r_i^{-1}\zeta & \text{if } k = -\rho_i - 1, \\ 0 & \text{if } k < -\rho_i - 1. \end{cases}$$

As in [20, proof of Theorem 5.2], we construct a map $\Phi\colon \mathsf{F}_i^N \mathsf{E}_i^N \operatorname{Id}_\nu \to \bigoplus_{[-\rho_i]_{q_i}} \operatorname{Id}_\nu$ induced by the morphism of bimodules



for all $x, y \in R_{\mathfrak{p}}^N(\nu)$. Then we compute



Therefore, $\Phi \circ \delta$ is given by a triangular matrix with invertible elements on the diagonal, and thus is an isomorphism. In particular, δ is left invertible, concluding the proof. □

Corollary 5.18. *If $i \in I_f$, then $1_\nu \mathsf{E}_i$ and $\mathsf{F}_i 1_\nu$ are biadjoint (up to shift).*

Proof. By the results in [7], we know the splitting map $\mathsf{E}_i^N \mathsf{F}_i^N \operatorname{Id}_\nu \to \mathsf{F}_i^N \mathsf{E}_i^N \operatorname{Id}_\nu$ of Theorem 5.17 together with the unit and counit of the adjunction $\mathsf{F}_i \dashv \mathsf{E}_i$ allow to construct a unit and counit for the adjunction $\mathsf{E}_i \dashv \mathsf{F}_i$. □

Proposition 5.19. *For each $i, j \in I$ there is a natural isomorphism*

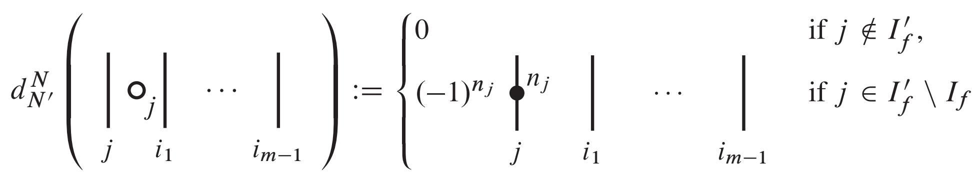
$$\bigoplus_{a=0}^{\lfloor \frac{d_{ij}+1}{2} \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a \end{bmatrix}_{q_i} (\mathsf{F}_i^N)^{2a} \mathsf{F}_j^N (\mathsf{F}_i^N)^{d_{ij}+1-2a} \cong \bigoplus_{a=0}^{\lfloor \frac{d_{ij}}{2} \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a+1 \end{bmatrix}_{q_i} (\mathsf{F}_i^N)^{2a+1} \mathsf{F}_j^N (\mathsf{F}_i^N)^{d_{ij}-2a}.$$

By adjunction, the same isomorphism exists for the $\mathsf{E}_i^N, \mathsf{E}_j^N$.

Proof. This follows from Proposition 5.3. □

In particular, there is a strong 2-action of the 2-Kac–Moody algebra of [24, 39] associated to $\langle E_i, F_i, K_i \rangle_{i \in I_f}$ on $\bigoplus_{\nu \in X^+} R_{\mathfrak{p}}^N(\nu)$-mod through $\mathsf{F}_i^N, \mathsf{E}_i^N$.

5.5. A differential on $R_{\mathfrak{p}}^N$. We fix a subset $I_f \subset I'_f \subset I$ and consider the parabolic subalgebras $U_q(\mathfrak{p}) \subset U_q(\mathfrak{p}') \subset U_q(\mathfrak{g})$. For each $j \in I'_f \setminus I_f$ we choose a weight $n'_j \in \mathbb{N}$. For $j \in I_f$ we take $n'_j := n_j \in N$, and we write $N' := \{n'_j\}_{j \in I'_f}$. Then we equip the cyclotomic $\mathfrak{p}$-KLR algebra $R_{\mathfrak{p}}^N(m)$ with a differential $d_{N'}^N$ which is zero on dots and crossings and



As before, we extend using the graded Leibniz rule, and verifying that $d_{N'}^N$ is well-defined is straightforward.

Theorem 5.20. *The dg-algebra $(R_{\mathfrak{p}}^N(m), d_{N'}^N)$ is formal with homology*

$$H(R_{\mathfrak{p}}^N(m), d_{N'}^N) \cong R_{\mathfrak{p}'}^{N'}(m).$$

Proof. We have $R_{\mathfrak{p}}^N(m) \cong H(R_{\mathfrak{b}}(m), d_N)$ and $R_{\mathfrak{p}'}^{N'}(m) \cong H(R_{\mathfrak{b}}(m), d_{N'})$ by Theorem 4.4. Moreover, $d_{N'}^N$ can be lifted to $R_{\mathfrak{b}}(m)$. We split the homological grading of $R_{\mathfrak{b}}(m)$ in three: a first one that counts the amount of floating dots with subscript in I_f, a second one for the floating dots with subscript in $I'_f \setminus I_f$, and a third one for $I \setminus I'_f$ that we ignore for the moment. Then we have that $d_{N'}^N$ has degree $(0, -1)$ and d_N has degree $(-1, 0)$, and they commute with each other. Thus we have a (bounded) double complex $(R_{\mathfrak{b}}, d_N, d_{N'}^N)$ with total complex being $(R_{\mathfrak{b}}, d_{N'})$, since $d_{N'} = d_N + d_{N'}^N$. In particular, there is a spectral sequence from $H(R_{\mathfrak{p}}^N(m), d_{N'}^N)$ to $H(R_{\mathfrak{b}}, d_{N'}) \cong R_{\mathfrak{p}'}^{N'}(m)$. Now, Theorem 4.4 tells us that $H(R_{\mathfrak{b}}, d_N)$ is concentrated in homological degree zero (for the first homological grading). Thus, the spectral sequence converges at the second page, and in particular $H(R_{\mathfrak{p}}^N(m), d_{N'}^N) \cong R_{\mathfrak{p}'}^{N'}(m)$. □

We interpret this result as a categorical version of the fact that if there is an arrow from a parabolic Verma module $M^{\mathfrak{p}}(\Lambda, N)$ to $M^{\mathfrak{p}'}(\Lambda', N')$ (see Section 2.2), then there is a surjection $M^{\mathfrak{p}}(\Lambda, N) \twoheadrightarrow M^{\mathfrak{p}'}(\Lambda', N')$. Indeed, in that case there is a surjective quasi-isomorphism

$(R_{\mathfrak{p}}^{N}, d_{N'}) \xrightarrow{\simeq} (R_{\mathfrak{p}'}^{N'}, 0)$, inducing equivalences of derived categories that commute up to quasi-isomorphism with the categorical actions of $U_q(\mathfrak{g})$.

6. The categorification theorems

Recall that the $\Bbbk$-algebra of formal Laurent series $\Bbbk(\!(x_1, \dots, x_n)\!)$ (as constructed in [3], see also [33, Section 5]) is given by first choosing a total additive order $\prec$ on $\mathbb{Z}^n$. One says that a cone $C := \{\alpha_1 v_1 + \cdots + \alpha_n v_n \mid \alpha_i \in \mathbb{R}_{\geq 0}\} \subset \mathbb{R}^n$ is compatible with $\prec$ whenever $0 \prec v_i$ for all $i \in \{1, \dots, n\}$. Then we set

$$\Bbbk(\!(x_1, \dots, x_n)\!) := \bigcup_{\boldsymbol{e} \in \mathbb{Z}^n} x^{\boldsymbol{e}}\, \Bbbk_{\prec}[\![x_1, \dots, x_n]\!],$$

where $\Bbbk_{\prec}[\![x_1, \dots, x_n]\!]$ consists of formal Laurent series in $\Bbbk[\![x_1, \dots, x_n]\!]$ such that the terms are contained in a cone compatible with $\prec$. We will also write $\Bbbk_{\prec}^{+}[\![x_1, \dots, x_n]\!]$ for the elements in $\Bbbk_{\prec}[\![x_1, \dots, x_n]\!]$ with terms contained in a cone without the 0 element (i.e. series with the degree zero term being zero). We obtain a ring by equipping $\Bbbk(\!(x_1, \dots, x_n)\!)$ with the usual addition and multiplication of series. Requiring that all series are contained in cones compatible with $\prec$ ensures that the product of two elements in $\Bbbk(\!(x_1, \dots, x_n)\!)$ is well-defined. Indeed, under these conditions, any coefficient in the product can be determined by summing only a finite amount of terms.

6.1. C.b.l.f. derived category. We fix an arbitrary additive total order $\prec$ on $\mathbb{Z}^n$. We say that a $\mathbb{Z}^n$-graded $\Bbbk$-vector space $M = \bigoplus_{\boldsymbol{g} \in \mathbb{Z}^n} M_{\boldsymbol{g}}$ is *c.b.l.f. (cone bounded, locally finite) dimensional* if

- $\dim M_{\boldsymbol{g}} < \infty$ for all $\boldsymbol{g} \in \mathbb{Z}^n$,
- there exists a cone $C_M \subset \mathbb{R}^n$ compatible with $\prec$ and $\boldsymbol{e} \in \mathbb{Z}^n$ such that $M_{\boldsymbol{g}} = 0$ whenever $\boldsymbol{g} - \boldsymbol{e} \notin C_M$.

In other words, M is c.b.l.f. dimensional if and only if

$$\operatorname{gdim}_q M := \sum_{\boldsymbol{g} \in \mathbb{Z}^n} x^{\boldsymbol{g}} \dim(M_{\boldsymbol{g}}) \in x^{\boldsymbol{e}}\, \Bbbk_{\prec}[\![x_1, \dots, x_n]\!].$$

Let (A, d) be a $\mathbb{Z}^n$-graded dg-$\Bbbk$-algebra, where $A = \bigoplus_{(h,\boldsymbol{g}) \in \mathbb{Z} \times \mathbb{Z}^n} A_{\boldsymbol{g}}^{h}$, $d(A_{\boldsymbol{g}}^{h}) \subset A_{\boldsymbol{g}}^{h-1}$. Suppose that (A, d) is concentrated in non-negative homological degrees, that is $A_{\boldsymbol{g}}^{h} = 0$ whenever $h < 0$. Let $\mathcal{D}(A, d)$ be the derived category of (A, d). Let $\mathcal{D}^{\mathrm{lf}}(A, d)$ be the full triangulated subcategory of $\mathcal{D}(A, d)$ consisting of (A, d)-modules having homology being c.b.l.f. dimensional for the $\mathbb{Z}^n$-grading. We call $\mathcal{D}^{\mathrm{lf}}(A, d)$ the *c.b.l.f. derived category of* (A, d).

Definition 6.1 ([33]). We say that (A, d) is a *positive c.b.l.f.-dimensional dg-algebra* if

(i) A is c.b.l.f.dimensional for the $\mathbb{Z}^n$-grading,

(ii) A is non-negative for the homological grading,

(iii) $A_{0}^{h} = 0$ for $h > 0$,

(iv) (A, d) decomposes into a direct sum of shifted copies of relatively projective modules $P_i := Ae_i$ for some idempotent $e_i \in A$, such that P_i is non-negative for the $\mathbb{Z}^n$-grading and $A_0^0 P_i$ is semisimple.

