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# An approach to categorification of Verma modules

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Abstract

We give a geometric categorification of the Verma modules $M(\lambda)$ for quantum $\mathfrak{sl}_2$.

*Contents*



## 1. *Introduction*

After the pioneer works of Frenkel–Khovanov–Stroppel [**13**] and Chuang–Rouquier [**11**], there have been various developments in higher representation theory in different directions and with different flavors. One popular approach consists of using structures from geometry to construct categorical actions of Lie algebras. This is already present in the foundational papers [**11**, **13**], where cohomologies of finite-dimensional Grassmannians and partial flag varieties play an important role in the categorification of finite-dimensional irreducible representations of quantum $\mathfrak{sl}_2$. In the context of algebraic geometry Cautis, Kamnitzer and Licata [**9**] have defined and studied geometric categorical $\mathfrak{sl}_2$-actions, and Zheng gave a categorification of integral representations of quantum groups [**51**] and of tensor products of $\mathfrak{sl}_2$-modules [**50**].

In a remarkable series of papers, Khovanov and Lauda, and independently Rouquier, constructed categorifications of all quantum Kac–Moody algebras [**28–30**, **34–36**, **46**] and some of their 2-representations [**46**]. In Khovanov and Lauda's formulation the categorified quantum group is a 2-category $\dot{\mathcal{U}}$, defined diagrammatically by generators and relations. Khovanov and Lauda conjectured that certain quotients of $\dot{\mathcal{U}}$ categorify integrable representations of these quantum Kac–Moody algebras. This conjecture was first proved in finite type $A_n$ by Brundan and Kleshshev [**6**]. Based on Khovanov and Lauda, Rouquier and Zheng's work, Webster gave in [**49**] a diagrammatic categorification of tensor products of integrable representations of symmetrizable quantum Kac–Moody algebras and used it to categorify the Witten–Reshetikhin–Turaev link invariant. Moreover, he proved Khovanov and Lauda's conjecture

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on categorification of integrable representations for all quantum Kac–Moody algebras. This was also done independently by Kang and Kashiwara in [20].

All these constructions share one common feature: they only categorify finite-dimensional representations in the finite type (representations in other types are locally finite-dimensional when restricted to any simple root quantum $\mathfrak{sl}_2$). In this paper we make a step towards a categorification of infinite-dimensional and non-integrable representations of quantum Kac–Moody algebras. We start with the simplest case by proposing a framework for categorification of the Verma modules for quantum $\mathfrak{sl}_2$. This paper is the first of a series which continues with [41], where the authors study the algebra $A_n$ introduced below. Then, the construction is generalized to all Kac–Moody algebras in [42]. In addition, in [43] the authors explain a connexion between Verma modules and the HOMFLY-PT polynomial. Then, they use the theory of categorified Verma modules to recover Khovanov–Rozansky link homology.

This paper was motivated by the geometric categorification of finite-dimensional irreducible representations of $\mathfrak{sl}_2$ by Frenkel, Khovanov and Stroppel in [13, § 6] and the subsequent constructions in [34, 35]. In particular, the inspiration from Lauda's description in [34, 35] should be clear.

Here are some of the main differences to these: First we use infinite-dimensional Grassmannians and introduce an exterior part carrying an extra grading and a parity (this is somehow similar to what is done to go from $\mathfrak{sl}_n$-link homology to HOMFLY-PT link homology, see [26, 31, 32]). Moreover, we drop the biadjunction hypothesis on the functors realizing the categorical action, keeping only an adjunction. Finally, to deal with the occurrence of a polynomial fraction in the commutator relation we introduce a different notion of Grothendieck group from the usual, that allow canceling infinite relations. Despite being different from the one introduced by Achar and Stroppel in [1] we call it a topological Grothendieck group. This notion is explained in Appendix.

Without further delays we now pass to describe our construction.

## 1.1. *Sketch of the construction*

1.1.1. *Verma modules.* Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{sl}_2$, and $\lambda = q^c$ for some $c$ either formal or integer. Denote by $V_\lambda$ the one-dimensional $U_q(\mathfrak{b})$-module of weight $\lambda$, with $E$ acting trivially. The universal Verma module $M(\lambda)$ with the highest weight $\lambda$ is the induced module

$$M(\lambda) = U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{b})} V_\lambda.$$

We follow the notation in [18] (cf. [18, §§ 2.2 and 2.4]), but in the special case when $\lambda = q^n$ for an integer number $n$, we write $M(n)$ instead of $M(q^n)$. The Verma module $M(\lambda)$ is irreducible unless $\lambda = q^n$ with $n$ a non-negative integer. In the latter case $M(n)$ contains $M(-n-2)$ as a unique non-trivial proper submodule and the quotient $M(n)/M(-n-2)$ is isomorphic to the irreducible $U_q(\mathfrak{sl}_2)$-module $V(n)$ of dimension $n+1$. Throughout this paper, we will treat $\lambda$ as a formal parameter and we will think of $M(\lambda)$ as a module over $\mathbb{Q}(\!(q,\lambda)\!)$. Here $\mathbb{Q}(\!(q,\lambda)\!)$ means the field of formal Laurent series in the variables $q$ and $\lambda$. We will also consider $M_A(\lambda)$ and $M_A^*(\lambda)$, where we replace the ground field $\mathbb{Q}(\!(q,\lambda)\!)$ by the ring $A = \mathbb{Q}(\!(q)\!)[\lambda,\lambda^{-1}]$. They are Verma modules over $\dot{U}_\lambda$, the shifted idempotented quantum $\mathfrak{sl}_2$ defined below in § 2.1.1, and are given, respectively, by the canonical and dual canonical basis of $M(\lambda)$, also presented in § 2.1.1.

1.1.2. *Categorification of the weight spaces of $M(\lambda q^{-1})$.* We work over the field of rationals $\mathbb{Q}$ and $\otimes$ means $\otimes_{\mathbb{Q}}$. Let $G_k$ be the Grassmannian of $k$-planes in $\mathbb{C}^\infty$ and $H(G_k)$ its cohomology ring with rational coefficients. It is a graded algebra freely generated by the Chern classes $\underline{x}_k = (x_1,\dots,x_k)$ with $\deg(x_i) = 2i$ [35, § 3.1.1] (see also [15; 16, § 3] for more about cohomology of flag varieties). The ring $H(G_k)$ has a unique irreducible module up to isomorphism and grading shift, which is isomorphic to $\mathbb{Q}$. Let $\mathrm{Ext}_{H(G_k)}(\mathbb{Q},\mathbb{Q})$ denote the algebra of self-extensions of $\mathbb{Q}$ (this can be seen as the opposite algebra of the Koszul dual of $H(G_k)$) and for $k \geqslant 0$, define

$$\Omega_k = H(G_k) \otimes \mathrm{Ext}_{H(G_k)}(\mathbb{Q},\mathbb{Q}).$$

We have

$$\Omega_k \cong \mathbb{Q}[\underline{x}_k] \otimes \textstyle\bigwedge^\bullet(\underline{s}_k),$$

which we regard as a $\mathbb{Z}\times\mathbb{Z}$-graded superring, with even generators $x_i$ having degree $\deg(x_i) = (2i, 0)$, and odd generators $s_i$ with $\deg(s_i) = (-2i, 2)$ (the first grading is quantum and the second cohomological). We denote by $\langle r, s\rangle$ the grading shift up by $r$ units on the quantum grading and by $s$ units on the cohomological grading. In the sequel we use the term bigrading for a $\mathbb{Z}\times\mathbb{Z}$-grading.

The superring $\Omega_k$ has a unique irreducible supermodule up to isomorphism and (bi)grading shift, which is isomorphic to $\mathbb{Q}$ and denoted $S_k$, and a unique projective indecomposable supermodule, again up to isomorphism and (bi)grading shift, which is isomorphic to $\Omega_k$.

In the Appendix we develop several versions of 'topological' Grothendieck groups. The topological split Grothendieck group $K_0(\Omega_k)$ and topological Grothendieck group $G_0(\Omega_k)$ are one-dimensional modules over $\mathbb{Z}_\pi[\![q]\!][q^{-1},\lambda^{\pm1}]$, where $\mathbb{Z}_\pi = \mathbb{Z}[\pi]/(\pi^2-1)$, and generated, respectively, by the class of $\Omega_k$, and by the class of $S_k$. In another version, the topological Grothendieck group $\widehat{G}_0(\Omega_k) = \boldsymbol{G}_0(\Omega_k\text{-smod}_{\mathrm{lf}})$ is a one-dimensional module over $\mathbb{Z}_\pi(\!(q,\lambda)\!)$, and is generated either by $[\Omega_k]$, either by $[S_k]$.

For each non-negative integer $k$ we define $\mathcal{M}_k = \Omega_k\text{-smod}_{\mathrm{lf}}$ and take $\mathcal{M}_k$ as a categorification of the $(\lambda q^{-1-2k})$-weight space.

1.1.3. *The categorical $\mathfrak{sl}_2$-action.* To construct functors $\mathsf{F}$ and $\mathsf{E}$ that move between categories $\mathcal{M}_k$ we look for superrings $\Omega_{k+1,k}$ and (natural) maps

$$\begin{tikzcd} & \Omega_{k+1,k} & \\ \Omega_{k+1} \arrow[ur, "\psi^*_{k+1}"] & & \Omega_k \arrow[ul, "\phi^*_k"'] \end{tikzcd}$$

that turn $\Omega_{k+1,k}$ into a $(\Omega_{k+1},\Omega_k)$-superbimodule such that, up to an overall shift,

- $\Omega_{k+1,k}$ is a free right $\Omega_k$-supermodule of bigraded superdimension
$$\frac{\lambda q^{-k-1} - \lambda^{-1}q^{k+1}}{q-q^{-1}}$$
- $\Omega_{k+1,k}$ is a free left $\Omega_{k+1}$-supermodule of bigraded superdimension $[k+1]$.

Remark 1.1. The superring $H(G_{k,k+1}) \otimes \mathrm{Ext}_{H(G_{k,k+1})}(\mathbb{Q},\mathbb{Q})$ does not have these properties.

Let $G_{k,k+1}$ be the infinite partial flag variety

$$G_{k,k+1} = \{(U_k, U_{k+1}) |\, \dim_{\mathbb{C}} U_k = k, \dim_{\mathbb{C}} U_{k+1} = k+1,\ 0 \subset U_k \subset U_{k+1} \subset \mathbb{C}^\infty\}.$$

Its rational cohomology is a graded ring generated by the Chern classes:

$$H(G_{k,k+1}) = \mathbb{Q}[\underline{x}_k, \xi], \quad \deg(x_i) = 2i, \quad \deg(\xi) = 2.$$

The forgetful maps

$$\begin{tikzcd} & G_{k,k+1} \arrow[dl, "p_{k+1}"'] \arrow[dr, "p_k"] & \\ G_{k+1} & & G_k \end{tikzcd}$$

induce maps in cohomology

$$\begin{tikzcd} & H(G_{k,k+1}) & \\ H(G_{k+1}) \arrow[ur, "\psi_{k+1}"] & & H(G_k) \arrow[ul, "\phi_k"'] \end{tikzcd}$$

which make $H(G_{k,k+1})$ an $(H(G_{k+1}), H(G_k))$-superbimodule. As a right $H(G_k)$-supermodule, the bimodule $H(G_{k,k+1})$ is a free, bigraded module isomorphic to $H(G_k) \otimes \mathbb{Q}[\xi]$.

