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A REMARK ON RASMUSSEN'S INVARIANT OF KNOTS

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For Louis Kauffman on the occasion of his sixtieth birthday

ABSTRACT

We show that Rasmussen's invariant of knots, which is derived from Lee's variant of Khovanov homology, is equal to an analogous invariant derived from certain other filtered link homologies.

Keywords: Rasmussen's invariant; Khovanov homology; link homology.

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1. Introduction

In [3] Khovanov introduced a completely new way to define link invariants. He associated a bigraded cochain complex to a given link diagram and if two diagrams represent the same link, then the associated complexes are homotopy equivalent. Thus by taking homology a link invariant is defined. One of the first variations on Khovanov's construction was the theory defined by Lee [5]. Her link homology, originally defined over $\mathbb{Q}$, is only singly graded with a filtration in place of what was the internal degree in Khovanov's theory. If one forgets about the filtration, then Lee's link homology is completely determined by the linking matrix of the link, which makes it a rather poor invariant compared to Khovanov's theory. However, by using the filtration Rasmussen [6] has defined an integer invariant of knots $s(K)$ which has many wonderful properties. For example he showed that the s-invariant yields a lower bound of the smooth slice genus which led to a new combinatorial proof of the Milnor conjecture concerning the slice genus of torus knots. Another consequence is that if the s-invariant of a knot is greater than zero, then the knot is not smoothly slice which is particularly interesting if the knot is already known to be topologically slice. The s-invariant was conjectured to be equal to twice the τ-invariant in Heegaard–Floer knot homology, however this is now known to be false in general [2]. Much of this is explained in the survey paper [7].

In [1] Bar-Natan introduced a new link homology theory defined over $\mathbb{F}_2[H]$ where H has internal degree -2. Setting $H = 1$ defines a singly graded theory which can be explicitly computed (see [9]) and like Lee's theory depends only on the linking matrix. The theory is again filtered and one can use Rasmussen's definitions to produce an analogous s-invariant. The question that motivated the current note was: is Rasmussen's original s-invariant defined using Lee theory the same as the s-invariant defined using Bar-Natan theory? In fact working over $\mathbb{Q}$ or $\mathbb{F}_p$, p a prime, one can define a family of link homology theories depending on two elements h and t, encompassing Lee's theory and Bar-Natan's theory. Many of these theories give for a knot a two dimensional vector space in degree zero and for such a theory one can define a Rasmussen-type invariant.

In Sec. 2, we define the family of link homology theories of interest to us. We choose the ground field $\mathbb{K}$ to be one of $\mathbb{Q}$ or $\mathbb{F}_p$, p a prime and the family depends on two parameters $h, t \in \mathbb{K}$. We present a couple of computational results and discuss integral theories. In Sec. 3, we recall Rasmussen's s-grading and show that this is preserved by twist equivalence of theories and by the universal coefficient theorem. In Sec. 4, we define Rasmussen's s-invariant $s(K, \mathbb{K})_{h,t}$ for any theory arising from a triple $(\mathbb{K}, h, t)$ for which $h^2 + 4t$ is non-zero. Letting $\widetilde{\mathbb{K}}$ be $\mathbb{Q}$ or $\mathbb{F}_p$ ($\mathbb{K}$ and $\widetilde{\mathbb{K}}$ possibly different) our main result is as follows.

Theorem 4.2. *Let K be a knot. Let $h, t \in \mathbb{K}$ and $\tilde{h}, \tilde{t} \in \widetilde{\mathbb{K}}$ be such that $h^2 + 4t \neq 0 \in \mathbb{K}$ and $\tilde{h}^2 + 4\tilde{t} \neq 0 \in \widetilde{\mathbb{K}}$. Then*

$$s(K, \mathbb{K})_{h,t} = s(K, \widetilde{\mathbb{K}})_{\tilde{h},\tilde{t}}.$$

2. A Family of Link Homology Theories

Let p be a prime and let $\mathbb{K}$ be $\mathbb{Q}$ or $\mathbb{F}_p$. Recall that a *Frobenius system* over $\mathbb{K}$ is a quadruple $(A, \iota, \Delta, \epsilon)$, where A is a commutative ring with unit 1, $\iota\colon \mathbb{K} \to A$ a unital injective ring homomorphism, $\Delta\colon A \to A \otimes A$ a cocommutative coassociative

A-bimodule map and $\epsilon: A \to \mathbb{K}$ a $\mathbb{K}$-linear map satisfying the additional condition

$$(\epsilon \otimes \mathrm{Id})\Delta = \mathrm{Id}.$$

Khovanov has explained in [4] how a rank two Frobenius system gives rise to a link homology theory and moreover that isomorphic Frobenius systems give rise to isomorphic link homology theories.