Remark 6.2. As explained in [33, Remark 9.5], condition (iii). cannot be respected whenever $P_i := Ae_i$ is acyclic. However, in this case there is a quasi-isomorphism

$$(A, d) \xrightarrow{\simeq} (A/Ae_iA, d)$$

and we can weaken hypothesis (iii). so that it is respected only after removing all acyclic P_i. This is the case of $(R_{\mathfrak{b}}, d_N)$.

6.1.1. Asymptotic Grothendieck group. As already observed in the work [1] (see also [34, Appendix]), one caveat of the usual definition of the Grothendieck group is that it does not allow to take into consideration infinite iterated extensions of objects. We need to introduce new relations in the Grothendieck groups to handle such situations. One solution is to use *asymptotic Grothendieck groups*, as introduced by the first author in [33].

Let $\mathcal{C}$ be a triangulated subcategory of some triangulated category $\mathcal{T}$. Suppose $\mathcal{T}$ admits countable products and coproducts, and these preserve distinguished triangles. Let $K_0^\Delta(\mathcal{C})$ be the triangulated Grothendieck group of $\mathcal{C}$.

Recall the *Milnor colimit* $\mathrm{MColim}_{r\geq 0}(f_r)$ of a collection of arrows $\{X_r \xrightarrow{f_r} X_{r+1}\}_{r\in\mathbb{N}}$ in $\mathcal{T}$ is the mapping cone fitting inside the following distinguished triangle:

$$\coprod_{r\in\mathbb{N}} X_r \xrightarrow{1-f_\bullet} \coprod_{r\in\mathbb{N}} X_r \to \mathrm{MColim}_{r\geq 0}(f_r) \to \ .$$

where the left arrow is given by the infinite matrix

$$1 - f_\bullet := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -f_0 & 1 & 0 & 0 & \cdots \\ 0 & -f_1 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

There is a dual notion of Milnor limit. Consider a collection of arrows $\{X_{r+1} \xrightarrow{f_r} X_r\}_{r\geq 0}$ in $\mathcal{T}$. The *Milnor limit* is the object fitting inside the distinguished triangle

$$\mathrm{MLim}_{r\geq 0}(f_r) \to \prod_{r\geq 0} X_r \xrightarrow{1-f_\bullet} \prod_{r\geq 0} X_r \to \ .$$

Definition 6.3. The *asymptotic triangulated Grothendieck group* of $\mathcal{C} \subset \mathcal{T}$ is given by

$$\boldsymbol{K}_0^\Delta(\mathcal{C}) := K_0^\Delta(\mathcal{C})/T(\mathcal{C}),$$

where $T(\mathcal{C})$ is generated by

$$[Y] - [X] = \sum_{r\geq 0} [E_r]$$

whenever both $\bigoplus_{r\geq 0} \mathrm{Cone}(f_r) \in \mathcal{C}$ and $\bigoplus_{r\geq 0} E_r \in \mathcal{C}$, and

$$Y \cong \mathrm{MColim}\big(X = F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} \cdots\big),$$

is a Milnor colimit, or

$$X \cong \mathrm{MLim}\big(\cdots \xrightarrow{f_1} F_1 \xrightarrow{f_0} F_0 = Y\big),$$

is a Milnor limit, and

$$[E_r] = [\mathrm{Cone}(f_r)] \in K_0^{\Delta}(\mathcal{C})$$

for all $r \geq 0$.

In a $\mathbb{Z}^n$-graded triangulated category $\mathcal{T}$, we define the notion of *c.b.l.f. direct sum* as follows:

- take a finite collection of objects $\{K_1, \dots, K_m\}$ in $\mathcal{T}$,
- consider a direct sum of the form

$$\bigoplus_{\boldsymbol{g} \in \mathbb{Z}^n} x^{\boldsymbol{g}} (K_{1,\boldsymbol{g}} \oplus \cdots \oplus K_{m,\boldsymbol{g}}) \quad \text{with } K_{i,\boldsymbol{g}} = \bigoplus_{j=1}^{k_{i,\boldsymbol{g}}} K_i[h_{i,j,\boldsymbol{g}}],$$

  where $k_{i,\boldsymbol{g}} \in \mathbb{N}$ and $h_{i,j,\boldsymbol{g}} \in \mathbb{Z}$ such that:

- there exists a cone C compatible with $\prec$, and $\boldsymbol{e} \in \mathbb{Z}^n$ such that for all j we have $k_{j,\boldsymbol{g}} = 0$ whenever $\boldsymbol{g} - \boldsymbol{e} \notin C$,
- there exists $h \in \mathbb{Z}$ such that $h_{i,j,\boldsymbol{g}} \geq h$ for all $i, j, \boldsymbol{g}$.

If $\mathcal{T}$ admits arbitrary c.b.l.f. direct sums, then the Grothendieck group $K_0(\mathcal{T})$ has a natural structure of $\mathbb{Z}(\!(x_1, \dots, x_n)\!)$-module with

$$\sum_{\boldsymbol{g} \in C} a_{\boldsymbol{g}} x^{\boldsymbol{e}+\boldsymbol{g}} [X] := \Big[\bigoplus_{\boldsymbol{g} \in C} x^{\boldsymbol{g}+\boldsymbol{e}} X^{\oplus a_{\boldsymbol{g}}} \Big],$$

where $X^{\oplus a_{\boldsymbol{g}}} = \bigoplus_{\ell=1}^{|a_{\boldsymbol{g}}|} X[\alpha_{\boldsymbol{g}}]$ and $\alpha_{\boldsymbol{g}} = 0$ if $a_{\boldsymbol{g}} \geq 0$ and $\alpha_{\boldsymbol{g}} = 1$ if $a_{\boldsymbol{g}} < 0$.

Theorem 6.4 ([33, Theorem 9.15]). *Let (A, d) be a positive c.b.l.f. dg-algebra, and let $\{P_j\}_{j \in J}$ be a complete set of indecomposable cofibrant (A, d)-modules that are pairwise non-isomorphic (even up to degree shift). Let $\{S_j\}_{j \in J}$ be the set of corresponding simple modules. There is an isomorphism*

$$\boldsymbol{K}_0^{\Delta}\big(\mathcal{D}^{\mathrm{lf}}(A, d)\big) \cong \bigoplus_{j \in J} \mathbb{Z}(\!(x_1, \dots, x_\ell)\!)[P_j],$$

and $\boldsymbol{K}_0^{\Delta}(\mathcal{D}^{\mathrm{lf}}(A, d))$ is also freely generated by the classes of $\{[S_j]\}_{j \in J}$.

Proposition 6.5 ([33, Proposition 9.18]). *Let (A, d) and (A', d') be two c.b.l.f. positive dg-algebras. Let B be a c.b.l.f.-dimensional (A', d')-(A, d)-bimodule. The derived tensor product functor*

$$F : \mathcal{D}^{\mathrm{lf}}(A, d) \to \mathcal{D}^{\mathrm{lf}}(A', d'), \quad F(X) := B \otimes^{\mathrm{L}}_{(A,d)} X,$$

induces a continuous map

$$[F] : \boldsymbol{K}_0^{\Delta}(\mathcal{D}^{\mathrm{lf}}(A, d)) \to \boldsymbol{K}_0^{\Delta}(\mathcal{D}^{\mathrm{lf}}(A', d')).$$

We will need the following definitions in Section 7.

Definition 6.6. Let $\{K_1, \dots, K_m\}$ be a finite collection of objects in $\mathcal{C}$, and let $\{E_r\}_{r\in\mathbb{N}}$ be a family of direct sums of $\{K_1, \dots, K_m\}$ such that $\bigoplus_{r\in\mathbb{N}} E_r$ is a c.b.l.f. direct sum of $\{K_1, \dots, K_m\}$. Let $\{M_r\}_{r\in\mathbb{N}}$ be a collection of objects in $\mathcal{C}$ with $M_0 = 0$, such that they fit in distinguished triangles

$$M_r \xrightarrow{f_r} M_{r+1} \to E_r \to \ .$$

Then we say that an object $M \in \mathcal{C}$ such that $M \cong_{\mathcal{T}} \mathrm{MColim}_{r\geq 0}(f_r)$ in $\mathcal{T}$ is a *c.b.l.f. iterated extension of* $\{K_1, \dots, K_m\}$.

Definition 6.7. We say that $\mathcal{V}$ is *c.b.l.f. generated by* $\{X_j\}_{j\in J}$ for some collection of elements $X_j \in \mathcal{V}$ if for any object Y in $\mathcal{V}$ we can take a finite set $\{Y_k\}_{k\in K}$ of retracts $Y_k \subset X_{j_k}$ such that Y is isomorphic to a c.b.l.f. iterated extension of $\{Y_k\}_{k\in K}$.

6.2. Categorification. In this subsection we assume that $R_{\mathfrak{b}}(\nu)$ is a $\Bbbk$-algebra over a field $\Bbbk$. We also choose an arbitrary order $\prec$ for constructing $\mathbb{Z}(\!(q,\Lambda)\!)$ such that $0 \prec q \prec \lambda_i$ for all formal $\lambda_i = q^{\beta_i} \in \Lambda$. We assume that the parabolic Verma module $M^{\mathfrak{p}}(\lambda, N)$ is constructed over the ground ring $R := \mathbb{Q}(\!(\Lambda, q)\!)$ (instead of $\mathbb{Q}(\Lambda, q)$).

Every idempotent of $R_{\mathfrak{b}}(\nu)$ is the image of an idempotent of the classical KLR algebra $R_{\mathfrak{g}}(\nu)$ under the obvious inclusion $R_{\mathfrak{g}}(\nu) \hookrightarrow R_{\mathfrak{b}}(\nu)$. Thanks to [23, Section 2.5] we know all the idempotents of $R_{\mathfrak{g}}(\nu)$. We define the element

$$e_{i,n} := \tau_{\vartheta_n} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} 1_{ii\cdots i} \in R_{\mathfrak{g}}(n),$$

where ϑ_n is the longest element in S_n. Let $\mathrm{Seqd}(\nu)$ be the set of expressions $i_1^{(m_1)} i_2^{(m_2)} \cdots i_r^{(m_r)}$ for different $r \in \mathbb{N}$ and $m_\ell \in \mathbb{N}$ such that $\sum_{\ell=1}^r m_\ell \cdot \alpha_{i_\ell} = \nu$. For each $\boldsymbol{i} \in \mathrm{Seqd}(\nu)$ we define the idempotent

$$e_{\boldsymbol{i}} := e_{i_1,m_1} \otimes e_{i_2,m_2} \otimes \cdots \otimes e_{i_r,m_r} \in R_{\mathfrak{g}}(\nu),$$

where $x \otimes y$ means we put the diagram of x at the left of the one of y. Identifying $e_{\boldsymbol{i}}$ with its image in $R_{\mathfrak{b}}(\nu)$, as in [23], we define a projective left $R_{\mathfrak{b}}(\nu)$-module

$$P_{\boldsymbol{i}} := R_{\mathfrak{b}}(\nu) e_{\boldsymbol{i}},$$

Then we put

$$\langle \boldsymbol{i} \rangle := -\sum_{\ell=1}^{r} \frac{m_\ell(m_\ell - 1)}{2} d_{i_a}.$$

and we define

$$\tilde{P}_{\boldsymbol{i}} := q^{-\langle \boldsymbol{i} \rangle} P_{\boldsymbol{i}}.$$

When writing $\dots \boldsymbol{i} \dots$ and $\dots \boldsymbol{j} \dots$ we mean we take two sequences $\boldsymbol{i}_1 \boldsymbol{i}\, \boldsymbol{i}_2$ and $\boldsymbol{i}_1 \boldsymbol{j}\, \boldsymbol{i}_2$ in $\mathrm{Seqd}(\nu)$ that coincide everywhere except on $\boldsymbol{i}$ and $\boldsymbol{j}$. From the decomposition of the nilHecke algebra [23, Section 2.2] we get an isomorphism of $R_{\mathfrak{p}}^N$-modules

$$\tilde{P}_{\dots i^m \dots} \cong \bigoplus_{([m]_{q_i}!)} \tilde{P}_{\dots i^{(m)} \dots}.$$

Mimicking the arguments in [23, Proposition 2.13] and [25, Proposition 6], we have the following result.