We take

$$\Omega_{k+1,k} = H(G_{k,k+1}) \otimes \mathrm{Ext}_{H(G_{k+1})}(\mathbb{Q}, \mathbb{Q}).$$

We put $\psi^*_{k+1} = \psi_{k+1} \otimes 1 \colon \Omega_{k+1} \to \Omega_{k+1,k}$ and define $\phi^*_k \colon \Omega_k \to \Omega_{k+1,k}$ as the map sending $x_i$ to $x_i$ and $s_i$ to $s_i + \xi s_{i+1}$:

$$\begin{tikzcd} & \Omega_{k+1,k} & \\ \Omega_{k+1} \arrow[ur, "\psi^*_{k+1}=\psi_{k+1}\otimes 1"] & & \Omega_k. \arrow[ul, "\phi^*_k"'] \end{tikzcd}$$

This gives $\Omega_{k+1,k}$ the structure of an $(\Omega_{k+1}, \Omega_k)$-superbimodule. We write $\Omega_{k,k+1}$ for $\Omega_{k+1,k}$ when seen as an $(\Omega_k, \Omega_{k+1})$-superbimodule. It is easy to see that up to an overall shift, the superring $\Omega_{k+1,k}$ has the desired properties.

For each $k \geqslant 0$ define exact functors[†] $\mathsf{F}_k \colon \mathcal{M}_k \to \mathcal{M}_{k+1}$ and $\mathsf{E}_k \colon \mathcal{M}_{k+1} \to \mathcal{M}_k$ by

$$\mathsf{F}_k(-) = \mathrm{Res}^{k+1,k}_{k+1} \circ \Omega_{k+1,k} \otimes_{\Omega_k} (-)\langle -k, 0\rangle,$$

and

$$\mathsf{E}_k(-) = \mathrm{Res}^{k+1,k}_{k} \circ \Omega_{k,k+1} \otimes_{\Omega_{k+1}} (-)\langle k+2, -1\rangle.$$

Functors $(\mathsf{F}, \mathsf{E})$ form an adjoint pair up to grading shifts, but $\mathsf{F}$ does not admit $\mathsf{E}$ as a left adjoint. We would like to stress that this is necessary to prevent us from falling in the situation of $Q$-strong 2-representations from [**10**]. In that case, the construction would lift to a 2-representation of $\dot{\mathcal{U}}$ and Rouquier's results in [**46**] would imply that if the functor $\mathsf{E}$ kills a highest weight then its biadjoint functor would kill a lowest weight (this can be proved with a clever trick using degree zero bubbles from [**34**] to tunnel from the lowest weight to the highest weight[‡]). We should not expect a biadjunction between the functors $\mathsf{E}$ and $\mathsf{F}$ since it can be interpreted as a categorification of the involution exchanging operators $E$ and $F$ (up to coefficients).

Denote by $\Omega_k[\xi]$ the polynomial ring in $\xi$ with coefficients in $\Omega_k$ and by $\mathsf{Q}_k$ the functor of tensoring on the left with the $(\Omega_k, \Omega_k)$-superbimodule $\Pi\Omega_k[\xi]\langle 1, 0\rangle$, where $\Pi$ is the parity change functor. The categorical $\mathfrak{sl}_2$-action is encoded in a short exact sequence of functors,

$$0 \longrightarrow \mathsf{F}_{k-1} \circ \mathsf{E}_{k-1} \longrightarrow \mathsf{E}_k \circ \mathsf{F}_k \longrightarrow \mathsf{Q}_k\langle -2k-1, 1\rangle \ \oplus \Pi\mathsf{Q}_k\langle 2k+1, -1\rangle \longrightarrow 0. \tag{1}$$

From the work of Lauda in [**34**, **35**] adjusted to our context it follows that for each $n \geqslant 0$ there is an action of the nilHecke algebra $\mathrm{NH}_n$ on $\mathsf{F}^n$ and on $\mathsf{E}^n$. As a matter of fact, there is an enlargement of $\mathrm{NH}_n$, which we denote as $A_n$, acting on $\mathsf{F}^n$ and $\mathsf{E}^n$, also admitting a nice diagrammatic description (see § 1.1.6 for a sketch).

We define $\mathcal{M}$ as the direct sum of all the categories $\mathcal{M}_k$ and functors $\mathsf{F}$, $\mathsf{E}$ and $\mathsf{Q}$ in the obvious way. One of the main results in this paper is the following.

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[†] All our functors are, in fact, superfunctors which we tend to see as functors between categories endowed with a $\mathbb{Z}/2\mathbb{Z}$-action, whence the use of the terminology *functor*.

[‡] We thank Aaron Lauda for explaining this to us.

THEOREM 5.12. *The functors* $\mathsf{F}$ *and* $\mathsf{E}$ *induce an action of quantum* $\mathfrak{sl}_2$ *on the Grothendieck groups* $K_0(\mathcal{M})$, $G_0(\mathcal{M})$ *and* $\widehat{G}_0(\mathcal{M})$, *after specializing* $\pi = -1$. *With this action there are* $\mathbb{Q}[\![q]\!][q^{-1}, \lambda^{\pm 1}]$*-linear isomorphisms*

$$K_0(\mathcal{M}) \cong M_A(\lambda), \quad G_0(\mathcal{M}) \cong M_A^*(\lambda),$$

*of* $\dot{U}_\lambda$ *module and a* $\mathbb{Q}(\!(q,\lambda)\!)$*-linear isomorphism*

$$\widehat{G}_0(\mathcal{M}) \cong M(\lambda),$$

*of* $U_q(\mathfrak{sl}_2)$*-representations. Moreover, these isomorphisms send classes of projective indecomposables to canonical basis elements and classes of simples to dual canonical elements, whenever this makes sense.*

Form the 2-category $\mathfrak{M}(\lambda q^{-1})$ which is the completion under extensions of the 2-category whose objects are the categories $\mathcal{M}_k$, the 1-morphisms are cone-bounded, locally finite direct sums (see the Appendix) of shifts of functors from $\{E_k, F_k, Q_k, \mathrm{Id}_k\}$ and the 2-morphisms are (grading-preserving) natural transformations of functors. In this case the 2-category $\mathfrak{M}(\lambda q^{-1})$ is an example of a 2-*Verma module* for $\mathfrak{sl}_2$.

1.1.4. *Categorification of the Verma module with integral highest weight.* Forgetting the cohomological degree on the superrings $\Omega_k$ and $\Omega_{k,k+1}$ defines a forgetful functor into the category of $\mathbb{Z}$-graded $\Omega_k(-1)$-supermodules, where $\Omega(-1)$ is the ring $\Omega$ with the cohomological degree collapsed. This defines a category $\mathcal{M}(-1)$. A direct consequence of Theorem 5.11 is that the Grothendieck group $K_0(\mathcal{M})$ is isomorphic to the Verma module $M(-1)$.

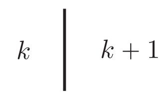
Our strategy to categorify $M(n)$ is to first define for each $n \in \mathbb{Z}$ certain sub-superrings $\Omega_k^n$ and $\Omega_{k,k+1}^n$ of $\Omega_k$ and $\Omega_{k,k+1}$, that agree with these for $n = -1$, and such that an immediate application of the procedure as before results in a categorification of the Verma module $M(\lambda q^n)$. We then apply the forgetful functor to define a categorification of $M(n)$.

As in the case of $\mathcal{M}$, for each $m \geqslant 0$ there is an action of the nilHecke algebra $\mathrm{NH}_m$ and of its enlargement $A_m$ on $\mathsf{F}^m$ and on $\mathsf{E}^m$.

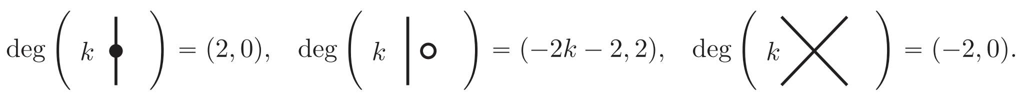
1.1.5. *A categorification of the* $(n+1)$*-dimensional irreducible representation from* $\mathcal{M}(\lambda)$. To recover the categorification of the finite-dimensional irreducible $V(n)$ from [**11**, **13**] we define for each $n \in \mathbb{N}$ a differential $d_n$ on the superrings $\Omega_k$ and $\Omega_{k,k+1}$, turning them in DG-algebras and DG-bimodules. These DG-algebras and DG-bimodules are quasi-isomorphic to the cohomologies of finite-dimensional Grassmannians and two-step flag manifolds in $\mathbb{C}^n$, as used in [**11**, **13**]. Moreover, the short exact sequence (1) can be turned into a short exact sequence of DG-bimodules that descends in the homology to the direct sums decompositions categorifying the $\mathfrak{sl}_2$-commutator relation in [**11**, **13**]. This means we can also interpret our construction as a DG-enhancement of [**11**, **13**]. The nilHecke algebra action descends to the usual nilHecke algebra action on integrable 2-representations of $\mathfrak{sl}_2$ (see [**10**, **34**, **35**, **46**]), and thus we call the enlargement of the nilHecke algebra $A_n$ the *enhanced nilHecke algebra*. The differential $d_n$ descends to the superrings $\Omega_k(n)$ and $\Omega_{k,k+1}(n)$ yielding the same result as in $\mathcal{M}(\lambda)$.

1.1.6. *A diagrammatic presentation for the enhanced nilHecke algebra.* The enhanced nilHecke algebra $A_n$ can be given a presentation in the spirit of KLR algebras [**28**, **34**, **46**] as isotopy classes of braid-like diagrams modulo some relations.