Example 2.1. Let $h, t \in \mathbb{K}$ and define

$$A_{h,t} = \mathbb{K}[x]/(x^2 - hx - t)$$

with coproduct and counit defined by

$$\begin{aligned} \Delta(1) &= 1 \otimes x + x \otimes 1 - h1 \otimes 1, & \Delta(x) &= x \otimes x + t1 \otimes 1 \\ \epsilon(1) &= 0, & \epsilon(x) &= 1. \end{aligned}$$

This is a rank two Frobenius system which in general is not bi-graded but has a filtration obtained by taking filtration degrees $\deg(x) = -1$ and $\deg(1) = 1$. This filtration induces a filtration on the associated link homology theory. Note that throughout we prefer to use the grading conventions in [3] rather than those in [4]. These theories are obtained from Khovanov's theory A_5 in [4] by specialisation of the variable h and t to elements of $\mathbb{K}$. When $h = t = 0$ the resulting theory is Khovanov's original link homology with coefficients in $\mathbb{K}$ which we denote $KH^*(-; \mathbb{K})$. In this case the theory is genuinely bi-graded. When $\mathbb{K} = \mathbb{Q}$, $h = 0$ and $t = 1$ one gets Lee's theory [5] and when $\mathbb{K} = \mathbb{F}_2$, $h = 1$ and $t = 0$ one gets Bar-Natan's theory [1]. We will denote the theory defined from $h, t \in \mathbb{K}$ by $U^*_{h,t}(L; \mathbb{K})$ for a link L.

There is one further idea from [4] that is important for us. Let A be a Frobenius system and let $\theta \in A$ be an invertible element. Then we can *twist A by θ* to obtain a new Frobenius system with the same product and unit map but a new coproduct and counit map defined by $\Delta'(a) = \Delta(\theta^{-1}a)$ and $\epsilon'(a) = \epsilon(\theta a)$. We call two Frobenius systems *twist equivalent* if one can be obtained from the other via an isomorphism and a twist. Khovanov [4] showed that two Frobenius systems related by twist equivalence give isomorphic link homology groups. It is important to note however that twisting may ruin nice functoriality properties with respect to link cobordisms. Actually one can repair things again by working with the projective spaces of the homologies, because only undesirable scalar factors are caused by twisting.

The following propositions are derived from the work of Lee [5], Shumakovitch [8] and Khovanov [4]. For this reason we only sketch the proofs here.

Proposition 2.2. *Let L be a link with n components and let $h, t, \tilde{h}, \tilde{t} \in \mathbb{K}$.*

(i) *If $h^2 + 4t = 0$, then there is an isomorphism $U^*_{h,t}(L; \mathbb{K}) \cong KH^*(L; \mathbb{K})$.*

(ii) *Suppose char$(\mathbb{K}) \neq 2$. If $h^2 + 4t \neq 0$ and $\tilde{h}^2 + 4\tilde{t} \neq 0$, then there exists an isomorphism $U^*_{h,t}(L; \mathbb{K}) \cong U^*_{\tilde{h},\tilde{t}}(L; \mathbb{K})$.*

Proof. For (i) let x be the generator of $A_{0,0}$ and y the generator of $A_{h,t}$. If $\mathrm{char}(\mathbb{K}) \neq 2$ then it can be checked by direct computation that the map defined by $1 \mapsto 1$, $y \mapsto x + \frac{h}{2}$ gives an isomorphism of Frobenius systems $A_{h,t} \to A_{0,0}$. In characteristic two $h^2 + 4t = 0$ if and only if $h = 0$, so the only non-trivial case is when $t = 1$ in which case the map $1 \mapsto 1$, $y \mapsto x + 1$ provides an isomorphism.

For (ii) suppose first that there exists a non-zero element $a \in \mathbb{K}$ such that

$$\frac{\tilde{h}^2 + 4\tilde{t}}{h^2 + 4t} = a^2.$$

Let x be the generator of $A_{h,t}$ and let y be the generator of $A_{\tilde{h},\tilde{t}}$. Let $b = \frac{1}{2}(\tilde{h} - ah)$ and let $A'_{h,t}$ be $A_{h,t}$ twisted by a^{-1}. Then by direct computation one sees that the map $A_{\tilde{h},\tilde{t}} \to A'_{h,t}$ given by $1 \mapsto 1$, $y \mapsto ax + b$ is an isomorphism of Frobenius systems.