Proposition 6.8. *There are isomorphisms*

$$\bigoplus_{a=0}^{\lfloor \frac{d_{ij}+1}{2} \rfloor} \tilde{P}_{\dots i^{(2a)} j i^{(d_{ij}+1-2a)} \dots} \cong \bigoplus_{a=0}^{\lfloor \frac{d_{ij}}{2} \rfloor} \tilde{P}_{\dots i^{(2a+1)} j i^{(d_{ij}-2a)} \dots}$$

for all $i \neq j \in I$.

Equipping $R_{\mathfrak{b}}(\nu)$ with d_N induces a differential on $\tilde{P}_{\boldsymbol{i}}$, and Proposition 6.8 holds for the dg-version $(\tilde{P}_{\boldsymbol{i}}, d_N)$. We put

$$\mathcal{M}^{\mathfrak{p}}(\Lambda, N) := \bigoplus_{m \geq 0} \mathcal{D}^{\mathrm{lf}}(R_{\mathfrak{b}}(m), d_N),$$

with the particular case $\mathcal{M}(\Lambda)$ meaning $\mathfrak{p} = \mathfrak{b}$ and $N = \emptyset$, and therefore $d_N = 0$. Note that $\mathcal{D}^{\mathrm{lf}}(R_{\mathfrak{b}}(m), d_N) \cong \mathcal{D}^{\mathrm{lf}}(R_{\mathfrak{p}}^N(m), 0)$. Let ${}_{\mathbb{Q}}\boldsymbol{K}_0^{\Delta}(-) := \boldsymbol{K}_0^{\Delta}(-) \otimes_{\mathbb{Z}(\!(q,\Lambda)\!)} \mathbb{Q}(\!(q, \Lambda)\!)$.

Proposition 6.9. *The $\mathbb{Z}^{1+|\Lambda|}$-graded dg-algebra $(R_{\mathfrak{b}}(m), d_N)$ is a positive c.b.l.f.-dimensional dg-algebra.*

Proof. Clearly, $R_{\mathfrak{b}}(m)$ is c.b.l.f. dimensional for the $\mathbb{Z}^{1+|\Lambda|}$-grading, and is non-negative for the homological grading. We can also assume we have applied Remark 6.2. Recall that the part in homological degree zero of $R_{\mathfrak{b}}(m)$ is isomorphic to the usual KLR algebra $R_{\mathfrak{g}}(m)$. As explained in [23, Section 3.3], for each monomial $f \in U_q^-(\mathfrak{g})$, we have a projective $R_{\mathfrak{g}}(m)$-module P_f (defined similarly as $P_{\boldsymbol{i}}$ for $f = F_{i_1}^{(m_1)} \cdots F_{i_r}^{(m_r)}$). Moreover, by [23, Proposition 3.22] extended for any $\mathfrak{g}$, P_f is indecomposable if and only if f is a canonical basis element. Also, the quadratic form in [30, Section 14.2] corresponds with the graded dimension of the graded hom-spaces between these projective $R_{\mathfrak{g}}(m)$-modules. The same applies for the homological degree zero part of the graded hom-spaces between our projective $R_{\mathfrak{b}}(m)$-modules $P_{\boldsymbol{i}}$. Then, by [30, Theorem 14.2.3], we obtain that

$$\operatorname{gdim} \operatorname{HOM}_{R_{\mathfrak{b}}(m)}(P_{\boldsymbol{i}}, P_{\boldsymbol{j}}) - \delta_{\boldsymbol{i},\boldsymbol{j}} \in \mathbb{Z}_{\prec}^{+}[\![q, \Lambda]\!],$$

which concludes the proof. □

Proposition 6.10. *There is an isomorphism of $\mathbb{Q}(\!(q, \Lambda)\!)$-modules*

$$U_q^-(\mathfrak{g}) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(\!(q, \Lambda)\!) \cong {}_{\mathbb{Q}}\boldsymbol{K}_0^{\Delta}(\mathcal{M}(\Lambda)),$$

and a $\mathbb{Q}(\!(q, \Lambda)\!)$-linear surjection

$$U_q^-(\mathfrak{g}) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(\!(q, \Lambda)\!) \twoheadrightarrow {}_{\mathbb{Q}}\boldsymbol{K}_0^{\Delta}(\mathcal{M}^{\mathfrak{p}}(\Lambda, N)),$$

both sending $F_{i_1}^{(m_1)} F_{i_2}^{(m_2)} \cdots F_{i_r}^{(m_r)}$ to $[(\tilde{P}_{\boldsymbol{i}}, d_N)]$ for $\boldsymbol{i} = i_1^{(m_1)} i_2^{(m_2)} \cdots i_r^{(m_r)}$.

Proof. Since projective modules of $R_{\mathfrak{b}}(\nu)$ are in bijection with the ones of the classical KLR algebra $R_{\mathfrak{g}}(\nu)$ and respect the categorified Serre relations (see Proposition 6.8), both claims are a direct consequence of the main results in [23, 25], together with Theorem 6.4. □

Consider the subring $P(\nu)$ of $R_{\mathfrak{b}}(\nu)$ consisting of dots on vertical strands (without floating dots). It admits an action of the symmetric group permuting the strands (with labels) and dots on them (not to be confused with the action of S_m on P_ν from Section 3.2). We write $\mathrm{Sym}(\nu) := P(\nu)^{S_m}$ for the subring of invariants under this action. Clearly it lies in the center of $R_{\mathfrak{b}}(\nu)$ but this inclusion is strict (see [35] or [4] for a study of the center in the case of $\mathfrak{sl}_2$).

The supercenter of $R_{\mathfrak{b}}(\nu)$ contains $\mathrm{Sym}(\nu) \otimes \bigotimes_{i\in I} \bigwedge^\bullet \langle \tilde\omega_i^0, \dots, \tilde\omega_i^{\nu_i-1}\rangle$, where $\tilde\omega_i^a$ is a floating dot with subscript i, superscript a and placed in the rightmost region:

$\tilde\omega_i^a :=$  .

We conjecture that the supercenter contains no other elements.

Conjecture 6.11. There is an isomorphism of rings

$$Z(R_{\mathfrak{b}}(\nu)) \cong \mathrm{Sym}(\nu) \otimes \bigotimes_{i\in I} \bigwedge\nolimits^\bullet \langle \tilde\omega_i^0, \dots, \tilde\omega_i^{\nu_i-1}\rangle,$$

where $Z(R_{\mathfrak{b}}(\nu))$ is the supercenter of $R_{\mathfrak{b}}(\nu)$.

In general $R_{\mathfrak{p},\mu}(\nu)$ is not a free module over $\mathrm{Sym}(\nu) \otimes \bigotimes_{i\in I} \bigwedge^\bullet \langle \tilde\omega_i^0, \dots, \tilde\omega_i^{\nu_i-1}\rangle$, but we have the following.

Proposition 6.12. *The module $R_{\mathfrak{b}}(\nu)$ is a free module over* $\mathrm{Sym}(\nu)$ *of rank $2^m(m!)^2$.*

Proof. It follows from Theorem 3.16 and the fact $P(\nu)$ is a free module of rank $m!$ on $\mathrm{Sym}(\nu)$. □

Since $\mathrm{Sym}(\nu)$ lies in the center of $R_{\mathfrak{b}}(\nu)$, any simple $R_{\mathfrak{b}}(\nu)$-module is annihilated by $\mathrm{Sym}^+(\nu)$, where $\mathrm{Sym}^+(\nu)$ consists of the elements in $\mathrm{Sym}(\nu)$ with non-zero degree. In particular, a simple $R_{\mathfrak{b}}(\nu)$-module must be a finite-dimensional $R_{\mathfrak{b}}(\nu)/\mathrm{Sym}^+(\nu)R_{\mathfrak{b}}(\nu)$-module. Since $R_{\mathfrak{b}}(\nu)/\mathrm{Sym}^+(\nu)R_{\mathfrak{b}}(\nu)$ has finite dimension over $\Bbbk$, we only have finitely many simple modules, up to shift and isomorphism. For each $\boldsymbol{i} \in \mathrm{Seqd}(\nu)$ such that $P_{\boldsymbol{i}}$ is indecomposable, we let $S_{\boldsymbol{i}}$ be the unique simple quotient of $P_{\boldsymbol{i}}$. We put $\tilde S_{\boldsymbol{i}} := q^{-\langle \boldsymbol{i}\rangle} S_{\boldsymbol{i}}$. If $(P_{\boldsymbol{i}}, d_N)$ is not acyclic, then it lifts automatically to a dg-version $(\tilde S_{\boldsymbol{i}}, 0)$ because of Proposition 6.9.

By Lemma 4.5 and Proposition 5.15 we know that $\mathsf{E}_i\,\mathrm{Id}_\nu$ and $\mathsf{F}_i\,\mathrm{Id}_\nu$ are exact. Moreover, they respect the conditions of Proposition 6.5. Therefore, they induce maps

$$[\mathsf{E}_i\,\mathrm{Id}_\nu] : \boldsymbol{K}_0\big(\mathcal{D}^{\mathrm{lf}}(R_{\mathfrak{b}}(\nu), d_N)\big) \to \boldsymbol{K}_0^\Delta\big(\mathcal{D}^{\mathrm{lf}}(R_{\mathfrak{b}}(\nu - i), d_N)\big),$$
$$[\mathsf{F}_i\,\mathrm{Id}_\nu] : \boldsymbol{K}_0\big(\mathcal{D}^{\mathrm{lf}}(R_{\mathfrak{b}}(\nu), d_N)\big) \to \boldsymbol{K}_0^\Delta\big(\mathcal{D}^{\mathrm{lf}}(R_{\mathfrak{b}}(\nu + i), d_N)\big).$$

Theorems 4.4, 5.17 and Proposition 5.19 tell us that ${}_{\mathbb{Q}}\boldsymbol{K}_0^\Delta(\mathcal{M}^{\mathfrak{p}}(\Lambda, N))$ is an $U_q(\mathfrak{g})$-weight module. By Proposition 6.10 we know that ${}_{\mathbb{Q}}\boldsymbol{K}_0^\Delta(\mathcal{M}^{\mathfrak{p}}(\Lambda, N))$ is cyclic as $U_q(\mathfrak{g})$-module, with highest weight generator given by the class of $(R_{\mathfrak{b}}(0), d_N) \cong (\Bbbk, 0)$. Thus ${}_{\mathbb{Q}}\boldsymbol{K}_0^\Delta(\mathcal{M}^{\mathfrak{p}}(\Lambda, N))$ is a highest weight module.