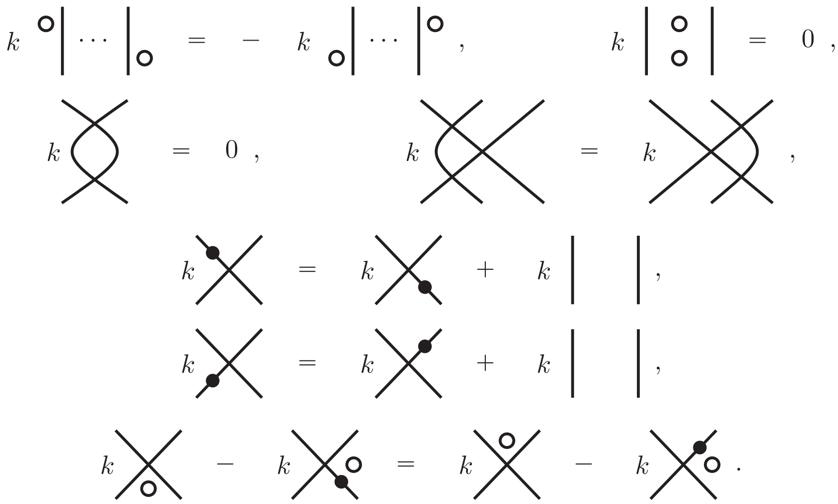
Our diagrams are isotopy classes of KLR diagrams with some extra structure. Besides the KLR dots we have another type of dot we call a *floating dot* (we keep the name *dot* for the KLR dots), which lives inside the regions between instead of on the strands, with condition that there is no floating dot in the leftmost region. Floating dots are made to satisfy the exterior algebra relations (see below). The enhanced nilHecke algebra is in fact a bigraded superalgebra, where nilHecke generators are even and floating dots are odd. Moreover, regions in a diagram are labeled with integer numbers, and crossing a strand from left to right increases the label by 1:




Fix a base ring $\Bbbk$. The $\Bbbk$-superalgebra $A_n$ consists of $\Bbbk$-linear combinations of $n$-strand diagrams as described above. The multiplication is given by concatenation of diagrams whenever the labels of the regions agree and zero otherwise. The superalgebra $A_n$ is bigraded with the $q$-degree of the floating dot given by minus two times the label of the region containing it:




The generators are subject to the following local relations:



All other isotopies are allowed (for example, switching the relative height of a dot and a floating dot, or a distant crossing and a floating dot). The relations above respect the bigrading as well as the parity.

We define $A_n(m)$ as the sub-superalgebra consisting of all diagrams with label $m$ at the leftmost region and

$$A(m) = \bigoplus_{n \geqslant 0} A_n(m).$$

The usual inclusion $A_n(m) \hookrightarrow A_{n+1}(m)$ that adds a strand at the right of a diagram from $A_n(m)$ gives rise to induction and restriction functors on $A(m)$-smod$_{\mathrm{lf}}$ that satisfy the $\mathfrak{sl}_2$-relations. Our results imply that together with these functors, $A(m)$-smod$_{\mathrm{lf}}$ categorifies the Verma module $M(\lambda q^{m-1})$. See the following paper [41] for more details about this construction and the combinatorics of $A_n$. The categorification of Verma modules with integral highest weight using the algebras $A_n$ follows as a consequence of our results. Moreover, we define for each $m \in \mathbb{N}$ a differential on $A_n(m)$ turning it into a DG-algebra, which is quasi-isomorphic to a cyclotomic quotient of the nilHecke algebra.

## 2. $U_q(\mathfrak{sl}_2)$ *and its representations*

### 2.1. *Forms of quantum* $\mathfrak{sl}_2$

The notions below are well known and can be found, for example, in [18, 38].

DEFINITION 2.1. The *quantum algebra* $U_q(\mathfrak{sl}_2)$ is the unital associative algebra over $\mathbb{Q}(q)$ with generators $E$, $F$, $K$ and $K^{-1}$ subject to the relations:

$$KK^{-1} = 1 = K^{-1}K,$$

$$KE = q^2 EK,$$

$$KF = q^{-2} FK,$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

We denote by $U_q(\mathfrak{b})$ the subalgebra of $U_q(\mathfrak{sl}_2)$ generated by $E$, $K$ and $K^{-1}$.

Define the quantum integer $[a] = (q^a - q^{-a})/(q - q^{-1})$, the quantum factorial $[a]! = [a][a-1]!$ with $[0]! = 1$, and the quantum binomial coefficient $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = [a]!/([b]![a-b]!)$ for $0 \leqslant b \leqslant a$, and put $\{a\} = q^{a-1}[a]$. For $a \geqslant 0$ define also the divided powers

$$E^{(a)} = \frac{E^a}{[a]!} \quad \text{and} \quad F^{(a)} = \frac{F^a}{[a]!}.$$

Following [34, § 2.1 and 2.2] we now introduce some important algebra (anti)automorphisms on $U_q(\mathfrak{sl}_2)$. Let $^-$, $\psi$, $\tau$ and $\rho$ be as follows:

(1) $^-$ is the $\mathbb{Q}$-linear involution that maps $q$ to $q^{-1}$.
(2) $\psi$ is the $\mathbb{Q}(q)$-antilinear algebra automorphism of $U_q(\mathfrak{sl}_2)$ given by

$$\psi(E) = E, \quad \psi(F) = F, \quad \psi(K) = K^{-1},$$

$$\psi(pX) = \overline{p}\psi(X), \quad \text{for } p \in \mathbb{Q}(q) \text{ and } X \in U_q(\mathfrak{sl}_2).$$

(3) $\tau\colon U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)^{op}$ is the $\mathbb{Q}(q)$-antilinear isomorphism given by

$$\tau(E) = q^{-1}K^{-1}F, \quad \tau(F) = q^{-1}KE, \quad \tau(K) = K^{-1}, \tag{2}$$

and

$$\tau(pX) = \overline{p}\tau(X), \quad \text{for } p \in \mathbb{Q}(q) \text{ and } X \in U_q(\mathfrak{sl}_2),$$

$$\tau(XY) = \tau(Y)\tau(X), \quad \text{for } X, Y \in U_q(\mathfrak{sl}_2).$$

(4) $\rho$ is the $\mathbb{Q}(q)$-linear algebra anti-involution defined by

$$\rho(E) = q^{-1}K^{-1}F, \quad \rho(F) = q^{-1}KE, \quad \rho(K) = K, \tag{3}$$

and

$$\rho(pX) = p\rho(X), \quad \text{for } p \in \mathbb{Q}(q) \text{ and } X \in U_q(\mathfrak{sl}_2),$$

$$\rho(XY) = \rho(Y)\rho(X), \quad \text{for } X, Y \in U_q(\mathfrak{sl}_2).?$$

The inverse of $\tau$ is given by $\tau^{-1}(E) = q^{-1}FK$, $\tau^{-1}(F) = q^{-1}EK^{-1}$, and $\tau^{-1}(K) = K^{-1}$.

REMARK 2.2. The $\rho$ defined above should be $\psi\rho\psi$ in the notations from [34].

2.1.1. *Deformed idempotented $U_q(\mathfrak{sl}_2)$.* For $c$ either integer or formal parameter, the shifted weight lattice is given by $c+\mathbb{Z}$. For $n \in \mathbb{Z}$ we denote by $e_n$ the idempotent corresponding to the projection onto the $(\lambda q^n)$-weight space. On this weight space, $K$ acts as multiplication by $\lambda q^n$:

$$e_n K = K e_n = \lambda q^n e_n. \tag{4}$$

In the spirit of Lusztig [**38**, Chapter 23] we now adjoin to $U_q(\mathfrak{sl}_2)$ the idempotents $e_n$ for all $n \in \mathbb{Z}$. Denote by $I$ the ideal generated by the relations (4) above together with

$$e_n e_m = \delta_{n,m} e_n, \quad E e_n = e_{n+2} E, \quad F e_n = e_{n-2} F. \tag{5}$$

Definition 2.3. Define the *shifted idempotented quantum* $\mathfrak{sl}_2$ as the $\mathbb{Q}(q)[\lambda^{\pm 1}]$-algebra

$$\dot{U}_\lambda = \left( \bigoplus_{m,n\in\mathbb{Z}} e_n \left(U_q(\mathfrak{sl}_2)\right) e_m \right) / I.$$

In this deformed version the main $\mathfrak{sl}_2$-relation becomes

$$EFe_n - FEe_n = \frac{\lambda q^n - \lambda^{-1} q^{-n}}{q - q^{-1}} e_n = [\lambda, n] e_n. \tag{6}$$

In the special case $\lambda = q^n$, we will see $\dot{U}_\lambda = \dot{U}_n$ as a $\mathbb{Q}(q)$-algebra.

The involution $^-$ and the algebra maps $\psi$, $\tau$ and $\rho$ introduced above extend to $\dot{U}_\lambda$ if we put

$$\overline{\lambda} = \lambda^{-1}, \quad \psi(e_n) = e_n, \quad \tau(e_n) = e_n, \quad \rho(e_n) = e_n.$$

The extended versions of $\psi$, $\tau$ and $\rho$ then take the form

$$\psi(q^s e_{n+2} E e_n) = q^{-s} e_{n+2} E e_n, \qquad \psi(q^s e_n F e_{n+2}) = q^{-s} e_n F e_{n+2},$$

$$\tau(q^s e_{n+2} E e_n) = \lambda^{-1} q^{-s-1-n} e_n F e_{n+2}, \qquad \tau(q^s e_n F e_{n+2}) = \lambda q^{-s+1+n} e_{n+2} E e_n,$$

and

$$\rho(q^s e_{n+2} E e_n) = \lambda^{-1} q^{s-1-n} e_n F e_{n+2}, \quad \rho(q^s e_n F e_{n+2}) = \lambda q^{s+1+n} e_{n+2} E e_n.$$

2.2. *Representations*

We only consider modules of type I in this paper and we follow the notations from [**18**]. As before, we write $\lambda = q^c$ for $c$ formal, and treat it as a formal parameter itself. Then there is an infinite-dimensional $U_q(\mathfrak{sl}_2)$ highest weight-module $M(\lambda)$ with the highest weight $\lambda$, called the *universal Verma module* (as explained in § 1.1.1 we follow the notation in [**18**] for Verma modules for quantum groups). Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{sl}_2$ and let $\mathbb{C}_\lambda = \mathbb{Q}((q,\lambda)) v_\lambda$ be a one-dimensional representation of $U_q(\mathfrak{b})$ with $E$ acting trivially while $K v_\lambda = \lambda v_\lambda$. The Verma module $M(\lambda)$ with the highest weight $\lambda$ is the induced module

$$M(\lambda) = U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{b})} \mathbb{C}_\lambda.$$

It has basis $m_0, m_1, \ldots, m_k, \ldots$, such that for all $i \geqslant 0$

$$\begin{aligned} K m_i &= \lambda q^{-2i} m_i, \\ F m_i &= [i+1] m_{i+1}, \\ E m_i &= \begin{cases} 0 & \text{if } i = 0, \\ \dfrac{\lambda q^{-i+1} - \lambda^{-1} q^{i-1}}{q - q^{-1}} m_{i-1} & \text{otherwise.} \end{cases} \end{aligned} \tag{7}$$

We call this basis the *canonical basis* of $M(\lambda)$. The change of basis $m'_i = [i]!m_i$ gives $M(\lambda)$ the following useful presentation of $M(\lambda)$:

$$\begin{aligned} Km'_i &= \lambda q^{-2i} m'_i, \\ Fm'_i &= m'_{i+1}, \\ Em'_i &= \begin{cases} 0 & \text{if } i = 0, \\ [i]\dfrac{\lambda q^{-i+1} - \lambda^{-1} q^{i-1}}{q - q^{-1}} m'_{i-1} & \text{else.} \end{cases} \end{aligned} \tag{8}$$