If

$$\frac{\tilde{h}^2 + 4\tilde{t}}{h^2 + 4t} = b$$

is not a square in $\mathbb{K}$, consider the quadratic extension $\mathbb{K}(\sqrt{b}) = \mathbb{K}[X]/\left(X^2 - b\right)$. By the previous arguments we have

$$U^*_{h,t}(L; \mathbb{K}(\sqrt{b})) \cong U^*_{\tilde{h},\tilde{t}}(L; \mathbb{K}(\sqrt{b})).$$

Since $\mathbb{K}(\sqrt{b}) \cong \mathbb{K} \oplus \mathbb{K}\sqrt{b}$ is a free $\mathbb{K}$-module, the universal coefficient theorem implies that we get

$$U^*_{h,t}(L; \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}(\sqrt{b}) \cong U^*_{\tilde{h},\tilde{t}}(L; \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}(\sqrt{b}).$$

Since $\dim_{\mathbb{K}} U^*_{h,t}(L; \mathbb{K}) = \dim_{\mathbb{K}(\sqrt{b})} U^*_{h,t}(L; \mathbb{K}(\sqrt{b}))$ and similarly for $\tilde{h}$ and $\tilde{t}$ we conclude that

$$U^*_{h,t}(L; \mathbb{K}) \cong U^*_{\tilde{h},\tilde{t}}(L; \mathbb{K}).$$ □

Note that when $h = 0$ and $t = 1$ the above result says that Lee theory over $\mathbb{F}_2$ is isomorphic to Khovanov's original theory over $\mathbb{F}_2$, a fact that was proved in [4].

Proposition 2.3. *Let L be a link with n components and $h, t \in \mathbb{K}$. If $h^2 + 4t \neq 0$ then*

$$\dim(U^*_{h,t}(L; \mathbb{K})) = 2^n.$$

All generators lie in even degree and for a knot both generators lie in degree zero.

Proof. Suppose $\mathrm{char}(\mathbb{K}) \neq 2$. If there exists a non-zero element $\gamma \in \mathbb{K}$ such that $h^2 + 4t = \gamma^2$, we can change basis to write $A_{h,t} = \mathbb{K}\{\alpha, \beta\}$ where

$$\alpha = x - \frac{1}{2}(h - \gamma),$$

$$\beta = x - \frac{1}{2}(h + \gamma).$$

Courtesy of the condition $h^2 + 4t = \gamma^2 \neq 0$ this change of basis diagonalises the multiplication:

$$\alpha^2 = \gamma\alpha, \quad \beta^2 = -\gamma\beta, \quad \alpha\beta = \beta\alpha = 0.$$

The rest of the proof is identical to Lee's proof in [5] in which the details of the special case $\mathbb{K} = \mathbb{Q}$, $h = 0$, $t = 1$ and $\gamma = 2$ are provided.

If $h^2 + 4t = b$ is not a square in $\mathbb{K}$, then the arguments above prove the claim for

$$U^*_{h,t}(L; \mathbb{K}(\sqrt{b})) \cong U^*_{h,t}(L; \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}(\sqrt{b}),$$

which shows that the claim holds for $U^*_{h,t}(L; \mathbb{K})$ as well, because

$$\dim_{\mathbb{K}} U^*_{h,t}(L; \mathbb{K}) = \dim_{\mathbb{K}(\sqrt{b})} U^*_{h,t}(L; \mathbb{K}(\sqrt{b})).$$

In characteristic two, if $h = 1$ and $t = 0$ then the basis change which diagonalizes the Frobenius system is $\alpha = x$ and $\beta = x + 1$ which was used in [9]. If $h = t = 1$, then $\mathbb{K}' = \mathbb{F}_2[y]/(y^2 + y + 1)$ is a quadratic extension of $\mathbb{F}_2$ since $y^2 + y + 1$ is irreducible modulo 2. In this case

$$\begin{aligned} \alpha &= x + y, \\ \beta &= x + y^2 \end{aligned}$$

diagonalizes the Frobenius system $A_{1,1}$ with coefficients in $\mathbb{K}'$. As above it follows from Lee's work that $\dim_{\mathbb{K}'} U_{1,1}(L, \mathbb{K}') = 2^n$. Since $\mathbb{K}' \cong \mathbb{F}_2 \oplus \mathbb{F}_2 y$ is a free $\mathbb{F}_2$-module, we get, by the universal coefficient theorem,

$$U^*_{1,1}(L, \mathbb{K}') \cong U^*_{1,1}(L, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{K}',$$

and hence $\dim_{\mathbb{F}_2} U^*_{1,1}(L, \mathbb{F}_2) = \dim_{\mathbb{K}'} U^*_{1,1}(L, \mathbb{K}') = 2^n$, which proves the claims in the proposition for this case too.