As in the paper [23], let $\psi : R_{\mathfrak{b}}(\nu) \to R_{\mathfrak{b}}(\nu)^{\mathrm{op}}$ be the map that takes the mirror image of diagrams along the horizontal axis. Given a left $(R_{\mathfrak{b}}(\nu), d_N)$-module M, we obtain a right $(R_{\mathfrak{b}}(\nu), d_N)$-module M^ψ with action given by

$$m^\psi \cdot r := (-1)^{\deg_h(r)\deg_h(m)} \psi(r) \cdot m$$

for $m \in M$ and $r \in R_{\mathfrak{b}}(\nu)$. Then we define the bifunctor

$$(-,-) : \mathcal{M}^{\mathfrak{p}}(\Lambda, N) \times \mathcal{M}^{\mathfrak{p}}(\Lambda, N) \to \mathcal{D}^{\mathrm{lf}}(\Bbbk, 0), \quad (M, M') := M^{\psi} \otimes^{\mathrm{L}}_{(R_{\mathfrak{b}}, d_N)} M',$$

where $\otimes^{\mathrm{L}}$ is the derived tensor product.

Proposition 6.13. *The bifunctor defined above respects:*

- $((R_{\mathfrak{b}}(0), d_N), (R_{\mathfrak{b}}(0), d_N)) \cong (\Bbbk, 0)$,
- $(\mathrm{Ind}_m^{m+i} M, M') \cong (M, \mathrm{Res}_m^{m+i} M')$ *for all* $M, M' \in \mathcal{M}^{\mathfrak{p}}(\Lambda, N)$,
- $(\bigoplus_f M, M') \cong (M, \bigoplus_f M') \cong \bigoplus_f (M, M')$ *for all* $f \in \mathbb{Z}(\!(q, \Lambda)\!)$.

Proof. Straightforward. □

Comparing Proposition 6.13 with Definition 2.5, we deduce that $(-,-)$ is a categorification of the Shapovalov form on $\boldsymbol{K}_0^{\Delta}(\mathcal{M}^{\mathfrak{p}}(\Lambda, N))$. Moreover, it turns $\tilde{S}_{\boldsymbol{i}}$ into the dual of $\tilde{P}_{\boldsymbol{i}}$ for each $\boldsymbol{i} \in \mathrm{Seqd}(\nu)$ such that $\tilde{P}_{\boldsymbol{i}}$ is indecomposable. Recall $M^{\mathfrak{p}}(\Lambda, N)$ is the parabolic Verma module, and we assume $\Lambda = \{q^{\beta_i} \mid i \in I_r\}$ contains only formal weights.

Theorem 6.14. *The asymptotic Grothendieck group*

$$_{\mathbb{Q}}\boldsymbol{K}_0^{\Delta}(\mathcal{M}^{\mathfrak{p}}(\Lambda, N))$$

is a $U_q(\mathfrak{g})$*-weight module, with action of* E_i, F_i *given by* $[\mathsf{E}_i], [\mathsf{F}_i]$*. Moreover, there is an isomorphism of* $U_q(\mathfrak{g})$*-modules*

$$_{\mathbb{Q}}\boldsymbol{K}_0^{\Delta}(\mathcal{M}^{\mathfrak{p}}(\Lambda, N)) \cong M^{\mathfrak{p}}(\Lambda, N).$$

Proof. We already proved the first claim above. Because of Proposition 4.3, for $i \in I_f$, both $[\mathsf{F}_i]$ and $[\mathsf{E}_i]$ act as locally nilpotent operators. In particular, the $U_q(\mathfrak{l})$-submodule of $_{\mathbb{Q}}\boldsymbol{K}_0^{\Delta}(\mathcal{M}^{\mathfrak{p}}(\Lambda, N))$ given by

$$U_q(\mathfrak{l}) \otimes_{U_q(\mathfrak{g})} [(R_{\mathfrak{b}}(0), d_N)],$$

is an integrable module for the Levi factor $U_q(\mathfrak{l})$. Since it is an integrable cyclic weight module, it must be isomorphic to $V(\Lambda, N)$ (see [30]). Therefore, there is a surjective $U_q(\mathfrak{g})$-module morphism

$$\gamma : M^{\mathfrak{p}}(\Lambda, N) \twoheadrightarrow {}_{\mathbb{Q}}\boldsymbol{K}_0^{\Delta}(\mathcal{M}^{\mathfrak{p}}(\Lambda, N)).$$

Since $M^{\mathfrak{p}}(\Lambda, N)$ is irreducible and γ is non-zero, it must be an isomorphism. □

Let $m_r = F_{\boldsymbol{i}} v_{\Lambda,N}$ be an induced basis element of $M^{\mathfrak{p}}(\Lambda, N)$ with $\boldsymbol{i} \in \mathrm{Seq}(\nu)$. Then the isomorphism of Theorem 6.14 identifies m_r with the class $[(R_{\mathfrak{b}}(\nu), d_N)1_{\boldsymbol{i}}]$. Similarly, let $m'_s = F_{\boldsymbol{j}} v_{\Lambda,N}$ for $j \in \mathrm{Seqd}(\nu)$ be a canonical basis element, and let m^s be its dual in the dual canonical basis. Then the isomorphism identifies m'_s with $[(\tilde{P}_{\boldsymbol{j}}, d_N)]$ and m^s with $[(\tilde{S}_{\boldsymbol{j}}, d_N)]$. Moreover, computing the c.b.l.f. composition series of $\tilde{P}_{\boldsymbol{i}}$ (see [33, Section 7]) or taking a certain cofibrant replacement of $\tilde{S}_{\boldsymbol{i}}$ (see [33, Section 9]) gives a categorical version of the change of basis between canonical and dual canonical basis elements.

7. 2-Verma modules

Let $\Bbbk$ be a field of characteristic 0. Let $\mathcal{V} \in \text{dg-cat}_{\Bbbk}$ be a $\mathbb{Z}$-graded pretriangulated dg-category (see Definition A.23). Let $\mathcal{E}nd_{\text{Hqe}}(\mathcal{V}) := \mathcal{RH}om_{\text{Hqe}}(\mathcal{V}, \mathcal{V})$ be the dg-category of quasi-endofunctors on $\mathcal{V}$ (see Section A.5.1).

Remark 7.1. For example, $\mathcal{V}$ could be the dg-category $\mathcal{D}_{\text{dg}}(R, d)$ of cofibrant dg-modules over a dg-algebra (R, d) (see Definition A.15). Then the subcategory of $\mathcal{E}nd_{\text{Hqe}}(\mathcal{V})$ consisting of coproduct preserving quasi-functors would be given by the dg-enhanced derived category of dg-bimodules $\mathcal{D}_{\text{dg}}((R, d)^{\text{op}} \otimes (R, d))$ (see Theorem A.21).

Let $\mathsf{Q}_i := \bigoplus_{\ell \geq 0} q_i^{1+2\ell} \,\text{Id}$. It is a categorification of $q_i/(1-q_i^2) = 1/(q_i^{-1} - q_i)$. We start by introducing a notion of dg-categorical action and dg-2-representation.

Definition 7.2. A *weak dg-categorical $U_q(\mathfrak{g})$-action* on $\mathcal{V}$ is a collection of quasi-endofunctors $\mathsf{F}_i, \mathsf{E}_i, \mathsf{K}_\gamma \in Z^0(\mathcal{E}nd_{\text{Hqe}}(\mathcal{V}))$ for all $i \in I$ and $\gamma \in Y^\vee$ such that

- there are isomorphisms
$$\mathsf{K}_0 \cong \text{Id}, \quad \mathsf{K}_\gamma \mathsf{K}_{\gamma'} \cong \mathsf{K}_{\gamma+\gamma'}, \quad \mathsf{K}_\gamma \mathsf{E}_i \cong q^{\gamma(\alpha_i)} \mathsf{E}_i \mathsf{K}_\gamma, \quad \mathsf{K}_\gamma \mathsf{F}_i \cong q^{-\gamma(\alpha_i)} \mathsf{F}_i \mathsf{K}_\gamma,$$
where q denotes the shift in the q-grading,
- there is a quasi-isomorphism
$$\text{Cone}\big(\mathsf{F}_i \mathsf{E}_j \xrightarrow{u_{ij}} \mathsf{E}_j \mathsf{F}_i\big) \xrightarrow{\simeq} \delta_{ij}\, \text{Cone}\big(\mathsf{Q}_i \mathsf{K}_i \xrightarrow{h_i} \mathsf{Q}_i \mathsf{K}_i^{-1}\big), \tag{7.1}$$
where $\mathsf{K}_i := \mathsf{K}_{\alpha_i^\vee}$,
- there are isomorphisms
$$\bigoplus_{a=0}^{\lfloor \frac{d_{ij}+1}{2} \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a \end{bmatrix}_{q_i} \mathsf{F}_i^{2a} \mathsf{F}_j \mathsf{F}_i^{d_{ij}+1-2a} \cong \bigoplus_{a=0}^{\lfloor \frac{d_{ij}}{2} \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a+1 \end{bmatrix}_{q_i} \mathsf{F}_i^{2a+1} \mathsf{F}_j \mathsf{F}_i^{d_{ij}-2a},$$
$$\bigoplus_{a=0}^{\lfloor \frac{d_{ij}+1}{2} \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a \end{bmatrix}_{q_i} \mathsf{E}_i^{2a} \mathsf{E}_j \mathsf{E}_i^{d_{ij}+1-2a} \cong \bigoplus_{a=0}^{\lfloor \frac{d_{ij}}{2} \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a+1 \end{bmatrix}_{q_i} \mathsf{E}_i^{2a+1} \mathsf{E}_j \mathsf{E}_i^{d_{ij}-2a},$$
for all $i \neq j \in I$.

We say a weak dg-categorical $U_q(\mathfrak{g})$-action on $\mathcal{V}$ is a *dg-categorical action* if in addition

- F_i is left adjoint to $q_i^{-1} \mathsf{K}_i \mathsf{E}_i$ in $Z^0(\mathcal{E}nd_{\text{Hqe}}(\mathcal{V}))$,
- there is a map of algebras
$$R_{\mathfrak{g}}(\boldsymbol{i}) \to Z^0(\text{END}(\mathsf{F}_{\boldsymbol{i}})) := \bigoplus_{z \in \mathbb{Z}} Z^0(\text{Hom}(\mathsf{F}_{\boldsymbol{i}}, q^z \mathsf{F}_{\boldsymbol{i}}))$$
with $\mathsf{F}_{\boldsymbol{i}} := \mathsf{F}_{i_1} \cdots \mathsf{F}_{i_m}$, for all $\boldsymbol{i} \in \text{Seq}(m)$, inducing a surjection
$$R_{\mathfrak{g}}(\boldsymbol{i}) \otimes_{\Bbbk} Z^0(\text{END}_{\mathcal{V}}(M)) \twoheadrightarrow Z^0(\text{END}_{\mathcal{V}}(\mathsf{F}_{\boldsymbol{i}} M))$$
for all $M \in \mathcal{V}$,
- $\mathcal{V}$ is dg-triangulated (i.e. $H^0(\mathcal{V})$ is idempotent complete).