We denote by $M_\alpha$ the one-dimensional weight spaces of weight $\alpha$. We can picture $M(\lambda)$ as the following diagram:

$$\begin{tikzcd} \cdots & M_{\lambda-2k} \arrow[r, bend left, "E"] & \cdots \arrow[l, bend left, "F"] & M_{\lambda-4} \arrow[r, bend left, "E"] & M_{\lambda-2} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & M_\lambda \arrow[l, bend left, "F"] \end{tikzcd} .$$

The Verma module $M(\lambda)$ is the unique infinite-dimensional module of the highest weight $\lambda$, and it is irreducible unless $c \in \mathbb{N}$. To keep the notation simple we write $M(n)$ instead of $M(q^n)$ whenever $c = n \in \mathbb{Z}$. In this case $\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(M(n'), M(n))$ is zero unless $n' = n$ or $n' = -n-2$, and there is a monomorphism $\phi\colon M(-n-2) \to M(n)$, uniquely determined up to scalar multiples. Moreover, the quotient $M(n)/M(-n-2)$ is isomorphic to the irreducible $U_q(\mathfrak{sl}_2)$-module $V(n)$ of dimension $n+1$, and all finite-dimensional irreducibles can be obtained this way. Under this quotient, the canonical basis of $M(n)$ descends to a particular case of Lusztig–Kashiwara canonical basis in finite-dimensional irreducible representations of quantum groups introduced in [**37**] and independently in [**24**].

The Verma module $M(\lambda)$ is *universal* in the sense that any given Verma module with integral highest weight can be obtained from $M(\lambda)$. This means that for each $n \in \mathbb{Z}$ there is an *evaluation map* $\mathrm{ev}_n\colon M(\lambda) \to M(n)$ which is a surjection (see [**45**] and also [**19**, **23**] for details).

Throughout this paper we will take $M(\lambda q^{-1})$ as the universal Verma module and we will call $M(\lambda q^{-1+n})$ ($n \in \mathbb{Z}$) the *shifted Verma modules* (see [**45**] for details). In our conventions, evaluating $M(\lambda q^{-1})$ at $n$ means putting $\lambda = q^{n+1}$. The evaluation map $\mathrm{ev}_n$ is then the composite of a shift with $\mathrm{ev}_{-1}$.

## 2.3. *Bilinear form*

The *universal Shapovalov form* $(-,-)_\lambda$ is the bilinear form on $M(\lambda q^{-1})$ such that for any $m, m' \in M(\lambda q^{-1})$, $u \in U_q(\mathfrak{sl}_2)$, and $f \in \mathbb{Q}(\!(q,\lambda)\!)$ we have

- $(m_0, m_0)_\lambda = 1$;
- $(um, m')_\lambda = (m, \rho(u)m')_\lambda$, where $\rho$ is the $\mathbb{Q}(q)$-linear antiautomorphism defined in equation (3);
- $f(m, m')_\lambda = (fm, m')_\lambda = (m, fm')_\lambda$.

The involution $^-$ does not extend to $\mathbb{Q}(\!(q,\lambda)\!)$ (for example, $\sum_{k\geqslant 0} q^k$ would be sent to $\sum_{k\geqslant 0} q^{-k}$ which is not an element of $\mathbb{Q}(\!(q,\lambda)\!)$, see Appendix A.3 for more details about $\mathbb{Q}(\!(q,\lambda)\!)$). However, when restricting the ground field to $\mathbb{Q}(q,\lambda)$ instead of $\mathbb{Q}(\!(q,\lambda)\!)$ (we write $M_{\mathbb{Q}(q,\lambda)}(\lambda q^{-1})$ in this case) there is another form we can define. We refer to it as the *twisted Shapovalov form*, and it is the sesquilinear form uniquely defined by

- $\langle m_0, m_0\rangle_\lambda = 1$;
- $\langle um, m'\rangle_\lambda = \langle m, \tau(u)m'\rangle_\lambda$, where $\tau$ is the $q$-antilinear antiautomorphism defined in equation (2);

- $f\langle m,m'\rangle_\lambda = \langle \bar{f}m,m'\rangle_\lambda = \langle m, fm'\rangle_\lambda$, where $^-$ is the $\mathbb{Q}$-linear involution of $\mathbb{Q}(q,\lambda)$ which maps $q$ to $q^{-1}$ and $\lambda$ to $\lambda^{-1}$;

for any $m,m' \in M_{\mathbb{Q}(q,\lambda)}(\lambda q^{-1})$, $u \in U_q(\mathfrak{sl}_2)$, and $f \in \mathbb{Q}(q,\lambda)$.

For example,

$$\langle F^n m_0, F^n m_0\rangle_\lambda = \lambda^n q^{-n(1+n)}[n]![\lambda,-1][\lambda,-2]\cdots[\lambda,-n],$$

the notation $[\lambda,m]$ being introduced in (6).

Evaluation of $M(\lambda q^{-1})$ at $n$ reduces to the well-known $\mathbb{Q}(q)$-valued bilinear form (see [**45**, **47**] for the original definition in the non-quantum context as well as a proof of uniqueness). The *$q$-Shapovalov form* $(-,-)_n$ is the unique bilinear form on $M(n)$ such that for any $m,m' \in M(n)$, $u \in U_q(\mathfrak{sl}_2)$, and $f \in \mathbb{Q}(q)$ we have

- $(v_0,v_0)_n = 1$;
- $(um,m')_n = (m,\rho(u)m')_n$, where $\rho$ is the $q$-linear antiautomorphism defined in equation (3);
- $f(m,m')_n = (fm,m')_n = (m,fm')_n$.

For $n \geqslant 0$ the radical of $(-,-)_n$ is the maximal proper submodule $M(-n-2)$ of $M(n)$, and hence we have $V(n) = M(n)/\mathrm{Rad}(-,-)_n$ and the $q$-Shapovalov form descends to a bilinear form on $V(n)$.

Using the Shapovalov form we define the *dual canonical basis*$\{m^i\}_{i\in\mathbb{N}_0}$ of $M(\lambda)$ by

$$(m_i', m^j)_\lambda = \delta_{i,j}.$$

Define $[\lambda,j]!$ recursively by

$$[\lambda,0]! = 1, \quad [\lambda,j]! = [\lambda,j-1]![\lambda,j].$$

Then

$$m^k = \frac{[k]!}{[\lambda,-k]!\lambda^k q^{-k(k+1)}} m_k,$$

and the action of $F$, $E$ and $K$ on the dual canonical basis is

$$\begin{aligned} K^{\pm1}m^k &= (\lambda q^{-2k})^{\pm1} m^k, \\ Fm^k &= \tfrac{\lambda q^{-k} - \lambda q^k}{q-q^{-1}} \lambda q^{-2k-1} m^{k+1}, \\ Em^k &= [k]\lambda^{-1}q^{2k-1}m^{k-1}. \end{aligned} \tag{9}$$

The above reduces without any changes to the case of $M(n)$ for $n \notin \mathbb{N}_0$. For $n \in \mathbb{N}_0$ the procedure cannot be applied on $M(n)$. However, it can be used in the finite-dimensional quotient $V(n)$ yielding the usual dual canonical basis of finite-dimensional representations, up to a normalization (see, for example, the exposition in [**13**, § 1.2]).

Restricting the ground field to the ring $\mathbb{Q}[\![q]\!][q^{-1},\lambda^{\pm1}]$ yields two $\dot{U}_\lambda$-modules, one given by the basis $\{m_k\}_{k\in\mathbb{N}}$ and the other one by $\{m^k\}_{k\in\mathbb{N}}$, which are non-isomorphic. We denote them, respectively, $M_A(\lambda)$ and $M_A^*(\lambda)$, where $A = \mathbb{Q}[\![q]\!][q^{-1},\lambda^{\pm1}]$.

## 3. *The geometry of the infinite Grassmannian*

### 3.1. *Grassmannians and their* Ext *algebras*

Let $G_k$ be the Grassmannian variety of $k$-planes in $\mathbb{C}^\infty$. This space classifies $k$-dimensional complex vector bundles over a manifold $\mathcal{N}$, in the sense that there is a tautological bundle over $G_k$, and every $k$-dimensional vector bundle over $\mathcal{N}$ is a pull-back of the tautological bundle by some map from $\mathcal{N}$ to $G_k$. Since this pull-back is invariant under homotopy we actually study homotopy classes of maps from $\mathcal{N}$ to $G_k$. The cohomology ring of $G_k$ is generated by the Chern classes (see, for example, [40, Chapter 14] for details),

$$H(G_k) \cong \mathbb{Q}[x_{1,k}, \dots, x_{k,k}, Y_{1,k}, \dots, Y_{i,k}, \dots]/I_{k,\infty},$$

where $I_{k,\infty}$ is the ideal generated the homogeneous components in $t$ satisfying the equation

$$(1 + x_{1,k}t + \cdots + x_{k,k}t^k)(1 + Y_{1,k}t + \cdots + Y_{i,k}t^i + \cdots) = 1. \tag{10}$$

This ring is $\mathbb{Z}$-graded with $\deg_q(x_{i,k}) = \deg_q(Y_{i,k}) = 2i$. Note that (10) yields recursively

$$Y_{i,k} = -\sum_{\ell=1}^{i} x_{\ell,k} Y_{i-\ell,k}, \tag{11}$$

where $Y_{i,k} = 0$ if $i < 0$, $Y_{0,k} = 1$ and $x_{j,k} = 0$ for $j > k$. Since every $Y_{i,k}$ can be written as a combination of $x_{j,k}$, we have

$$H(G_k) \cong \mathbb{Q}[x_{1,k}, \dots, x_{k,k}].$$

Now let $G_{k,k+1}$ be the infinite partial flag variety

$$G_{k,k+1} = \{(U_k, U_{k+1}) | \dim_{\mathbb{C}} U_k = k, \dim_{\mathbb{C}} U_{k+1} = k+1,\ 0 \subset U_k \subset U_{k+1} \subset \mathbb{C}^\infty\}.$$