The statement about the degree of the generators follows once again from Lee's proof. □

Khovanov's original link homology was defined integrally and each of the theories discussed so far also has an integral version. Indeed, the Frobenius system in Example 2.1 can also be defined over $\mathbb{Z}$ resulting in the link homology we denote by $U^*_{h,t}(L; \mathbb{Z})$.

Proposition 2.4. *Let L be a link with n components and let $h, t \in \mathbb{Z}$ satisfy $h^2 + 4t \neq 0$.*

(i) *There is an isomorphism*

$$U^*_{h,t}(L; \mathbb{Z}) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2^n} \oplus T^*$$

where T^* is all torsion.

(ii) *If $h, t < p$ and $h^2 + 4t \neq 0$ mod p where p is a prime, then $U^*_{h,t}(L; \mathbb{Z})$ has no p-torsion.*

Proof. If A is the Frobenius system giving $U^*_{h,t}(-;\mathbb{Z})$ then $A\otimes_{\mathbb{Z}}\mathbb{Q}$ is the Frobenius system giving $U^*_{h,t}(-;\mathbb{Q})$. By the construction of link homology this means that each chain group in the rational theory is the integral chain group tensored with $\mathbb{Q}$. Thus the universal coefficient theorem gives

$$\begin{aligned} U^i_{h,t}(L;\mathbb{Q}) &\cong U^i_{h,t}(L;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}\oplus \mathrm{Tor}^{\mathbb{Z}}(U^{i+1}_{h,t}(L;\mathbb{Z}),\mathbb{Q}) \\ &= U^i_{h,t}(L;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}. \end{aligned}$$

Thus by Proposition 2.3

$$\dim(U^*_{h,t}(L;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}) = \dim(U^*_{h,t}(L;\mathbb{Q})) = 2^n$$

from which part (i) follows.

For part (ii) we will prove by induction on i that $U^i_{h,t}(L;\mathbb{Z})$ has no p-torsion under the hypotheses given. Suppose that $U^i_{h,t}(L;\mathbb{Z})$ has no p-torsion for $i\le N$ and now claim the same holds true for $i = N+1$. Note that $U^i_{h,t}(L;\mathbb{Z})$ is non-trivial only for finitely many values of i so the induction has a base case. By the universal coefficient theorem we have

$$U^N_{h,t}(L;\mathbb{F}_p)\cong U^N_{h,t}(L;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{F}_p\oplus \mathrm{Tor}^{\mathbb{Z}}(U^{N+1}_{h,t}(L;\mathbb{Z}),\mathbb{F}_p).$$

If N is odd, then the left-hand side is trivial since it follows from Proposition 2.3 that all generators are in even homological degree. Hence $\mathrm{Tor}^{\mathbb{Z}}(U^{N+1}_{h,t}(L;\mathbb{Z}),\mathbb{F}_p) = 0$ showing there is no p-torsion in $U^{N+1}_{h,t}(L;\mathbb{Z})$. If N is even, by Proposition 2.3 we know the number of copies of $\mathbb{F}_p$ on the left and moreover that the same number occurs in the first summand on the right, so the Tor group is again trivial and $U^{N+1}_{h,t}(L;\mathbb{Z})$ does not have p-torsion. □

For integral Bar-Natan theory one can do slightly better. The change of basis $\alpha = x$, $\beta = x-1$ in fact diagonalises the theory so in this case T^* is trivial. For integral Lee theory part (ii) above shows that the only possible torsion is 2-torsion.

3. Rasmussen's s-Grading

As we noted above the theories we are concerned with are not in general bi-graded but instead possess a filtration. Let $C^*(L)$ be the complex formed using the Frobenius system $A_{h,t}$ over $\mathbb{K}$, i.e. whose homology is $U^*_{h,t}(L;\mathbb{K})$. As above $\mathbb{K}$ is one of $\mathbb{Q}$ or $\mathbb{F}_p$ for p a prime and we are assuming $h^2+4t\neq 0$.

Define $p\colon C^*(L)\to\mathbb{Z}$ as follows. Set $p(1)=1$ and $p(x)=-1$ and for any element $w = w_1\otimes w_2\otimes\cdots\otimes w_m\in C^*(L)$, where $w_i\in\{1,x\}$, set $p(w) = p(w_1)+\cdots+p(w_m)$. An arbitrary $w\in C^*(L)$ is not homogeneous with respect to p but can be written as $w = w^1+w^2+\cdots+w^l$, where w^j is homogeneous for all j. We define

$$p(w) = \min\left\{p(w^j)\,|\, j=1,\ldots,l\right\}.$$

Now for any $w \in C^i(L)$, define

$$q(w) = p(w) + i + c^+ - c^-,$$

where c^+ and c^- are the numbers of positive and negative crossings respectively in L. The filtration grading of an element w is $q(w)$.