Such a $\mathcal{V}$ carrying a dg-categorical action is called a *dg-2-representation* of $U_q(\mathfrak{g})$.

The following notions are dg-2-categorical lifts of the notions of weight module and integrable module.

Definition 7.3. We say that a dg-2-representation $\mathcal{V}$ is a *weight dg-2-representation* if there is a map

$$\lambda : Y^\vee \to \mathcal{E}nd_{\mathrm{Hqe}}(\mathcal{V}),$$

where $\lambda(\gamma)$ commutes with the grading shift q for all $\gamma \in Y^\vee$ and $\lambda(\gamma) \circ \lambda(\gamma') \cong \lambda(\gamma + \gamma')$ such that

$$\mathcal{V} \cong \bigoplus_{y \in Y} \mathcal{V}_{\lambda,y}, \quad \mathsf{K}_\gamma|_{\mathcal{V}_{\lambda,y}}(-) = \lambda(\gamma) q^{\gamma(y)}(-).$$

Definition 7.4. We say that a weight dg-2-representation $\mathcal{V}$ is *i-integrable* if

- $\lambda(\alpha_i^\vee) = q^{n_i}$ for some $n_i \in \mathbb{N}$,
- there is a quasi-isomorphism

$$\operatorname{Cone}\big(\mathsf{Q}_i \mathsf{K}_i \mathcal{V}_{\lambda,y} \xrightarrow{h_i} \mathsf{Q}_i \mathsf{K}_i^{-1} \mathcal{V}_{\lambda,y}\big) \xrightarrow{\simeq} \bigoplus_{[n_i - \alpha_i^\vee(y)]_{q_i}} \mathrm{Id}, \tag{7.2}$$

  where $\bigoplus_{[m]_{q_i}} \mathrm{Id} := \bigoplus_{[-m]_{q_i}} \mathrm{Id}[1]$ whenever $m < 0$,
- F_i and E_i are locally nilpotent.

Under some mild hypothesis, this definition recovers the notion of integrable 2-representation from [39] and [11].

Proposition 7.5. *Suppose $\mathcal{V}$ is i-integrable for all $i \in I$. Also suppose that there is some $M \in \mathcal{V}_{\lambda,0}$ such that $\mathsf{E}_i M = 0$ for all $i \in I$ and $\mathrm{End}_{\mathcal{V}}(M) \cong (\Bbbk, 0)$, and $H^0(\mathcal{V})$ is c.b.l.f. generated by $\{\mathsf{F}_{\boldsymbol{i}} M\}_{\boldsymbol{i} \in \mathrm{Seq}(I)}$. Then $H^0(\mathcal{V})$ carries an integrable categorical $U_q(\mathfrak{g})$-action in the sense of* [39].

Proof. First, by adjunction, equations (7.1) and (7.2), we have

$$\operatorname{gdim}_q H^0(\mathrm{END}_{\mathcal{V}}(\mathsf{F}_{\boldsymbol{i}} M)) \cong \operatorname{gdim}_q R_{\mathfrak{g}}^N(\boldsymbol{i})$$

for all $\boldsymbol{i} \in \mathrm{Seq}(I)$. In particular, we have that $x_1^{n_i} 1_i$ acts by 0 on $H^0(\mathrm{End}_{\mathcal{V}}(\mathsf{F}_i M))$ for all $i \in I$, and $x_1 1_i$ acts non-trivially whenever $n_i > 1$. Thus, there is a map

$$\gamma : R_{\mathfrak{g}}^N(\boldsymbol{i}) \to H^0(\mathrm{END}_{\mathcal{V}}(\mathsf{F}_{\boldsymbol{i}} M)).$$

Since γ is surjective, we obtain

$$R_{\mathfrak{g}}^N(\boldsymbol{i}) \cong H^0(\mathrm{END}_{\mathcal{V}}(\mathsf{F}_{\boldsymbol{i}} M)),$$

and the result follows from Theorem 5.17. □

For a $\mathbb{Z}^n$-graded dg-algebra (A, d), we put $\mathcal{D}_{\mathrm{dg}}^{\mathrm{lf}}(A, d)$ for the dg-category having as objects the one in $\mathcal{D}^{\mathrm{lf}}(A, d) \cap \mathcal{D}_{\mathrm{dg}}(A, d)$ and the hom-spaces inherited from $\mathcal{D}_{\mathrm{dg}}(A, d)$. It is a dg-enhancement of the c.b.l.f. derived category of (A, d).

Definition 7.6. A *parabolic* 2*-Verma module* for $\mathfrak{p}$ is a $\mathbb{Z} \times \mathbb{Z}^{|I_r|}$-graded weight dg-2-representation $\mathcal{V}$ such that

- the highest weight space $\mathcal{V}_\lambda := \mathcal{V}_{\lambda,\mathbf{0}} \cong \mathcal{D}^{\mathrm{lf}}_{\mathrm{dg}}(\Bbbk, 0)$,
- there exists $M \cong (\Bbbk, 0) \in \mathcal{V}_\lambda$ such that $\mathcal{V}_{\lambda,y}$ is c.b.l.f. generated by $\{F_{\boldsymbol{i}} M\}_{\boldsymbol{i} \in \mathrm{Seq}(y)}$ for all $-y \in X^+$, and $\mathcal{V}_{\lambda,y} = 0$ otherwise,
- $\mathcal{V}$ is i-integrable for all $i \in I_f$,
- $h_j = 0$ and $\lambda_{\alpha_j^\vee} = \lambda_j$ (the degree shift) for all $j \notin I_f$,
- for each $j \notin I_f$, $n_j \in \mathbb{N}$ and $\boldsymbol{i} \in \mathrm{Seq}(I)$, after specializing $\lambda_j = q^{n_j}$, there exists a differential d_{n_j} anticommuting with the differential d of $\big(\mathrm{End}_{\mathcal{V}}(F_{\boldsymbol{i}} M), d\big)$ such that the triangulated dg-category generated by c.b.l.f. iterated extension of the representable modules of $\mathcal{V}^{n_j} := \bigoplus_{\boldsymbol{i} \in \mathrm{Seq}(I)} (\mathrm{End}_{\mathcal{V}}(F_{\boldsymbol{i}} M), d + d_{n_j})$ is j-integrable with $\lambda(\alpha_j^\vee) = q^{n_j}$.

Proposition 7.7. *Let $\mathcal{V}$ be a parabolic* 2*-Verma module. There is an isomorphism*

$$(R_{\mathfrak{b}}(\boldsymbol{i}), d_N) \cong \mathrm{END}_{\mathcal{V}}(\mathsf{F}_{\boldsymbol{i}} M)$$

in $\mathcal{D}(\Bbbk, 0)$ for $M \cong (\Bbbk, 0) \in \mathcal{V}_\lambda$.

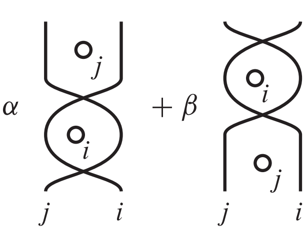
Proof. First, by adjunction together with equations (7.1) and (7.2) we have

$$\mathrm{END}_{\mathcal{V}}(\mathsf{F}_i M) \cong \mathrm{HOM}_{\mathcal{V}}(M, q_i^{-1}\mathsf{K}_i \mathsf{E}_i \mathsf{F}_i M) \cong R^N_{\mathfrak{p}}(i) \tag{7.3}$$

in $\mathcal{D}(\Bbbk, 0)$ for all $i \in I$. Also,

$$\mathrm{gdim}_q\, H^*(\mathrm{END}_{\mathcal{V}}(\mathsf{F}_{\boldsymbol{i}} M)) = \mathrm{gdim}_q\, R^N_{\mathfrak{p}}(\boldsymbol{i}) \tag{7.4}$$

for all $\boldsymbol{i} \in \mathrm{Seq}(I)$. In particular, there is a relation up to homotopy

(7.5) α  $+\,\beta$ $= 0$

in $\mathrm{END}_{\mathcal{V}}(\mathsf{F}_i \mathsf{F}_j M)$ for all $i, j \in I_r$, identifying the diagrams with the image of the KLR elements under the surjection $R_{\mathfrak{g}}(ij) \twoheadrightarrow Z^0(\mathrm{END}_{\mathcal{V}}(F_i F_j M))$, and the floating dot coming from the isomorphism (7.3). Then the existence of d_{n_i} and d_{n_j} forces to have $\alpha = \beta$. Thus, by Corollary 3.17, there is an A_∞-map

$$(R_{\mathfrak{b}}(\boldsymbol{i}), d_N) \to \mathrm{END}_{\mathcal{V}}(\mathsf{F}_{\boldsymbol{i}} M).$$

By equation (7.4), we conclude it is a quasi-isomorphism. Thus, there exists an isomorphism $(R_{\mathfrak{b}}(\boldsymbol{i}), d_N) \cong \mathrm{END}_{\mathcal{V}}(\mathsf{F}_{\boldsymbol{i}} M)$ in $\mathcal{D}(\Bbbk, 0)$. □

Using Theorem A.21, we can think of F_i^N and E_i^N from Section 5.4 as quasi-endofunctors of $\mathcal{D}_{\mathrm{dg}}(R_{\mathfrak{b}}, d_n)$. By Proposition 5.5 we obtain immediately the following.

Corollary 7.8. *For all $i \in I$ there is a quasi-isomorphism of cones*

$$\mathrm{Cone}\big(\mathsf{F}_i^N \mathsf{E}_i^N \,\mathrm{Id}_\nu \to \mathsf{E}_i^N \mathsf{F}_i^N \,\mathrm{Id}_\nu\big) \xrightarrow{\simeq} \mathrm{Cone}\big(\mathsf{Q}_i \lambda_i q_i^{-\alpha_i^\vee(\nu)} \,\mathrm{Id}_\nu \to \mathsf{Q}_i \lambda_i^{-1} q^{\alpha_i^\vee(\nu)} \,\mathrm{Id}_\nu\big)$$

in $\mathcal{E}nd_{\mathrm{Hqe}}(\mathcal{D}_{\mathrm{dg}}(R_{\mathfrak{b}}, d_N))$.

Together with Proposition 5.19, it means that the dg-enhancement $\mathcal{M}^{\mathfrak{p}}_{\mathrm{dg}}(\Lambda, N)$ of the category $\mathcal{M}^{\mathfrak{p}}(\Lambda, N)$ (obtained by replacing $\mathcal{D}^{\mathrm{lf}}(R_{\mathfrak{b}}(m), d_N)$ with $\mathcal{D}^{\mathrm{lf}}_{\mathrm{dg}}(R_{\mathfrak{b}}(m), d_N)$) is a weight dg-2-representation of $U_q(\mathfrak{g})$, where

$$\lambda(\alpha_i^\vee) := \begin{cases} \lambda_i & \text{whenever } i \in I_r, \\ q^{n_i} & \text{whenever } i \in I_f. \end{cases}$$

Then, by Theorem 5.17, we obtain that $\mathcal{M}^{\mathfrak{p}}_{\mathrm{dg}}(\Lambda, N)$ is a parabolic 2-Verma module.