As it turns out, the infinite Grassmannian $G_k$ is homotopy equivalent to the classifying space $BU(k)$ of the unitary group $U(k)$, and we have a fibration

$$B \to BU(k) \times BU(1) \to BU(k+1)$$

induced by the inclusion $U(k) \times U(1) \to U(k+1)$. The fiber has the homotopy type of the quotient $U(k+1)/(U(k) \times U(1))$ and corresponds to $G_{k,k+1}$ in the sense that specifying $U_k \subset U_{k+1}$ in $\mathbb{C}^\infty$ corresponds to specifying $U_k$ in $\mathbb{C}^\infty$ and a one-dimensional Grassmannian in $U_k$. As a consequence, we get that the cohomology of $B$, and therefore of $G_{k,k+1}$, is generated by the Chern classes

$$H(G_{k,k+1}) \cong \mathbb{Q}[w_{1,k}, \dots, w_{k,k}, \xi_{k+1}, Z_{1,k+1}, \dots, Z_{i,k+1}, \dots]/I_{k,k+1,\infty},$$

with $I_{k,k+1,\infty}$ given by the equation

$$(1 + w_{1,k}t + \cdots + w_{k,k}t^k)(1 + \xi_{k+1}t)(1 + Z_{1,k+1}t + \cdots + Z_{i,k+1}t^i + \cdots) = 1.$$

Without surprise, $H(G_{k,k+1})$ has a natural structure of a $\mathbb{Z}$-graded ring with

$$\deg_q(w_{i,k}) = \deg_q(Z_{i,k+1}) = 2i, \quad \deg_q(\xi_{k+1}) = 2.$$

Again, we can write every $Z_{i,k+1}$ as a combination of $w_{j,k}$ and $\xi_{k+1}$ to get

$$H(G_{k,k+1}) \cong \mathbb{Q}[w_{1,k}, \dots, w_{k,k}, \xi_{k+1}].$$

The ring $H(G_k)$ is a graded positive noetherian ring which has a unique simple module, up to isomorphism and grading shift, $H(G_k)/H(G_k)_+ \cong \mathbb{Q}$, where $H(G_k)_+$ is the submodule of $H(G_k)$ generated by the elements of non-zero degree. Let $\mathrm{Ext}_{H(G_k)}(\mathbb{Q}, \mathbb{Q})$ be the algebra of self-extensions of $\mathbb{Q}$, which is an exterior algebra in $k$ variables,

$$\mathrm{Ext}_{H(G_k)}(\mathbb{Q}, \mathbb{Q}) \cong {\textstyle\bigwedge}^\bullet(s_{1,k}, \dots, s_{k,k}).$$

It is a $\mathbb{Z} \times \mathbb{Z}$-graded ring with $\deg_q(s_{i,k}) = -2i$ and $\deg_\lambda(s_{i,k}) = 2$. The first grading is induced by the grading in $H(G_k)$ and we call it quantum, while the second grading is cohomological. Sometimes we write $\deg_{q,\lambda}(x)$ for the ordered pair $(\deg_q(x), \deg_\lambda(x))$.

In another way of looking at this we note that $H(G_k)$ is a Koszul algebra and therefore quadratic. Indeed let $V_k$ be the $\mathbb{Q}$-vector space with basis $\langle x_{1,k}, \dots, x_{k,k}\rangle$ and $R =$

$\{x_{i,k}x_{j,k} - x_{j,k}x_{i,k} | 1 \leqslant i,j \leqslant k\}$, then $H(G_k) \cong T(V_k)/(R)$ and the Koszul dual of $H(G_k)$ coincides with the quadratic dual $H(G_k)^! = T(V_k^*)/(R^\perp)$, with $R^\perp = \{f \in V_k^* \otimes V_k^* | f(R) = 0\}$ (see [**4**, § 2.10]). An easy exercise shows that $H(G_k)^! \cong \bigwedge^\bullet(s_{1,k}, \dots, s_{k,k})$, where we identify $s_{i,k}$ with $x_{i,k}^* : V_k \to \mathbb{Q}$. In conclusion we have an isomorphism $H(G_k) \otimes H(G_k)^! \cong H(G_k) \otimes (H(G_k)^!)^{op} \cong H(G_k) \otimes \mathrm{Ext}_{H(G_k)}(\mathbb{Q},\mathbb{Q})$.

Definition 3.1. For each $k \in \mathbb{N}$ we form the bigraded rings

$$\Omega_k = H(G_k) \otimes \mathrm{Ext}_{H(G_k)}(\mathbb{Q},\mathbb{Q})$$

and

$$\Omega_{k,k+1} = H(G_{k,k+1}) \otimes \mathrm{Ext}_{H(G_{k+1})}(\mathbb{Q},\mathbb{Q}).$$

Note that we do not use extensions of $H(G_{k,k+1})$-modules and also $\Omega_k$ is isomorphic to the Hochschild cohomology of $H(G_k)$. In order to fix some notation and avoid any possibility of confusion in future computations we fix presentations of these rings as

$$\Omega_k = \mathbb{Q}[\underline{x}_k, \underline{s}_k], \quad \text{and} \quad \Omega_{k,k+1} = \mathbb{Q}[\underline{w}_k, \xi_{k+1}, \underline{\sigma}_{k+1}],$$

where we write $\underline{t}_m$ for an array $(t_{1,m}, \dots, t_{m,m})$ of $m$ variables and where it is abusively implied that the variables $s_i$ and $\sigma_i$ are anticommutative.

Rings $\Omega_k$ and $\Omega_{k,k+1}$ are in fact (supercommutative rings) superrings with an inherent $\mathbb{Z}_2$-grading, called *parity*, given by

$$p(x_{i,k}) = 0, \quad p(s_{i,k}) = 1,$$

for $x_{i,k}$, $s_{i,k} \in \Omega_k$, and

$$p(w_{i,k}) = p(\xi_{k+1}) = 0, \quad p(\sigma_{i,k+1}) = 1,$$

for $w_i$, $\xi_{k+1}$ and $\sigma_i \in \Omega_{k,k+1}$.

## 3.2. *Superbimodules*

Let $R$ be a superring. A left (respectively, right) $R$-supermodule is a $\mathbb{Z}_2$-graded left (respectively, right) $R$-module. A left supermodule map $f : M \to N$ is a homogeneous group homomorphism that supercommutes with the action of $R$,

$$f(r \bullet m) = (-1)^{p(f)p(r)} r \bullet f(m),$$

for all $r \in R$ and $m \in M$. A right supermodule map is a homogeneous right module homomorphism. An $(R,R')$-superbimodule is both a left $R$-supermodule and a right $R'$-supermodule, with compatible actions. A superbimodule map is both a left supermodule map and a right supermodule map.

Then, if $R$ has a supercommutative ring structure and if we view it as an $(R,R)$-superbimodule, multiplying at the left by an element of $R$ gives rise to a superbimodule endomorphism.

Let $M$ and $N$ be, respectively, an $(R',R)$ and an $(R,R'')$-superbimodules. One form their tensor product over $R$ in the usual way for bimodules, giving a superbimodule. Given two superbimodule maps $f : M \to M'$ and $g : N \to N'$, we can form the tensor product $f \otimes g : M \otimes N \to M' \otimes N'$, which is defined by

$$(f \otimes g)(b \otimes m) = (-1)^{p(g)p(b)} f(b) \otimes g(m),$$

and gives a superbimodule map.

Now define the *parity shift* of a supermodule $M$, denoted $\Pi M = \{\pi(m) | m \in M\}$, where $\pi(m)$ is the element $m$ with the parity inversed, and if $M$ is a left supermodule (or superbimodule) with left action given by

$$r \bullet \pi(m) = (-1)^{p(r)} \pi(r \bullet m),$$

for $r \in R$ and $m \in M$. The action on the right remains the same.

In this context, the map $R \to \Pi R$ defined by $r \mapsto \pi(ar)$ for some odd element $a \in R$ is a $\mathbb{Z}/2\mathbb{Z}$-grading-preserving homomorphism of $(R, R)$-superbimodules.

Let $\pi : M \to \Pi M$ denote the change of parity map $x \mapsto \pi(x)$. It is a supermodule map with parity 1 and satisfies $\pi^2 = \mathrm{Id}$. The map $\pi \otimes \pi : \Pi M \otimes N \to M \otimes \Pi N$ is $\mathbb{Z}/2\mathbb{Z}$-grading-preserving and such that $(\pi \otimes \pi)^2 = -\mathrm{Id}$, thus

$$\Pi(M \otimes N) \cong \Pi M \otimes N \cong M \otimes \Pi N$$

are isomorphisms of supermodules. All the above are presented with a more categorical flavor in [**12**, **21**] (see also [**22**]), showing that the supermodules and superbimodules give supercategories. Of course, all the above extend to the case when $R$ has additional gradings, making it a multigraded superring.

### 3.3. *Graded dimensions*

Recall that a $\mathbb{Z} \times \mathbb{Z}$-graded supermodule

$$M = \bigoplus_{i,j,k \,\in, \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_2} M_{i,j,k}$$

is *locally of finite rank* if each $M_{i,j,k}$ has finite rank. The same notion applies for bigraded vector spaces. We denote $M\langle r, s\rangle$ the supermodule with the $q$-grading shifted up by $r$ and the $\lambda$-grading shifted up by $s$.

In the context of locally finite rank supermodules and vector spaces it makes sense to talk about graded ranks and graded dimensions. In the cases under consideration the graded dimension of $M$ is the Poincaré series

$$\mathrm{gdim}(M) = \sum_{i,j,k \,\in\, \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_2} \pi^k \lambda^j q^i \dim(M_{i,j,k}) \in \mathbb{Z}_\pi[\![q^{\pm 1}, \lambda^{\pm 1}]\!], \tag{12}$$

where $\mathbb{Z}_\pi = \mathbb{Z}[\pi]/(\pi^2 - 1)$. For example,

$$\mathrm{gdim}(\Omega_k) = \prod_{s=1}^{k} (1 + \pi\lambda^2 q^{-2s})(1 + q^{2s} + q^{4s} + \cdots).$$

In this case, we can view $\mathrm{gdim}(\Omega_k)$ as living inside of $\mathbb{Z}_\pi(\!(q, \lambda)\!)$ (see Appendix A.3 for more details about $\mathbb{Z}(\!(q, \lambda)\!)$) and it gives

$$\mathrm{gdim}(\Omega_k) = \prod_{s=1}^{k} \frac{1 + \pi\lambda^2 q^{-2s}}{1 - q^{2s}}.$$

When we refer to the (graded) superdimension, we will mean we specialize $\pi = -1$ in the graded dimension, giving a series in $\mathbb{Z}[\![q^{\pm 1}, \lambda^{\pm 1}]\!]$. We denote it $\mathrm{sdim}(M)$.

Sometimes it is useful to consider a direct sum of objects (for example, supermodules or superbimodules) where the $q$-degree of each summand has been shifted by a different amount. In this case we use the notion of shifting an object by a Laurent polynomial: given $f = \sum f_j q^j \in \mathbb{N}[q, q^{-1}]$ we write $\oplus_f M$ or $M^{\oplus f}$ for the direct sum over $j \in \mathbb{Z}$ of $f_j$ copies of $M\langle j, 0\rangle$. In a further notational simplification will write $M\langle j\rangle$ for $M\langle j, 0\rangle$ whenever convenient.