As Rasmussen explains in [6] this determines a grading s on the homology. For $\alpha \in U^*_{h,t}(L;\mathbb{K})$ define

$$s(\alpha, \mathbb{K})_{h,t} = \max\{q(w) \mid w \in C^*(L), [w] = \alpha\}\,.$$

If there is no confusion we will supress h and t from the notation writing $s(\alpha, \mathbb{K})$ for $s(\alpha, \mathbb{K})_{h,t}$. Note that we can define the linear subspaces

$$F^m U^*_{h,t}(L;\mathbb{K}) = \left\{\alpha \in U^*_{h,t}(L;\mathbb{K}) \mid s(\alpha) \geq m\right\},$$

which form a filtration

$$0 \subseteq F^i U^*_{h,t}(L;\mathbb{K}) \subseteq F^{i-1} U^*_{h,t}(L;\mathbb{K}) \subseteq \cdots \subseteq F^j U^*_{h,t}(L;\mathbb{K}) = U^*_{h,t}(L;\mathbb{K}),$$

where i and j are the maximal and the minimal s-value respectively.

For integral theories we define $s(\alpha, \mathbb{Z})$ in a similar manner by restricting the definition to classes α in the torsion-free part of $U^*_{h,t}(L;\mathbb{Z})$.

The s-grading satisfies some important properties given in the following two propositions.

Proposition 3.1. *Suppose char*$(\mathbb{K}) \neq 2$. *If* $h^2 + 4t \neq 0$ *and* $\tilde{h}^2 + 4\tilde{t} \neq 0$ *then there exists an isomorphism*

$$U^*_{h,t}(L;\mathbb{K}) \cong U^*_{\tilde{h},\tilde{t}}(L;\mathbb{K})$$

which preserves the s*-grading.*

Proof. *Case I*: there exists a non-zero element $a \in \mathbb{K}$ such that

$$\frac{\tilde{h}^2 + 4\tilde{t}}{h^2 + 4t} = a^2.$$

In this case we can use the isomorphism in the proof of Proposition 2.2(ii). Recall that if x is the generator of $A_{h,t}$ and y is the generator of $A_{\tilde{h},\tilde{t}}$, then the isomorphism is induced by twisting $A_{h,t}$ by a^{-1} and using the isomorphism of Frobenius systems $A_{\tilde{h},\tilde{t}} \to A'_{h,t}$ defined by $1 \mapsto 1$, $y \mapsto ax + b$ where $b = \frac{1}{2}(\tilde{h} - ah)$. The latter induces an isomorphism $\psi_*\colon U^*_{\tilde{h},\tilde{t}}(L;\mathbb{K}) \to U^*_{h,t}(L;\mathbb{K})$.

It is clear that the twist preserves s so we only need to consider ψ_*. Let $C^*_{h,t}(L)$ be the complex whose homology is $U^*_{h,t}(L;\mathbb{K})$ and similarly let $C^*_{\tilde{h},\tilde{t}}(L)$ be the complex giving $U^*_{\tilde{h},\tilde{t}}(L;\mathbb{K})$. Let $\psi\colon C^*_{\tilde{h},\tilde{t}}(L) \to C^*_{h,t}(L)$ be induced by the isomorphism of Frobenius systems above. We claim that ψ preserves the filtration degree q. We can write $w \in C^*_{\tilde{h},\tilde{t}}(L)$ as

$$w = \sum \lambda_I \epsilon_I(y)$$

where each $\epsilon_I(y) = \epsilon_1 \otimes \epsilon_2 \otimes \cdots$ with $\epsilon_j \in \{1, y\}$. By the definition of ψ we have

$$\psi(\epsilon_I(y)) = a^{r(I)}\epsilon_I(x) + \text{terms of higher filtration}$$

where $r(I)$ is the number of y's in $\epsilon_I(y)$. From this it follows that $q(\psi(w)) = q(w)$ since any term ϵ_I with $q(\epsilon_I) = q(w)$ also appears in $\psi(w)$.