Corollary 7.9. *Let $\mathcal{V}$ be a parabolic 2-Verma module. There is a quasi-equivalence*

$$\mathcal{M}^{\mathfrak{p}}_{\mathrm{dg}}(\Lambda, N) \xrightarrow{\simeq} \mathcal{V}.$$

Proof. Since $\mathcal{V}_{\lambda,y}$ is c.b.l.f. generated by $\bigoplus_{\boldsymbol{i} \in \mathrm{Seq}(y)} F_{\boldsymbol{i}} M$, we have that $\mathcal{V}_{\lambda,y}$ is completely determined as dg-category by $\mathrm{END}_{\mathcal{V}}(F_{\boldsymbol{i}} M)$. Thus, we conclude by using Proposition 7.7. □

Remark 7.10. A parabolic 2-Verma module can also be given a "2-categorical" interpretation as an $(\infty, 2)$-category where the hom-spaces are stable $(\infty, 1)$-categories. For this, it is enough to see $\mathcal{D}_{\mathrm{dg}}(R_{\mathfrak{b}}(\nu), d_N)$ as 0-cells in the $(\infty, 2)$-category of A_∞-categories constructed by Faonte [14], and replace $\mathcal{H}om_{\mathrm{Hqe}}$ by the dg-nerve of Lurie [29]. Thanks to [15], we know that this is a stable $(\infty, 1)$-category.

A. Summary on the homotopy category of dg-categories and pretriangulated dg-categories

We gather some useful results on the homotopy category of dg-categories. References for this section are [21] and [42]. We also suggest [22] and [43] for nice surveys on the subject.

Our goal is to recall how to construct a category of dg-categories up to quasi-equivalence, so that the space of functors between two "triangulated categories" is "triangulated".

A.1. Dg-categories.

Recall the definition of a dg-category:

Definition A.1. A *dg-category* $\mathcal{A}$ is a $\Bbbk$-linear category such that

- $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is a $\mathbb{Z}$-graded $\Bbbk$-vector space,
- the composition

$$\mathrm{Hom}_{\mathcal{A}}(Y, Z) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{A}}(X, Y) \xrightarrow{-\circ-} \mathrm{Hom}_{\mathcal{A}}(X, Z),$$

preserves the $\mathbb{Z}$-degree,

- there is a differential $d : \mathrm{Hom}_{\mathcal{A}}(X,Y)^i \to \mathrm{Hom}_{\mathcal{A}}(X,Y)^{i-1}$ such that

$$d^2 = 0, \quad d(f \circ g) = df \circ g + (-1)^{|f|} f \circ dg.$$

Remark A.2. We use a differential of degree -1 to match the conventions used in the rest of the paper.

Example A.3. Any dg-algebra (A,d) can be seen as a dg-category $\boldsymbol{BA}$ with a single abstract object $\star$ and $\mathrm{Hom}_{\boldsymbol{BA}}(\star,\star) := (A,d)$.

Example A.4. Let $\mathcal{C}$ be an abelian, Grothendieck, $\Bbbk$-linear category. Consider the category $C(\mathcal{C})$ of complexes in $\mathcal{C}$, and define $C_{\mathrm{dg}}(\mathcal{C})$ as the category, where

- objects are complexes in $\mathcal{C}$,
- hom-spaces are homogeneous maps of $\mathbb{Z}$-graded modules,
- the differential $d : \mathrm{Hom}_{C_{\mathrm{dg}}(\mathcal{C})}(X^\bullet, Y^\bullet)^i \to \mathrm{Hom}_{C_{\mathrm{dg}}(\mathcal{C})}(X^\bullet, Y^\bullet)^{i-1}$ is given by

$$df := d_Y \circ f - (-1)^{|f|} f \circ d_X.$$

This data forms a dg-category.

Given a dg-category $\mathcal{A}$, one defines

(i) the *underlying category* $Z^0(\mathcal{A})$ as
  - having the same objects as $\mathcal{A}$,
  - $\mathrm{Hom}_{Z^0(\mathcal{A})}(X,Y) := \ker(\mathrm{Hom}_{\mathcal{A}}(X,Y)^0 \xrightarrow{d} \mathrm{Hom}_{\mathcal{A}}(X,Y)^{-1})$,

(ii) the *homotopy category* $H^0(\mathcal{A})$ (or $[\mathcal{A}]$) as
  - having the same objects as $\mathcal{A}$,
  - $\mathrm{Hom}_{H^0(\mathcal{A})}(X,Y) := H^0(\mathrm{Hom}_{\mathcal{A}}(X,Y), d)$.

Example A.5. For $\mathcal{C}$ as in Example A.4, we have

$$Z^0(C_{\mathrm{dg}}(\mathcal{C})) \cong C(\mathcal{C}) \quad \text{and} \quad H^0(C_{\mathrm{dg}}(\mathcal{C})) \cong \mathrm{Kom}(\mathcal{C})$$

the homotopy category of complexes in $\mathcal{C}$.

A.2. Category of dg-categories.

Definition A.6. A *dg-functor* $F : \mathcal{A} \to \mathcal{B}$ is a functor between two dg-categories such that $F(d_{\mathcal{A}} f) = d_{\mathcal{B}}(Ff)$. We write $[F] : H^0(\mathcal{A}) \to H^0(\mathcal{B})$ for the induced functor.

We write dg-cat for the *category of dg-categories*, where objects are dg-categories and hom-spaces are given by dg-functors.

Let $F, G : \mathcal{A} \to \mathcal{B}$ be a pair of dg-functors between dg-categories. Then one defines $\mathcal{H}om(F,G)$ as the $\mathbb{Z}$-graded $\Bbbk$-module of homogeneous natural transformations equipped with the differential induced by $d \in \mathrm{Hom}_{\mathcal{B}}(FX, GX)$ for all $X \in \mathcal{A}$. Then we put

$$\mathrm{Hom}(F,G) := Z^0(\mathcal{H}om(F,G)).$$

Definition A.7. A dg-functor $\mathcal{A} \to \mathcal{B}$ is a *quasi-equivalence* if

- $F : \mathrm{Hom}_{\mathcal{A}}(X, Y) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{B}}(FX, FY)$ is a quasi-isomorphism for all $X, Y \in \mathcal{A}$,
- $[F] : H^0(\mathcal{A}) \to H^0(\mathcal{B})$ is essentially surjective (thus an equivalence).

One defines the dg-category $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ of dg-functors between $\mathcal{A}$ and $\mathcal{B}$ as

- objects are dg-functors $\mathcal{A} \to \mathcal{B}$,
- hom-spaces are $\mathrm{Hom}_{\mathcal{H}om(\mathcal{A},\mathcal{B})}(F, G) := \mathcal{H}om(F, G)$.

There is also a notion of tensor product of dg-categories $\mathcal{A} \otimes \mathcal{B}$ defined as

- objects are pairs $X \otimes Y$ for all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$,
- hom-spaces are $\mathrm{Hom}_{\mathcal{A}\otimes\mathcal{B}}(X \otimes Y, X' \otimes Y') := \mathrm{Hom}_{\mathcal{A}}(X, X') \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{B}}(Y, Y')$ with composition
$$(f' \otimes g') \circ (f \otimes g) := (-1)^{|g'||f|}(f' \circ f) \otimes (g' \circ g),$$
- the differential is $d(f \otimes g) := df \otimes g + (-1)^{|f|} f \otimes dg$.

Then there is a bijection
$$\mathrm{Hom}_{\text{dg-cat}}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong \mathrm{Hom}_{\text{dg-cat}}(\mathcal{A}, \mathcal{H}om(\mathcal{B}, \mathcal{C})).$$

This defines a symmetric closed monoidal structure on dg-cat. However, the tensor product of dg-categories does not preserve quasi-equivalences.

A.3. Dg-modules. Let $\mathcal{A}$ be a dg-category. The opposite dg-category $\mathcal{A}^{\mathrm{op}}$ is given by

- same objects as in $\mathcal{A}$,
- $\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(X, Y) := \mathrm{Hom}_{\mathcal{A}}(Y, X)$,
- composition $g \circ_{\mathcal{A}^{\mathrm{op}}} f := (-1)^{|f||g|} f \circ_{\mathcal{A}} g$.

A *left (resp. right) dg-module* M over $\mathcal{A}$ is a dg-functor
$$M : \mathcal{A} \to C_{\mathrm{dg}}(\Bbbk) \quad (\text{resp. } N : \mathcal{A}^{\mathrm{op}} \to C_{\mathrm{dg}}),$$
where $C_{\mathrm{dg}}(\Bbbk)$ is the dg-category of $\Bbbk$-complexes. The *dg-category of (right) dg-modules* is $\mathcal{A}^{\mathrm{op}}\text{-mod} := \mathcal{H}om(\mathcal{A}^{\mathrm{op}}, C_{\mathrm{dg}}(\Bbbk))$. The *category of (right) dg-modules* is $C(\mathcal{A}) := Z^0(\mathcal{A}\text{-mod})$, and it is an abelian category. The *derived category* $\mathcal{D}(\mathcal{A})$ is the localization of $Z^0(\mathcal{A}^{\mathrm{op}}\text{-mod})$ along quasi-isomorphisms.

Moreover, for any $X \in \mathcal{A}$ there is a right dg-module
$$X^\wedge := \mathrm{Hom}_{\mathcal{A}}(-, X).$$

One calls such dg-module *representable*. Any dg-module quasi-isomorphic to a representable dg-module is called *quasi-representable*. It yields a dg-enriched Yoneda embedding
$$\mathcal{A} \to \mathcal{A}^{\mathrm{op}}\text{-mod}.$$

Example A.8. Let (A, d) be a dg-algebra. Then
$$Z^0(\boldsymbol{BA})\text{-mod} \cong (A, d)\text{-mod} \quad \text{and} \quad \mathcal{D}(\boldsymbol{BA}) \cong \mathcal{D}(A, d).$$

The unique representable dg-module $\mathrm{Hom}_{\boldsymbol{BA}}(-, \star)$ is equivalent to the free module (A, d).

A.4. Model categories. We recall the basics of model category theory from [17]. Model category theory is a powerful tool to study localization of categories. For example, we can use it to compute hom-spaces in a derived category. We will mainly use it to describe the homotopy category of dg-categories up to quasi-equivalence.

Let M be a category with limits and colimits.