3.4. *The superbimodules* $\Omega_{k,k+1}$

The forgetful maps

$$\begin{tikzcd} & G_{k,k+1} \arrow[dl, "p_{k+1}"'] \arrow[dr, "p_k"] & \\ G_{k+1} & & G_k \end{tikzcd}$$

induce maps in the cohomology

$$\begin{tikzcd} & H(G_{k,k+1}) & \\ H(G_{k+1}) \arrow[ur, "\psi_{k+1}"] & & H(G_k) \arrow[ul, "\phi_k"'] \end{tikzcd} \tag{13}$$

given by

$$\phi_k \colon H(G_k) \to H(G_{k,k+1}), \quad x_{i,k} \mapsto w_{i,k}, \quad Y_{i,k} \mapsto Z_{i,k+1} + \xi_{k+1} Z_{i-1,k+1},$$

and

$$\psi_{k+1} \colon H(G_{k+1}) \to H(G_{k,k+1}), \quad x_{i,k+1} \mapsto w_{i,k} + \xi_{k+1} w_{i-1,k}, \quad Y_{i,k+1} \mapsto Z_{i,k+1},$$

with the understanding that $w_{0,k} = Z_{0,k+1} = 1$ and $w_{k+1,k} = 0$.

These inclusions make $H(G_{k,k+1})$ an $(H(G_{k+1}), H(G_k))$-bimodule. As a right $H(G_k)$-module, $H(G_{k,k+1})$ is a free, graded module, isomorphic to $H(G_k) \otimes_{\mathbb{Q}} \mathbb{Q}[\xi_{k+1}]$.

To get a correspondence in terms of $\Omega_k$, $\Omega_{k+1}$ and $\Omega_{k,k+1}$ we use the maps $\phi_k$ and $\psi_{k+1}$ above to construct maps $\phi_k^*$ and $\psi_{k+1}^*$ between the various rings involved, as in (13),

$$\begin{tikzcd} & \Omega_{k,k+1} & \\ \Omega_{k+1} \arrow[ur, "\psi_{k+1}^*"] & & \Omega_k. \arrow[ul, "\phi_k^*"'] \end{tikzcd}$$

Let $V_{k,k+1}$ be the $\mathbb{Q}(\xi_{k+1})$-vector space with basis $\langle x_{1,k}, \dots, x_{k,k} \rangle$. The maps $\phi_k$ and $\psi_{k+1}$ induce $\mathbb{Q}$-linear injective maps $V_k \to V_{k,k+1}$ and $V_{k+1} \to V_{k,k+1}$. Now recall that we can view $s_{i,k}$ as the $\mathbb{Q}$-linear map $x_{i,k}^* : V_k \to \mathbb{Q}$ and $s_{i,k+1}$ as $x_{i,k+1}^* : V_{k+1} \to \mathbb{Q}$. They can both be extended to $\mathbb{Q}(\xi_{k+1})$-linear maps $\widetilde{x}_{i,k}^*, \widetilde{x}_{i,k+1}^* : V_{k,k+1} \to \mathbb{Q}(\xi_{k+1})$. We have $\psi_{k+1}(x_{i,k+1}) = \phi_k(x_{i,k}) + \xi_{k+1}\phi_k(x_{i-1,k})$, which gives $\phi(x_{i,k}) = \sum_{\ell=0}^{i} (-1)^\ell \xi_{k+1}^\ell \psi_{k+1}(x_{i-\ell,k+1})$, and thus

$$\widetilde{x}_{i,k}^* = \widetilde{x}_{i,k+1}^* + \xi_{k+1} \widetilde{x}_{i+1,k+1}^*.$$

Translated to the language of the elements $s_i$ we get that $s_{i,k}$ should be equivalent to $s_{i,k+1} + \xi_{k+1} s_{i+1,k+1}$. Hence we define the map $\phi_k^* \colon \Omega_k \to \Omega_{k,k+1}$ as

$$\phi_k^* \colon \Omega_k \to \Omega_{k,k+1}, \qquad \begin{cases} x_{i,k} \mapsto w_{i,k}, \\ s_{i,k} \mapsto \sigma_{i,k+1} + \xi_{k+1}\sigma_{i+1,k+1}, \end{cases} \tag{14}$$

and $\psi_{k+1}^* \colon \Omega_{k+1} \to \Omega_{k,k+1}$ as

$$\psi_{k+1}^* \colon \Omega_{k+1} \to \Omega_{k,k+1}, \qquad \begin{cases} x_{i,k+1} \mapsto w_{i,k} + \xi_{k+1} w_{i-1,k}, \\ s_{i,k+1} \mapsto \sigma_{i,k+1}, \end{cases} \tag{15}$$

with $w_{0,k+1} = 1$ and $w_{k+1,k} = 0$. Since every $\sigma_{i,k+1}$ and $w_{i,k}$ can be obtained from $s_{i,k+1}$ and $x_{i,k}$, we write $\Omega_{k,k+1}$ in this basis as

$$\Omega_{k,k+1} \cong \mathbb{Q}[\underline{x}_k, \xi_{k+1}, \underline{s}_{k+1}].$$

We will also write $Y_{i,k+1}$ for $Z_{i,k+1} = \psi_{k+1}^*(Y_{i,k+1})$ in $\Omega_{k,k+1}$.

REMARK 3.2. Note the elements $s_i$ and $\sigma_i$ behave like the elements $Y_{-i}$ and $Z_{-i}$ with a (supposed) negative index $-i$. This will be useful to recover the finite case, as we will see in § 7.3.

As expected, maps $\phi_k^*$ and $\psi_{k+1}^*$ give $\Omega_{k,k+1}$ the structure of an $(\Omega_k, \Omega_{k+1})$-superbimodule. Since these rings are supercommutative we can also think of $\Omega_{k,k+1}$ as an $(\Omega_{k+1}, \Omega_k)$-superbimodule which we denote by $\Omega_{k+1,k}$. When dealing with tensor products of superbimodules we simplify the notation and write $\otimes_k$ for $\otimes_{\Omega_k}$ and $\otimes$ for $\otimes_{\mathbb{Q}}$.

We use the notation smod-$\Omega_k$ and $\Omega_k$-smod for right and left $\Omega_k$-supermodules, respectively. As a right $\Omega_k$-supermodule $\Omega_{k+1,k} \cong \mathbb{Q}[\xi_{k+1}, s_{k+1}] \otimes \Omega_k$ is a free graded polynomial supermodule, which is of graded dimension

$$\mathrm{gdim}_{\text{smod-}\Omega_k}(\Omega_{k+1,k}) = \frac{1+\pi\lambda^2 q^{-2k-2}}{1-q^2}.$$

As a left $\Omega_{k+1}$-supermodule $\Omega_{k+1,k} \cong \oplus_{\{k+1\}}\Omega_{k+1}$ is a free graded supermodule, with $\{k+1\} = 1 + q^2 + \cdots + q^{2k}$, using the convention from § 3.3. Thus $\Omega_{k+1,k}$ is of graded dimension

$$\mathrm{gdim}_{\Omega_{k+1}\text{-smod}}(\Omega_{k+1,k}) = \{k+1\} = 1 + q^2 + \cdots + q^{2k}.$$

Due to the specific nature of our superrings and (super)categories, there are several notions and results that can be borrowed unchanged from the non-super case, as the notion of sweetness for bimodules below. Recall that a superbimodule is sweet if it is projective as a left supermodule and as a right supermodule. Tensoring with a superbimodule yields an exact functor that sends projectives to projectives if and only if the superbimodule is sweet. The superbimodule $\Omega_{k,k+1}$ is *sweet*.

Let $G_{k,k+1,\cdots,k+m}$ be the iterated flag variety

$$\{(U_k, U_{k+1}, \ldots, U_{k+m}) | \dim_{\mathbb{C}} U_{k+i} = k+i,\ 0 \subset U_k \subset U_{k+1} \subset \cdots \subset U_{k+m} \subset \mathbb{C}^\infty\}.$$

As in the cases of $G_k$ and $G_{k,k+1}$ the cohomology of $G_{k,\cdots,k+m}$ has a description in terms of Chern classes,

$$H(G_{k,\ldots,k+m}) \cong \mathbb{Q}[\underline{w}_k, \underline{\xi}_m],$$

with $\deg_q(w_{i,k}) = 2i$ and $\deg_q(\xi_{j,m}) = 2$. Paralleling the case of $\Omega_{k,k+1}$, we define the bigraded superring

$$\Omega_{k,\ldots,k+m} = H(G_{k,\ldots,k+m}) \otimes \mathrm{Ext}_{H(G_{k+m})}(\mathbb{Q},\mathbb{Q}) \cong \mathbb{Q}[\underline{w}_k, \underline{\xi}_m, \underline{\sigma}_{k+m}],$$

with $\deg_{\lambda,q}(\sigma_{j,k+m}) = (-2j, 2)$. In this case we also have maps

$$\phi_{k,m}^*\colon \Omega_k \to \Omega_{k,\ldots,k+m}, \qquad \begin{cases} x_{i,k} \mapsto w_{i,k}, \\ s_{i,k} \mapsto \sum\limits_{j=0}^{m} \sigma_{i+j,k+m} e_j(\underline{\xi}_m), \end{cases}$$

and

$$\psi_{k+m,m}^*\colon \Omega_{k+m} \to \Omega_{k,\ldots,k+m}, \qquad \begin{cases} x_{i,k+m} \mapsto \sum\limits_{j=0}^{i} w_{j,k} e_{i-j}(\underline{\xi}_m), \\ s_{i,k+m} \mapsto \sigma_{i,k+m}, \end{cases}$$

where $e_j(\underline{\xi}_m)$ is the $j$th elementary symmetric polynomial in the variables $\xi_1, \ldots, \xi_m$.

LEMMA 3.3. *The superring $\Omega_{k,k+1,\ldots,k+m}$ is a bigraded $(\Omega_k, \Omega_{k+m})$-superbimodule, which is isomorphic to $\Omega_{k,k+1} \otimes_{k+1} \Omega_{k+1,k+2} \otimes_{k+2} \cdots \otimes_{k+n-1} \Omega_{k+n-1,k+n}$.*

We can form more general superbimodules. For a sequence $k_1, \dots, k_m$ of non-negative integers we define the $(\Omega_{k_1}, \Omega_{k_m})$-superbimodule

$$\check{\Omega}_{k_1,\dots,k_m} = \Omega_{k_1,k_2} \otimes_{k_2} \Omega_{k_2,k_3} \otimes_{k_3} \cdots \otimes_{k_{m-1}} \Omega_{k_{m-1},k_m}.$$

This superbimodule has an interpretation in terms of the geometry of partial flag varieties. Consider the variety $G_{k_1,\dots,k_m}$ consisting of sequences $(U_{k_1}, \dots, U_{k_m})$ of linear subspaces of $\mathbb{C}^\infty$ such that $\dim(U_{k_i}) = k_i$ and $U_{k_i} \subset U_{k_{i+1}}$ if $k_i \leqslant k_{i+1}$ and $U_{k_i} \supset U_{k_{i+1}}$ if $k_i \geqslant k_{i+1}$. As before, the forgetful maps

$$\begin{tikzcd} & G_{k_1,\dots,k_m} \arrow[dl, "p_{k_1}"'] \arrow[dr, "p_{k_m}"] & \\ G_{k_1} & & G_{k_m} \end{tikzcd}$$

induce maps of the respective cohomology rings. Proceeding as above one can construct maps

$$\begin{tikzcd} & \Omega_{k_1,\dots,k_m} & \\ \Omega_{k_1} \arrow[ur, "\phi^*_{k_1}"] & & \Omega_{k_m}. \arrow[ul, "\psi^*_{k_m}"'] \end{tikzcd}$$

As expected, the $(\Omega_{k_1}, \Omega_{k_m})$-superbimodules $\check{\Omega}_{k_1,\dots,k_m}$ and $\Omega_{k_1,\dots,k_m}$ are isomorphic.