Next we claim that ψ_* preserves s, i.e. for $\alpha \in U^*_{\tilde{h},\tilde{t}}(L;\mathbb{K})$

$$s(\alpha,\mathbb{K})_{\tilde{h},\tilde{t}} = s(\psi_*(\alpha),\mathbb{K})_{h,t}. \tag{3.1}$$

Let $w \in C^*_{\tilde{h},\tilde{t}}(L)$ such that $[w] = \alpha$ and $q(w) = s(\alpha,\mathbb{K})_{\tilde{h},\tilde{t}}$. Then $\psi(w)$ represents $\psi_*(\alpha)$ and so

$$s(\psi_*(\alpha),\mathbb{K})_{h,t} \geq q(\psi(w)) = q(w) = s(\alpha,\mathbb{K})_{h,t}.$$

Conversely, let $v \in C^*_{h,t}(L)$ be such that $[v] = \psi_*(\alpha)$ and $q(v) = s(\psi_*(\alpha),\mathbb{K})_{h,t}$. Then $\psi^{-1}(v)$ represents α so

$$s(\alpha,\mathbb{K})_{\tilde{h},\tilde{t}} \geq q(\psi^{-1}(v)) = q(v) = s(\psi_*(\alpha),\mathbb{K})_{h,t}.$$

proving (3.1).

Case II: the element

$$\frac{\tilde{h}^2 + 4\tilde{t}}{h^2 + 4t} = b$$

is not a square in $\mathbb{K}$. In this case we will show

$$F^m U^*_{h,t}(L;\mathbb{K})/F^{m+1}U^*_{h,t}(L;\mathbb{K}) \cong F^m U^*_{\tilde{h},\tilde{t}}(L;\mathbb{K})/F^{m+1}U^*_{\tilde{h},\tilde{t}}(L;\mathbb{K})$$

for all $m \in \mathbb{Z}$.

Case I above shows that there exists an isomorphism

$$U^*_{h,t}(L;\mathbb{K}(\sqrt{b})) \cong U^*_{\tilde{h},\tilde{t}}(L;\mathbb{K}(\sqrt{b})) \tag{3.2}$$

which preserves the s-grading.

Let us now show that the inclusion

$$\iota\colon U^*_{h,t}(L;\mathbb{K}) \to U^*_{h,t}(L;\mathbb{K}(\sqrt{b})) \tag{3.3}$$

preserves the s-grading. The map ι is induced by the inclusion

$$\bar{\iota}\colon Z^*(L,\mathbb{K}) \to C^*(L,\mathbb{K}) \otimes \mathbb{K}(\sqrt{b}) = C^*(L,\mathbb{K}(\sqrt{b}))$$

given by

$$\bar{\iota}(w) = w \otimes 1,$$

which clearly preserves the q-values. Let $\alpha \in U^*_{h,t}(L;\mathbb{K})$ and let $w \in Z^*(L,\mathbb{K})$ be such that $\alpha = [w]$ with $s(\alpha) = q(w)$. Then we have

$$s(\alpha,\mathbb{K}) = q(w) = q(w \otimes 1) \leq s(\iota(\alpha),\mathbb{K}(\sqrt{b})).$$

Conversely, let $u \in C^*(L,\mathbb{K}(\sqrt{b}))$ be such that $\iota(\alpha) = [u]$ and $s(\iota(\alpha),\mathbb{K}(\sqrt{b})) = q(u)$. Then $[\pi(u)] = \alpha$, where π is induced by the projection $\mathbb{K}(\sqrt{b}) \cong \mathbb{K} \oplus \mathbb{K}\sqrt{b} \to \mathbb{K}$

which is the left inverse of the inclusion. Clearly $q(u) \leq q(\pi(u))$, so we get

$$s(\iota(\alpha), \mathbb{K}(\sqrt{b})) = q(u) \leq q(\pi(u)) \leq s(\alpha, \mathbb{K}).$$

Thus we have $s(\iota(\alpha), \mathbb{K}(\sqrt{b})) = s(\alpha, \mathbb{K})$, showing that ι preserves the s-grading.

It follows now that ι induces inclusions on filtration quotients

$$F^m U^*_{h,t}(L;\mathbb{K})/F^{m+1}U^*_{h,t}(L;\mathbb{K}) \to F^m U^*_{h,t}(L;\mathbb{K}(\sqrt{b}))/F^{m+1}U^*_{h,t}(L;\mathbb{K}(\sqrt{b}))$$

and hence inclusions

$$\begin{aligned}&[F^m U^*_{h,t}(L;\mathbb{K})/F^{m+1}U^*_{h,t}(L;\mathbb{K})] \otimes \mathbb{K}(\sqrt{b}) \\ &\quad \to F^m U^*_{h,t}(L;\mathbb{K}(\sqrt{b}))/F^{m+1}U^*_{h,t}(L;\mathbb{K}(\sqrt{b})).\end{aligned}$$