Definition A.9. A *model category* on M is the data of three classes of morphisms

- the *weak equivalences* W,
- the *fibrations* Fib,
- the *cofibrations* Cof

satisfying

- for $X \xrightarrow{f} Y \xrightarrow{g} Z \in M$, if two out of three terms in $\{f, g, g \circ f\}$ are in W, then so is the third,
- *stability along retracts:* W, Fib and Cof are stable along retracts, that is if we have a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"] \arrow[rr, bend left, "\mathrm{Id}_X"] & Y \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ X' \arrow[r] \arrow[rr, bend right, "\mathrm{Id}_{X'}"'] & Y' \arrow[r] & X' \end{tikzcd}$$

  and $f \in W$, Fib or Cof then so is g,
- *factorization:* any map $X \xrightarrow{f} Y$ factorizes as $p \circ i$, where $p \in$ Fib and $i \in$ Cof $\cap\, W$ or $p \in$ Fib $\cap\, W$ and $i \in$ Cof, and the factorization is functorial in f,
- *lifting property:* given a commutative square diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{Cof} \ni i"'] & X \arrow[d, "p \in \mathrm{Fib}"] \\ B \arrow[r] \arrow[ur, dashed, "\exists h"] & Y \end{tikzcd}$$

  with $i \in$ Cof and $p \in$ Fib, if either $i \in W$ or $p \in W$, then there exists $h : B \to X$ making the diagram commute.

We tend to think about fibrations as "nicely behaved surjections", and cofibrations as "nicely behaved injections".

The localization $\mathrm{Ho}(M) := W^{-1}M$ of M along weak equivalences is called the *homotopy category of M*. It has a nice description in terms of *homotopy classes* of maps between *fibrant* and *cofibrant* objects.

Definition A.10. If $\emptyset \to X \in$ Cof, then we say X is *cofibrant*. If $Y \to * \in$ Fib, then Y is *fibrant*.

One says that $f \sim g$, that is $f : X \to Y$ is *homotopy equivalent* to $g : X \to Y$, if there is a commutative diagram

$$\begin{array}{ccc} & \overset{\mathrm{Id}_X}{\circlearrowleft} & \\ & X & \\ {\scriptstyle \mathrm{Fib}\cap W \ni p} \uparrow & \downarrow {\scriptstyle i} & \searrow^{f} \\ & C(X) & \xrightarrow{h} Y, \\ {\scriptstyle p} \downarrow & \uparrow {\scriptstyle j} & \nearrow_{g} \\ & X & \\ & \underset{\mathrm{Id}_X}{\circlearrowright} & \end{array}$$

where $i \sqcup j : X \sqcup X \to C(X) \in \mathrm{Cof}$. One calls $C(X)$ the *cylinder object of* X. When X is cofibrant and Y fibrant, then $\sim$ is an equivalence relation on $\mathrm{Hom}_M(X, Y)$. Moreover, we have

$$\mathrm{Hom}_{\mathrm{Ho}(M)}(X, Y) \cong \mathrm{Hom}_M(X, Y)/\sim$$

whenever X is cofibrant and Y fibrant. Note that any $X \in M$ admits a cofibrant replacement QX since we have a commutative diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & X. \\ {\scriptstyle \mathrm{Cof} \ni i} \searrow & & \nearrow {\scriptstyle p \in \mathrm{Fib}\cap W} \\ & QX & \end{array}$$

Similarly, any $Y \in M$ admits a fibrant replacement RY.

Let M^{cf} be the full subcategory of M given by objects that are both fibrant and cofibrant. Let $M^{\mathrm{cf}}/\sim$ be the quotient of M^{cf} by identifying maps that are homotopy equivalent. Then the localization functor $M \to \mathrm{Ho}(M)$ restricts to M^{cf}, inducing an equivalence of categories

$$M^{\mathrm{cf}}/\sim \xrightarrow{\simeq} \mathrm{Ho}(M).$$

Example A.11. Let $C(\Bbbk)$ be the category of complexes of $\Bbbk$-modules. It comes with a model category structure where W is the quasi-isomorphisms, Fib is the surjective maps, and Cof is given by the maps respecting the lifting property. All objects are fibrant and the cofibrant objects are essentially the complexes of projective $\Bbbk$-modules. Then $\mathrm{Ho}(C(\Bbbk)) \cong \mathcal{D}(\Bbbk)$.

A model category on M is a $C(\Bbbk)$-*model category* if it is (strongly) enriched over $C(\Bbbk)$, and the models are compatible (see [43, Section 3.1] for a precise definition). This definition means that we have

- a tensor product $- \otimes - : C(\Bbbk) \times M \to M$,
- an enriched dg-hom-space $\mathcal{H}om_M(X, Y) \in C(\Bbbk)$ for any $X, Y \in M$ compatible with the tensor product:

$$\mathrm{Hom}_M(E \otimes X, Y) \cong \mathrm{Hom}_{C(\Bbbk)}(E, \mathcal{H}om_M(X, Y)),$$

- $\mathrm{Ho}(M)$ is enriched over $\mathcal{D}(\Bbbk) \cong \mathrm{Ho}(C(\Bbbk))$,

- a derived hom-functor
$$\mathcal{R}\mathcal{H}om_M(X,Y) := \mathcal{H}om_M(QX, RY) \in \mathcal{D}(\Bbbk),$$
where QX is a cofibrant replacement of X, and RY a fibrant replacement of Y,
- $\mathrm{Hom}_{\mathrm{Ho}(M)}(X,Y) \cong H^0(\mathcal{R}\mathcal{H}om_M(X,Y))$.

Note that in particular for $X, Y \in M^{\mathrm{cf}}$ we have $\mathrm{Hom}_{\mathrm{Ho}(M)}(X,Y) \cong H^0(\mathcal{H}om(X,Y))$.

Example A.12. Let $\mathcal{A}$ be a dg-category. There is a $C(\Bbbk)$-model category on $\mathcal{A}$-mod, where W is given by the quasi-isomorphisms, Fib are the surjective morphisms, and Cof is given by the maps respecting the lifting property. Then $\mathrm{Ho}(\mathcal{A}\text{-mod}) \cong \mathcal{D}(\mathcal{A})$.

Remark A.13. In the $C(\Bbbk)$-model category $\mathcal{A}$-mod, all objects are fibrant. Moreover, P is cofibrant if and only if for all surjective quasi-isomorphism $f : L \xrightarrow{\simeq} X$ (i.e. map in $W \cap \mathrm{Fib}$) then there exists $h : P \to L$ such that the following diagram commutes:

$$\begin{array}{ccc} \emptyset & \longrightarrow & L \\ \downarrow & \overset{\exists h}{\nearrow} & \downarrow \wr \\ P & \longrightarrow & X. \end{array}$$

Note that, in a practical way, cofibrant dg-modules are quasi-isomorphic to direct summand of dg-modules admitting a (possibly infinite) exhaustive filtration where all the quotients are free dg-modules.

Definition A.14. For M a $C(\Bbbk)$-model category, let $\underline{M}$ (resp. $\mathrm{Int}(M)$) be the dg-category with

- the same objects as M (resp. M^{cf}),
- $\mathrm{Hom}_{\underline{M}}(X,Y) := \mathcal{H}om_M(X,Y)$.

Then we have $H^0(\mathrm{Int}(M)) \cong \mathrm{Ho}(M)$, and we say that $\mathrm{Int}(M)$ is a *dg-enhancement* of $\mathrm{Ho}(M)$.

Definition A.15. We write
$$\mathcal{D}_{\mathrm{dg}}(\mathcal{A}) := \mathrm{Int}(\mathcal{A}\text{-mod})$$
for the *dg-enhanced derived category of* $\mathcal{A}$.

Note that $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ is a dg-enhancement of $\mathcal{D}(\mathcal{A})$ since we have $H^0(\mathcal{D}_{\mathrm{dg}}(\mathcal{A})) \cong \mathcal{D}(\mathcal{A})$.

Example A.16. Let R be a $\Bbbk$-algebra viewed as a dg-category with trivial differential. Then we have that $\mathcal{D}_{\mathrm{dg}}(R)$ is the dg-category of complexes of projective R-modules.

A.5. The model category of dg-categories. Let W be the collection of quasi-equivalences in dg-cat. Let Fib be the collection of dg-functors $F : \mathcal{A} \to \mathcal{B}$ in dg-cat such that

(i) $F_{X,Y} : \mathrm{Hom}_{\mathcal{A}}(X,Y) \twoheadrightarrow \mathrm{Hom}_{\mathcal{B}}(FX, FY)$ is surjective,

(ii) for every isomorphism $v : F(X) \xrightarrow{\simeq} Y \in H^0(\mathcal{B})$ there exists an isomorphism

$$u : X \xrightarrow{\simeq} Y_0 \in H^0(\mathcal{A})$$

such that $[F](u) = v$.

This defines a model structure on dg-cat where everything is fibrant. One calls

$$\mathrm{Hqe} := \mathrm{Ho}(\text{dg-cat})$$

the *homotopy category of dg-categories* (up to quasi-equivalence).

How can we compute $\mathrm{Hom}_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{B})$? It appears that constructing a cofibrant replacement for $\mathcal{A}$ is in general a difficult problem. However, we can do the following:

(i) replace $\mathcal{A}$ by a $\Bbbk$-*flat* quasi-equivalent dg-category $\mathcal{A}'$: meaning it is such that

$$\mathrm{Hom}_{\mathcal{A}'}(X, Y) \otimes_{\Bbbk} -$$

preserves quasi-isomorphisms (e.g. when $\mathrm{Hom}_{\mathcal{A}'}(X, Y)$ is cofibrant in $C(\Bbbk)$, i.e. a complex of projective $\Bbbk$-modules),

(ii) define $\mathrm{Rep}(\mathcal{A}, \mathcal{B})$ as the subcategory of $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ with $F \in \mathrm{Rep}(\mathcal{A}, \mathcal{B})$ if and only if for all $X \in \mathcal{A}$ there exists $Y \in \mathcal{B}$ such that

$$X \otimes^{\mathrm{L}} F \cong_{\mathcal{D}(\mathcal{B})} Y^{\vee}$$

(in other words, F is a dg-bimodule sending representable $\mathcal{A}$-modules to quasi-representable $\mathcal{B}$-modules),

(iii) then

$$\mathrm{Hom}_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{B}) \cong \mathrm{Iso}(\mathrm{Rep}(\mathcal{A}, \mathcal{B})),$$

where Iso means the set of objects up to isomorphism.

Remark A.17. Note that whenever $\Bbbk$ is a field, all dg-categories are $\Bbbk$-flat.

We refer to elements in $\mathrm{Rep}(\mathcal{A}, \mathcal{B})$ as *quasi-functors*. Since a quasi-functor $F : \mathcal{A} \to \mathcal{B}$ induces a functor

$$[F] : H^0(\mathcal{A}) \to H^0(\mathcal{B}),$$

we can think of $\mathrm{Rep}(\mathcal{A}, \mathcal{B})$ as the category of "representations up to homotopy" of $\mathcal{A}$ in $\mathcal{B}$.

A.5.1. Closed monoidal structure. If $\mathcal{A}$ is cofibrant, then $- \otimes \mathcal{A}$ preserves quasi-equivalences and one can define the bifunctor

$$- \otimes^{\mathrm{L}} - : \mathrm{Hqe} \times \mathrm{Hqe} \to \mathrm{Hqe}, \quad \mathcal{A} \otimes^{\mathrm{L}} \mathcal{B} := Q\mathcal{A} \otimes Q\mathcal{B},$$

where $Q\mathcal{A}$ and $Q\mathcal{B}$ are cofibrant replacements. Then, as proven by Toen [42], there exists an internal hom-functor $\mathcal{RH}om_{\mathrm{Hqe}}(-,-)$ such that

$$\mathrm{Hom}_{\mathrm{Hqe}}(\mathcal{A} \otimes^{\mathrm{L}} \mathcal{B}, \mathcal{C}) \cong \mathrm{Hom}_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{RH}om_{\mathrm{Hqe}}(\mathcal{B}, \mathcal{C})).$$

Therefore, Hqe is a symmetric closed monoidal category.