In particular, the isomorphism from $\Omega_{0,k}$ to $\check{\Omega}_{0,1,\dots,k}$ is explicitly given by

$$x_{i,k} \mapsto e_i(\underline{\xi}_k), \quad Y_{i,k} \mapsto (-1)^i h_i(\underline{\xi}_k), \quad s_{i,k} \mapsto \sigma_{i,k},$$

with $h_i(\underline{\xi}_k)$ being the $i$th complete homogeneous symmetric polynomial in variables $\xi_1, \dots, \xi_k$.

## 4. *The* 2-*category* $\mathbf{ExtFlag}_\lambda$

### 4.1. *The* 2-*category* $\mathbf{ExtFlag}_\lambda$

Let $\mathbf{Bim}^s$ denote the (super) 2-category of superbimodules, with objects given by superrings, 1-morphisms by superbimodules and 2-morphisms by degree-preserving superbimodule maps. The superbimodules introduced in the previous section can be used to define a locally full sub 2-category[†] of $\mathbf{Bim}^s$, which we now describe.

Definition 4.1. The 2-category $\mathbf{ExtFlag}_\lambda$ is defined as follows:

- Objects: the bigraded superrings $\Omega_k$ for each $k \in \mathbb{N}$.
- 1-Morphisms: generated by the graded $(\Omega_k, \Omega_k)$-superbimodules $\Omega_k$ and $\Omega_k^\xi = \Omega_k[\xi]$, the graded $(\Omega_k, \Omega_{k+1})$-superbimodule $\Omega_{k,k+1}$ and the graded $(\Omega_{k+1}, \Omega_k)$-superbimodule $\Omega_{k+1,k}$, together with their bidegree and parity shifts. The superbimodules $\Omega_k$ are the identity 1-morphisms. A generic 1-morphism from $\Omega_{k_1}$ to $\Omega_{k_m}$ is a direct sum of bigraded superbimodules of the form
$$\Pi^\pi \Omega_{k_m,k_{m-1}} \otimes_{k_{m-1}} \Omega_{k_{m-1},k_{m-2}} \otimes_{k_{m-2}} \cdots \otimes_{k_2} \Omega_{k_2,k_1} \otimes_{k_1} \Omega_{k_1}[\xi_1, \dots, \xi_\ell]\langle s, t\rangle$$
with $|k_{i+1} - k_i| = 1$ for all $1 \leqslant i \leqslant m$ and $\pi \in \{0, 1\}$.
- 2-Morphisms: degree-preserving superbimodule maps.

As in other instances of categorical $\mathfrak{sl}_2$-actions, the $(\Omega_k, \Omega_k)$-superbimodules $\Omega_{k,k+1} \otimes_{k+1} \Omega_{k+1,k}$ and $\Omega_{k,k-1} \otimes_{k-1} \Omega_{k-1,k}$ are related through a categorical version of the commutator

[†]This means it induces a full embedding between the corresponding Hom-categories.

relation (6). To make our formulas simpler when dealing with tensor products of superbimodules we write $\Omega_{k(k+1)k}$ instead of $\Omega_{k,k+1}\otimes_{k+1}\Omega_{k+1,k}$ and $\Omega_{k(k-1)k}$ instead of $\Omega_{k,k-1}\otimes_{k-1}\Omega_{k-1,k}$.

To be able to state and prove this categorical version of the commutator in $\mathbf{ExtFlag}_\lambda$, we need some preparation.

LEMMA 4.2. *In $\Omega_{k,k+1}$, the following identities hold for all $i,\ell\geqslant 0$:*

$$x_{\ell,k}=\sum_{p=0}^{\ell}(-1)^p\psi^*(x_{\ell-p,k+1})\xi_{k+1}^p, \tag{16}$$

$$Y_{\ell,(k+1)}=\sum_{p=0}^{\ell}(-1)^p\phi^*(Y_{\ell-p,k})\xi_{k+1}^p, \tag{17}$$

$$\xi_{k+1}^i=(-1)^i\sum_{\ell=0}^{i}x_{\ell,k}Y_{i-\ell,k+1}. \tag{18}$$

*Proof.* The three relations are obtained by induction on (14) and (15). □

LEMMA 4.3. *Each element of $\Omega_{k(k-1)k}$ decomposes uniquely as a sum*

$$(f_0\otimes_{k-1}g_0)+(f_1\xi_k\otimes_{k-1}g_1)+\cdots+(f_{k-1}\xi_k^{k-1}\otimes_{k-1}g_{k-1})$$

*with $f_i,g_i\in\psi_k^*(\Omega_k)$.*

*Proof.* From (18) we see that every element of $\Omega_{k-1,k}$ decomposes uniquely as a sum

$$\alpha_0x_{0,k-1}+\alpha_1x_{1,k-1}+\cdots+\alpha_{k-1}x_{k-1,k-1},$$

with $\alpha_i\in\psi_k^*(\Omega_k)$. Then, sliding every $x_{i,k-1}$ over the tensor product we get that every element of $\Omega_{k(k-1)k}$ can be written as

$$(h_0\otimes_{k-1}\alpha_0)+(h_1\otimes_{k-1}\alpha_1)+\cdots+(h_{k-1}\otimes_{k-1}\alpha_{k-1}),$$

with $h_i\in\Omega_{k,k-1}$. Moreover, by (16), every element of $\Omega_{k,k-1}$ can be decomposed as a sum

$$\beta_0+\beta_1\xi_k+\cdots+\beta_{k-1}\xi_k^{k-1}, \tag{19}$$

with $\beta_i\in\psi_k^*(\Omega_k)$. Using (19) to decompose every $h_i$ we get a decomposition as in the statement. □

PROPOSITION 4.4. *In $\Omega_{k(k+1)k}$, the following identity holds:*

$$\sum_{\ell=0}^{k}(-1)^\ell x_{l,k}\otimes_{k+1}\xi_{k+1}^{k-\ell}=\sum_{\ell=0}^{k}(-1)^\ell\xi_{k+1}^{k-\ell}\otimes_{k+1}x_{\ell,k}.$$

*Moreover, the $\xi_{k+1}$ slides over this sum and therefore over the tensor product:*

$$\xi_{k+1}\sum_{\ell=0}^{k}(-1)^\ell x_{\ell,k}\otimes_{k+1}\xi_{k+1}^{k-\ell}=\sum_{\ell=0}^{k}(-1)^\ell x_{\ell,k}\otimes_{k+1}\xi_{k+1}^{k-\ell+1}.$$

*Proof.* The same computations as in [**35**, § 3.2] can be used here since the polynomial side of $\Omega_{k(k+1)k}$ is the cohomology of the two-step flag manifold as in the reference. □

Definition 4.5. We construct injective superbimodule morphisms of degrees $(2k, 0)$ by setting

$$\iota : \Omega_k^\xi \hookrightarrow \Omega_{k(k+1)k}, \quad \xi^i \mapsto \xi_{k+1}^i \sum_{\ell=0}^{k} (-1)^\ell x_{\ell,k} \otimes_{k+1} \xi_{k+1}^{k-\ell},$$

$$\eta : \Omega_k \hookrightarrow \Omega_{k(k+1)k}, \quad 1 \mapsto \sum_{\ell=0}^{k} (-1)^\ell x_{\ell,k} \otimes_{k+1} \xi_{k+1}^{k-\ell},$$

and extending by the $(\Omega_k, \Omega_k)$-superbimodule structure (14).

Note that these maps are superbimodule morphisms since the variable $\xi_{k+1}$ slides over the tensor product thanks to Proposition 4.4, and thus multiplying at the left or at the right gives the same result. Injectivity is a straightforward consequence of the fact that all our superbimodules are free as $\mathbb{Q}$-modules.

Proposition 4.6. *The left inverse of $\iota$ is given by*

$$\pi : \Omega_{k(k+1)k} \twoheadrightarrow \Omega_k^\xi, \quad \begin{cases} \xi_{k+1}^i \otimes_{k+1} \xi_{k+1}^j & \mapsto (-1)^{i+j-k} Y^\xi_{i+j-k,k}, \\ \xi_{k+1}^i \otimes_{k+1} \xi_{k+1}^j s_{k+1,k+1} & \mapsto 0, \end{cases}$$

*with $Y^\xi_{m,k} = 0$ for $m < 0$, $Y^\xi_{0,k} = 1$, and $Y^\xi_{i,k}$ is defined recursively by $Y^\xi_{i,k} = (-\xi)^i - \sum_{\ell=1}^{i} x_{\ell,k} Y^\xi_{i-\ell,k}$.*

*Proof.* We observe that for all $i \geqslant 0$ we have

$$\begin{aligned}(\pi \circ \iota)(\xi^i) &= \pi\left(\xi_{k+1}^i \cdot \sum_{\ell=0}^{k} (-1)^\ell \xi_{k+1}^{k-\ell} \otimes_{k+1} x_{\ell,k}\right) \\ &= \sum_{\ell=0}^{k} (-1)^i x_{\ell,k} Y^\xi_{i-\ell,k} = (-1)^i Y^\xi_{i,k} + (-1)^i \sum_{\ell=1}^{k} x_{\ell,k} Y^\xi_{i-\ell,k} \\ &= \xi^i - (-1)^i \sum_{\ell=1}^{i} x_{\ell,k} Y^\xi_{i-\ell,k} + (-1)^i \sum_{\ell=1}^{k} x_{\ell,k} Y^\xi_{i-\ell,k} = \xi^i,\end{aligned}$$

with the last equality coming from the fact that $Y_{i-\ell,k} = 0$ for $\ell > i$ and $x_{\ell,k} = 0$ for $\ell > k$. $\square$

Remark 4.7. Note that $Y^\xi_{i,k}$ has the same expression in $x_{r,k}$ as $Z_{i,k+1}$ in $w_{r,k}$ when we identify $\xi_{k+1}$ with $\xi$. Indeed we have

$$Z_{i,k+1} = -\sum_{\ell=1}^{i} \left(w_{\ell,k} Z_{i-\ell,k+1} + \xi_{k+1} w_{\ell-1,k} Z_{i-\ell,k+1}\right) = (-\xi_{k+1})^i - \sum_{\ell=1}^{i} w_{\ell,k} Z_{i-\ell,k+1}.$$

Definition 4.8. We also define a surjective morphism of degree $(-2k+2, 0)$ by

$$\epsilon : \Omega_{k(k-1)k} \twoheadrightarrow \Omega_k, \quad \xi_k^i \otimes_{k-1} \xi_k^j \mapsto (-1)^{i+j-k+1} Y_{(i+j-k+1),k}.$$

Remark 4.9. We see that

$$(\epsilon \otimes_{k+1} \mathrm{Id}_{\Omega_{k+1,k}}) \circ (\mathrm{Id}_{\Omega_{k+1,k}} \otimes_k \eta) = \mathrm{Id}_{\Omega_{k+1,k}},$$

and

$$(\mathrm{Id}_{\Omega_{k+1,k}} \otimes_k \epsilon) \circ (\eta \otimes_{k+1} \mathrm{Id}_{\Omega_{k+1,k}}) = \mathrm{Id}_{\Omega_{k+1,k}},$$

by a computation similar to the one in [**35**, Lemma 4.5].