Thus by taking dimensions (of $\mathbb{K}$-modules) we have

$$\begin{aligned}&2\dim_{\mathbb{K}}[F^m U^*_{h,t}(L;\mathbb{K})/F^{m+1}U^*_{h,t}(L;\mathbb{K})] \\ &\quad \leq \dim_{\mathbb{K}}[F^m U^*_{h,t}(L;\mathbb{K}(\sqrt{b}))/F^{m+1}U^*_{h,t}(L;\mathbb{K}(\sqrt{b}))].\end{aligned}$$

Now if L has n components then by Proposition 2.3, we have

$$\sum_m 2\dim_{\mathbb{K}}[F^m U^*_{h,t}(L;\mathbb{K})/F^{m+1}U^*_{h,t}(L;\mathbb{K})] = 2^{n+1}$$

and

$$\sum_m \dim_{\mathbb{K}}[F^m U^*_{h,t}(L;\mathbb{K}(\sqrt{b}))/F^{m+1}U^*_{h,t}(L;\mathbb{K}(\sqrt{b}))] = 2^{n+1}.$$

Thus we can conclude

$$\begin{aligned}&2\dim_{\mathbb{K}}[F^m U^*_{h,t}(L;\mathbb{K})/F^{m+1}U^*_{h,t}(L;\mathbb{K})] \\ &\quad = \dim_{\mathbb{K}}[F^m U^*_{h,t}(L;\mathbb{K}(\sqrt{b}))/F^{m+1}U^*_{h,t}(L;\mathbb{K}(\sqrt{b}))].\end{aligned}$$

Similar equations hold using $\tilde{h}$ and $\tilde{t}$ and combining these with (3.2) gives

$$\dim_{\mathbb{K}} F^m U^*_{h,t}(L;\mathbb{K})/F^{m+1}U^*_{h,t}(L;\mathbb{K}) = \dim_{\mathbb{K}} F^m U^*_{\tilde{h},\tilde{t}}(L;\mathbb{K})/F^{m+1}U^*_{\tilde{h},\tilde{t}}(L;\mathbb{K}),$$

for any $m \in \mathbb{Z}$, which proves the proposition. □

The next property involves the maps in the universal coefficient theorem. Recall that the universal coefficient theorem provides a short exact sequence

$$0 \longrightarrow U^*_{h,t}(L;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{K} \xrightarrow{\phi} U^*_{h,t}(L;\mathbb{K}) \longrightarrow \mathrm{Tor}^{\mathbb{Z}}(U^{*+1}_{h,t}(L;\mathbb{Z}), \mathbb{K}) \longrightarrow 0\ .$$

Proposition 3.2. *If $h, t \in \mathbb{Z}$ are such that $h^2 + 4t \neq 0$ in $\mathbb{K}$ then*

$$\phi\colon U^*_{h,t}(L;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{K} \to U^*_{h,t}(L;\mathbb{K})$$

is an isomorphism that preserves the s-grading.

Proof. It is an isomorphism since the Tor group is trivial: over $\mathbb{Q}$ always and over $\mathbb{F}_p$ courtesy of part (ii) of Proposition 2.4.

Recall that ϕ is induced by the inclusion

$$\overline{\phi}\colon Z^*(L,\mathbb{Z})\otimes\mathbb{K}\to C^*(L,\mathbb{Z})\otimes\mathbb{K}=C^*(L,\mathbb{K})$$

which clearly preserves the filtration grading q.

To show ϕ preserves s we must show that given $\alpha\in U^*_{h,t}(L;\mathbb{Z})/\mathrm{Tors}$ we have

$$s(\alpha,\mathbb{Z})=s(\phi(\alpha\otimes 1),\mathbb{K}). \tag{3.4}$$

Let $w\in Z^*(L,\mathbb{Z})$ be a representative of α such that $q(w)=s(\alpha,\mathbb{Z})$. Then $\overline{\phi}(w\otimes 1)$ represents $\phi(\alpha\otimes 1)$ and so

$$s(\phi(\alpha\otimes 1),\mathbb{K})\geq q(\overline{\phi}(w\otimes 1))=q(w)=s(\alpha,\mathbb{Z}).$$

Conversely, let $u\in Z^*(L,\mathbb{K})$ represent $\phi(\alpha\otimes 1)$ such that $q(u)=s(\phi(\alpha\otimes 1),\mathbb{K})$. We may write $u=\sum v_i\otimes\lambda_i\in Z^*(L,\mathbb{Z})\otimes\mathbb{K}$. When $\mathbb{K}=\mathbb{Q}$ let λ be the least common multiple of the denominators of the λ_i and when $\mathbb{K}=\mathbb{F}_p$ let $\lambda=1$. Define $v\in Z^*(L,\mathbb{Z})$ by

$$\lambda\sum v_i\otimes\lambda_i=v\otimes 1\in Z^*(L,\mathbb{Z})\otimes\mathbb{K}.$$