Remark A.18. Note that the internal hom can not simply be the derived hom functor (because tensor product of cofibrant dg-categories is not cofibrant in general).

Define the *dg-category of quasi-functors* $\mathrm{Rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$ as

- the objects in $\mathrm{Rep}(\mathcal{A}, \mathcal{B}) \cap (\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}\text{-mod})^{\mathrm{cf}}$,
- the dg-homs $\mathcal{H}om(X, Y)$ of $\mathrm{Int}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}\text{-mod})$.

In other words, $\mathrm{Rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$ is the full subcategory of quasi-functors in $\mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$, thus of cofibrant dg-bimodules that preserves quasi-representable modules. It is a dg-enhancement of $\mathrm{Rep}(\mathcal{A}, \mathcal{B})$.

If $\mathcal{A}$ is $\Bbbk$-flat, then

$$\mathcal{RH}om_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{B}) \cong_{\mathrm{Hqe}} \mathrm{Rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B}).$$

Thus $H^0(\mathcal{RH}om_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{B})) \cong \mathrm{Hom}_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{B})$.

Remark A.19. If $\Bbbk$ is a field of characteristic 0, then the dg-category $\mathcal{RH}om_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{B})$ is equivalent to the A_∞-category of strictly unital A_∞-functors [14].

Example A.20. We have $\mathrm{Rep}_{\mathrm{dg}}(\mathcal{A}, \mathrm{Int}(C(\Bbbk))) \cong \mathrm{Int}(\mathcal{A}^{\mathrm{op}}\text{-mod}) \cong \mathcal{D}_{\mathrm{dg}}(\mathcal{A})$.

Recall that classical Morita theory says that for A and B being $\Bbbk$-algebras, there is an equivalence

$$\mathrm{Hom}^{\mathrm{cop}}(A\text{-mod}, B\text{-mod}) \cong A^{\mathrm{op}} \otimes_{\Bbbk} B\text{-mod},$$

where $\mathrm{Hom}^{\mathrm{cop}}$ is given by the functors that preserve coproducts.

Similarly, we put

$$\mathrm{Rep}_{\mathrm{dg}}^{\mathrm{cop}}(\mathcal{D}_{\mathrm{dg}}(\mathcal{A}), \mathcal{D}_{\mathrm{dg}}(\mathcal{B}))$$

for the subcategory of $\mathrm{Rep}_{\mathrm{dg}}(\mathcal{D}_{\mathrm{dg}}(\mathcal{A}), \mathcal{D}_{\mathrm{dg}}(\mathcal{B}))$ where $F \in \mathrm{Rep}_{\mathrm{dg}}^{\mathrm{cop}}(\mathcal{D}_{\mathrm{dg}}(\mathcal{A}), \mathcal{D}_{\mathrm{dg}}(\mathcal{B}))$ if and only if $[F] : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ preserves coproducts.

Theorem A.21. *If $\mathcal{A}$ is $\Bbbk$-flat, then we have*

$$\mathcal{RH}om_{\mathrm{Hqe}}^{\mathrm{cop}}(\mathcal{D}_{\mathrm{dg}}(\mathcal{A}), \mathcal{D}_{\mathrm{dg}}(\mathcal{B})) := \mathrm{Rep}_{\mathrm{dg}}^{\mathrm{cop}}(\mathcal{D}_{\mathrm{dg}}(\mathcal{A}), \mathcal{D}_{\mathrm{dg}}(\mathcal{B})) \cong_{\mathrm{Hqe}} \mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}).$$

Under the hypothesis of Theorem A.21, the internal composition of dg-quasifunctors preserving coproducts is given by taking a cofibrant replacement of the derived tensor product over $\mathcal{A}$.

A.6. Pretriangulated dg-categories. Basically, a triangulated dg-category is a dg-category such that its homotopy category is canonically triangulated. But before being able to give a precise definition, we need to do a detour through Quillen exact categories, Frobenius categories and stable categories.

A.6.1. Frobenius structure on $C(\mathcal{A})$. Recall that a Quillen exact category [37] is an additive category with a class of short exact sequences

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0,$$

called *conflations*, which are pairs of ker-coker, where f is called an *inflation* and g a *deflation*, respecting some axioms:

- the identity is a deflation,
- the composition of deflations is a deflation,
- deflations (resp. inflations) are stable under base (resp. cobase) change.

A *Frobenius* category is a Quillen exact category having enough injectives and projectives, and where injectives coincide with projectives. The *stable category* $\underline{\mathcal{C}}$ of a Frobenius category $\mathcal{C}$ is given by modding out the maps that factor through an injective/projective object. It carries a canonical triangulated structure, where

- the suspension functor S is obtained by taking the target of a conflation
$$0 \to X \to IX \to SX \to 0,$$
where IA is an injective hull of X, for all $X \in \mathcal{C}$,
- the distinguished triangles are equivalent to standard triangles
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} SX,$$
obtained from conflations by the following commutative diagram:
$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & X & \longrightarrow & IX & \longrightarrow & SX & \longrightarrow & 0. \end{array}$$

Example A.22. Let $\mathcal{A}$ be a small dg-category. One can put a Frobenius structure on $C(\mathcal{A})(:= Z^0(\mathcal{A}^{\mathrm{op}}\text{-mod}))$ by using split short exact sequences as class of conflations. Then there is an equivalence $\underline{C(\mathcal{A})} \cong H^0(\mathcal{A}^{\mathrm{op}}\text{-mod})$, and the suspension functor coincides with the usual homological shift. Moreover, $\mathcal{D}(\mathcal{A})$ inherits the triangulated structure from $H^0(\mathcal{A}\text{-mod})$, where distinguished triangles are equivalent to distinguished triangles obtained from all short exact sequences in $C(\mathcal{A})$.

A.6.2. Pretriangulated dg-categories. Remark that for any dg-category $\mathcal{A}$ there is a Yoneda functor
$$Z^0(\mathcal{A}) \to C(\mathcal{A}), \quad X \mapsto \mathrm{Hom}_{\mathcal{A}}(-, X).$$

Definition A.23. A dg-category $\mathcal{T}$ is *pretriangulated* if the image of the Yoneda functor is stable under translations and extensions (for the Quillen exact structure on $C(\mathcal{T})$ described in Example A.22).

This definition implies that

- $Z^0(\mathcal{T})$ is a Frobenius subcategory of $C(\mathcal{T})$,
- $H^0(\mathcal{T})$ inherits a triangulated structure, called *canonical triangulated structure*, from $H^0(\mathcal{T}\text{-mod})$.

Example A.24. Let $\mathcal{A}$ be a dg-category. We have that $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$ is pretriangulated with $Z^0(\mathcal{D}_{\mathrm{dg}}(\mathcal{A})) \cong C(\mathcal{A})^{\mathrm{cf}}$. Moreover, the canonical triangulated structure of $H^0(\mathcal{D}_{\mathrm{dg}}(\mathcal{A}))$ coincides with the usual on $\mathcal{D}(\mathcal{A})$.

Then it is possible to show that

- any dg-category $\mathcal{A}$ admits a pretriangulated hull pretr($\mathcal{A}$) such that

$$\mathcal{RH}om_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{T}) \xrightarrow{\simeq} \mathcal{RH}om_{\mathrm{Hqe}}(\mathrm{pretr}(\mathcal{A}), \mathcal{T})$$

  for all pretriangulated dg-category $\mathcal{T}$,
- $\mathcal{RH}om_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{T})$ is pretriangulated whenever $\mathcal{T}$ is pretriangulated,
- any dg-functor $F : \mathcal{T} \to \mathcal{T}'$ between pretriangulated dg-categories induces a triangulated functor $[F] : H^0(\mathcal{T}) \to H^0(\mathcal{T}')$.

For $\mathcal{A}$ being $\mathbb{k}$-flat, the pretriangulated structure of $\mathcal{RH}om_{\mathrm{Hqe}}(\mathcal{D}_{\mathrm{dg}}(\mathcal{A}), \mathcal{D}_{\mathrm{dg}}(\mathcal{B}))$ restricts to the one of $\mathcal{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ (viewed as sub-dg-category). In particular, we obtain distinguished triangles of quasi-functors from short exact sequences of dg-bimodules.

Definition A.25. For a morphism $f : X \to Y \in Z^0(\mathcal{T})$ in the underlying category of pretriangulated dg-category $\mathcal{T}$, one calls *mapping cone* an object $\mathrm{Cone}(f) \in \mathcal{T}$ such that

$$\mathrm{Cone}(f)^\wedge \cong \mathrm{Cone}(X^\wedge \xrightarrow{\bar{f}} Y^\wedge) \in H^0(\mathcal{T}\text{-mod}).$$

A.6.3. Dg-Morita equivalences.

Definition A.26. A dg-functor $F : \mathcal{A} \to \mathcal{B}$ is a dg-Morita equivalence if it induces an equivalence

$$\boldsymbol{L}F : \mathcal{D}(\mathcal{A}) \xrightarrow{\simeq} \mathcal{D}(\mathcal{B}), \quad X \mapsto F(QX),$$

where QX is a cofibrant replacement of X.

Example A.27. In particular, a quasi-equivalence is a dg-Morita equivalence and the functor that sends dg-categories to their pretriangulated hull $\mathcal{A} \mapsto \mathrm{pretr}(\mathcal{A})$ is a dg-Morita equivalence.

Theorem A.28 ([41]). *There is a model structure* dg-cat$_{\mathrm{mor}}$ *on* dg-cat*, where the weak-equivalences are the dg-Morita equivalences and the fibrations are the same as before.*

Definition A.29. We say that $\mathcal{T}$ is triangulated if it is fibrant in dg-cat$_{\mathrm{mor}}$.

Equivalently, $\mathcal{T}$ is triangulated if and only if the Yoneda functor induces an equivalence $H^0(\mathcal{T}\text{-mod}) \xrightarrow{\simeq} \mathcal{D}^c(\mathcal{T})$ (i.e. every compact object is quasi-representable). Also equivalently, $\mathcal{T}$ is triangulated if and only if $\mathcal{T}$ is pretriangulated and $H^0(\mathcal{T}\text{-mod})$ is idempotent complete.

In particular, any category admits a triangulated hull tr($\mathcal{A}$) (i.e. fibrant replacement). It is given by

$$\mathrm{tr}(\mathcal{A}) := \mathcal{D}^c_{\mathrm{dg}}(\mathcal{A}),$$

the dg-category of compact objects in $\mathcal{D}_{\mathrm{dg}}(\mathcal{A})$.

Example A.30. Let R be a $\Bbbk$-algebra viewed as a dg-category. Then $\mathcal{D}^c_{\mathrm{dg}}(R)$ is the dg-category of perfect complexes, i.e. bounded complexes of finitely generated projective R-modules.

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