Note that $q(v)=q(u)$ and moreover that since ϕ is an isomorphism $[v]=\lambda\alpha$. We also have $s(\lambda\alpha,\mathbb{Z})=s(\alpha,\mathbb{Z})$ and so

$$s(\alpha,\mathbb{Z})=s(\lambda\alpha,\mathbb{Z})\geq q(v)=q(u)=s(\phi(\alpha\otimes 1),\mathbb{K})$$

proving (3.4) and hence the claim. □

4. Rasmussen's Invariant

Let $\mathbb{K}$ be one of $\mathbb{Q}$ or $\mathbb{F}_p$ and let $h,t\in\mathbb{K}$ satisfy $h^2+4t\neq 0\in\mathbb{K}$. Let K be a knot and define

$$s_{\min}(K,\mathbb{K})_{h,t}=\min\{s(\alpha,\mathbb{K})_{h,t}\,|\,\alpha\in U^*_{h,t}(K;\mathbb{K}),\alpha\neq 0\}$$

and

$$s_{\max}(K,\mathbb{K})_{h,t}=\max\{s(\alpha,\mathbb{K})_{h,t}\,|\,\alpha\in U^*_{h,t}(K;\mathbb{K}),\alpha\neq 0\}.$$

Rasmussen's s-invariant for the theory $U^*_{h,t}(-;\mathbb{K})$ is defined as follows. The original definition in [6] is for the case $\mathbb{K}=\mathbb{Q}$.

Definition 4.1.

$$s(K,\mathbb{K})_{h,t}=\frac{s_{\min}(K,\mathbb{K})_{h,t}+s_{\max}(K,\mathbb{K})_{h,t}}{2}.$$

For integral theories we may make an analogous definition by using $s(\alpha, \mathbb{Z})$ which we recall restricts its definition to the the torsion-free part of $U^*_{h,t}(K;\mathbb{Z})$.

Here is our main result. Let $\mathbb{K}$ and $\widetilde{\mathbb{K}}$ be $\mathbb{Q}$ or $\mathbb{F}_p$ ($\mathbb{K}$ and $\widetilde{\mathbb{K}}$ possibly different).

Theorem 4.2. *Let K be a knot. Let $h, t \in \mathbb{K}$ and $\tilde{h}, \tilde{t} \in \widetilde{\mathbb{K}}$ be such that $h^2 + 4t \neq 0 \in \mathbb{K}$ and $\tilde{h}^2 + 4\tilde{t} \neq 0 \in \widetilde{\mathbb{K}}$. Then*

$$s(K, \mathbb{K})_{h,t} = s(K, \widetilde{\mathbb{K}})_{\tilde{h},\tilde{t}}.$$

Proof. If $\mathbb{K} = \widetilde{\mathbb{K}} = \mathbb{Q}$ then by Proposition 3.1

$$s(K, \mathbb{Q})_{h,t} = s(K, \mathbb{Q})_{\tilde{h},\tilde{t}}.$$

If $\mathbb{K} = \mathbb{F}_p$ and $\widetilde{\mathbb{K}} = \mathbb{Q}$ then we lift h and t to $\mathbb{Z}$ and apply Proposition 3.2 to give

$$s(K, \mathbb{K})_{h,t} = s(K, \mathbb{Z})_{h,t}.$$

Applying Proposition 3.2 once more and then Proposition 3.1 gives

$$s(K, \mathbb{Z})_{h,t} = s(K, \mathbb{Q})_{h,t} = s(K, \mathbb{Q})_{\tilde{h},\tilde{t}}$$

which proves the result in this case.

If $\mathbb{K} = \mathbb{F}_p$ and $\widetilde{\mathbb{K}} = \mathbb{F}_q$ then in a similar way to the above we can apply Propositions 3.1 and 3.2 to get

$$s(K, \mathbb{K})_{h,t} = s(K, \mathbb{Z})_{h,t} = s(K, \mathbb{Q})_{h,t} = s(K, \mathbb{Q})_{\tilde{h},\tilde{t}} = s(K, \mathbb{Z})_{\tilde{h},\tilde{t}} = s(K, \widetilde{\mathbb{K}})_{\tilde{h},\tilde{t}}. \qquad \square$$

In particular $s(K, \mathbb{F}_2)_{1,0} = s(K, \mathbb{Q})_{0,1}$ holds true, showing that the s-invariant from Bar-Natan's characteristic two theory is equal to Rasmussen's original s-invariant defined using Lee theory over $\mathbb{Q}$.

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