Definition 4.10. We define a surjective superbimodule morphism of degree $(2k+2,-2)$ by

$$\mu : \Omega_{k(k+1)k} \twoheadrightarrow \Pi\Omega_k^\xi, \qquad \begin{cases} \xi_{k+1}^i \otimes_{k+1} \xi_{k+1}^j & \mapsto 0, \\ \xi_{k+1}^i \otimes_{k+1} \xi_{k+1}^j s_{k+1,k+1} & \mapsto (-1)^{i+j} Y_{i+j,k}^\xi, \end{cases}$$

and extending to $\Omega_{k(k+1)k}$ using the superbimodule structure (14).

We now define maps which allow connecting our construction to the nilHecke algebra later on.

Definition 4.11. We define the *nilHecke maps* by

$$X^- : \Omega_{k,k+1,k+2} \to \Omega_{k,k+1,k+2},$$

$$\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j \mapsto \sum_{\ell=0}^{i-1} \xi_{k+1}^{i+j-1-\ell} \otimes_{k+1} \xi_{k+2}^\ell - \sum_{\ell=0}^{j-1} \xi_{k+1}^{i+j-1-\ell} \otimes_{k+1} \xi_{k+2}^\ell,$$

$$X^+ : \Omega_{k+2,k+1,k} \to \Omega_{k+2,k+1,k},$$

$$\xi_{k+2}^i \otimes_{k+1} \xi_{k+1}^j \mapsto \sum_{\ell=0}^{j-1} \xi_{k+2}^{i+j-1-\ell} \otimes_{k+1} \xi_{k+1}^\ell - \sum_{\ell=0}^{i-1} \xi_{k+2}^{i+j-1-\ell} \otimes_{k+1} \xi_{k+1}^\ell,$$

and extending using the right (for $X^-$) and the left (for $X^+$) supermodule structures. These maps are both of degree $(-2,0)$.

Lemma 4.12. *For all $i,j \geqslant 0$ we have*

$$\xi^i \otimes_{k+1} \xi^j = X^\pm(\xi^{i+1} \otimes_{k+1} \xi^j) - X^\pm(\xi^i \otimes_{k+1} \xi^j)\xi = \xi X^\pm(\xi^i \otimes_{k+1} \xi^j) - X^\pm(\xi^i \otimes_{k+1} \xi^{j+1}) \tag{20}$$

*and thus*

$$X^\pm(\xi^{i+1} \otimes_{k+1} \xi^{j+1}) = \xi X^\pm(\xi^i \otimes_{k+1} \xi^j)\xi.$$

*Proof.* The proof is a direct computation, which is done in [**34**, Lemma 7.10]. □

Proposition 4.13. *The maps $X^-$ and $X^+$ are superbimodule morphisms.*

*Proof.* Since by definition $X^-$ is a right supermodule morphism, we only need to prove that it is also a left supermodule morphism. This means we have to show that

$$X^-(x_{\alpha,k}\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j) = x_{\alpha,k} X^-(\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j),$$

$$X^-((s_{\alpha,k+1} + \xi_{k+1}s_{\alpha+1,k+1})\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j) = (s_{\alpha,k+1} + \xi_{k+1}s_{\alpha+1,k+1})X^-(\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j),$$

for all $i,j \geqslant 0$ and $\alpha \leqslant k$. Using Lemma 4.2 we compute

$$
\begin{aligned}
X^-(x_{\alpha,k}\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j) &\overset{(16)}{=} \sum_{\ell=0}^{\alpha}(-1)^\ell X^-(\xi_{k+1}^{i+\ell} \otimes_{k+1} \xi_{k+2}^j x_{\alpha-\ell,k+1}) \\
&\overset{(16)}{=} \sum_{\ell=0}^{\alpha}\sum_{p=0}^{\alpha-\ell}(-1)^{\ell+p} X^-(\xi_{k+1}^{i+\ell} \otimes_{k+1} \xi_{k+2}^{j+p})\psi^*(x_{\alpha-\ell-p,k+2}), \\
x_{\alpha,k}X^-(\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j) &\overset{(16)}{=} \sum_{\ell=0}^{\alpha}(-1)^\ell \xi_{k+1}^\ell X^-(\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j) x_{\alpha-\ell,k+1} \\
&\overset{(16)}{=} \sum_{\ell=0}^{\alpha}\sum_{p=0}^{\alpha-\ell}(-1)^{\ell+p} \xi_{k+1}^\ell X^-(\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j)\xi_{k+2}^p\psi^*(x_{\alpha-\ell-p,k+2}).
\end{aligned}
$$

These sums are equal by Lemma 4.12.

To prove the second relation in the statement we slide $s_{\alpha,k+1}$ and $s_{\alpha+1,k+1}$ to the right through the tensor products $\otimes_{k+1}$ to get

$$
\begin{aligned}
&X^-((s_{\alpha,k+1} + \xi_{k+1}s_{\alpha+1,k+1})\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j) \\
&\quad \overset{(14)}{=} X^-(\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j)s_{\alpha,k+2} + X^-(\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^{j+1})s_{\alpha+1,k+2} \\
&\qquad + X^-(\xi_{k+1}^{i+1} \otimes_{k+1} \xi_{k+2}^j)s_{\alpha+1,k+2} + X^-(\xi_{k+1}^{i+1} \otimes_{k+1} \xi_{k+2}^{j+1})s_{\alpha+2,k+2} \\
&\quad \overset{(20)}{=} X^-(\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j)s_{\alpha,k+2} + X^-(\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j)\xi_{k+2}s_{\alpha+1,k+2} \\
&\qquad + \xi_{k+1}X^-(\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j)s_{\alpha+1,k+2} + \xi_{k+1}X^-(\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j)\xi_{k+2}s_{\alpha+2,k+2} \\
&\quad \overset{(14)}{=} (s_{\alpha,k+1} + \xi_{k+1}s_{\alpha+1,k+1})X^-(\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j).
\end{aligned}
$$

The proof for $X^+$ is similar. ∎

Proposition 4.14. *There is an injective superbimodule map*

$$u : \Omega_{k(k-1)k} \hookrightarrow \Omega_{k(k+1)k},$$

*preserving the degree and given by*

$$
\begin{aligned}
u &= (\epsilon \otimes_k \mathrm{Id}) \circ (\mathrm{Id} \otimes_{k-1} X^- \otimes_{k+1} \mathrm{Id}) \circ (\mathrm{Id} \otimes_k \eta) \\
&= (\mathrm{Id} \otimes_k \epsilon) \circ (\mathrm{Id} \otimes_{k+1} X^+ \otimes_{k-1} \mathrm{Id}) \circ (\eta \otimes_k \mathrm{Id}).
\end{aligned}
$$

*Moreover, this morphism takes the form*

$$u(\xi_k^i \otimes_{k-1} \xi_k^j) = -\xi_{k+1}^j \otimes_{k+1} \xi_{k+1}^i, \tag{21}$$

*for all $i + j < k$.*

*Proof.* Thanks to Lemma 4.3, it is enough to show that the two superbimodule morphisms $(\epsilon \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes X^- \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \eta)$ and $(\mathrm{Id} \otimes \epsilon) \circ (\mathrm{Id} \otimes X^+ \otimes \mathrm{Id}) \circ (\eta \otimes \mathrm{Id})$ take the form (21). First, we suppose that $u = (\epsilon \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes X^- \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \eta)$ and we compute, for $i < k$,

$$u(\xi_k^i \otimes_{k-1} 1) = -\sum_{\ell=0}^{k}\sum_{p=0}^{k-\ell-1}(-1)^{i-p}\phi^*(Y_{i-\ell-p,k})\xi_{k+1}^p \otimes_{k+1} x_{\ell,k}.$$

By Lemma 4.2 we have

$$1 \otimes_{k+1} \xi_{k+1}^i \overset{(18),(17)}{=} (-1)^i \sum_{\ell=0}^{i} \sum_{p=0}^{i-\ell} (-1)^p \phi^*(Y_{i-\ell-p,k}) \xi_{k+1}^p \otimes_{k+1} x_{\ell,k}.$$

Since $Y_{i-\ell-p,k} = 0$ for $\ell + p > i$, we get $u(\xi_k^i \otimes_{k-1} 1) = -1 \otimes_k \xi_{k+1}^i$. Using this result together with Lemma 4.2, we compute

$$\xi_k^i \otimes_{k-1} \xi_k^j \overset{(18),(16)}{=} (-1)^j \sum_{\ell=0}^{j} \sum_{p=0}^{\ell} (-1)^p \psi^*(x_{\ell-p,k}) \xi_k^{i+p} \otimes_{k-1} Y_{j-\ell,k},$$

$$\begin{aligned} u(\xi_k^i \otimes_{k-1} \xi_k^j) &= -(-1)^j \sum_{\ell=0}^{j} \sum_{p=0}^{\ell} (-1)^p \phi^*(x_{\ell-p,k}) \otimes_{k+1} \xi_{k+1}^{i+p} \phi^*(Y_{j-\ell,k}) \\ &= -(-1)^j \sum_{r=0}^{j} \sum_{s=0}^{j-r} (-1)^s x_{r,k} \otimes_{k+1} \xi_{k+1}^{i+s} \phi^*(Y_{j-r-s,k}), \end{aligned}$$

$$-\xi_{k+1}^j \otimes_{k+1} \xi_{k+1}^i \overset{(18),(17)}{=} -(-1)^j \sum_{\ell=0}^{j} \sum_{p=0}^{j-\ell} (-1)^p x_\ell \otimes_{k+1} \phi^*(Y_{j-\ell-p,k}) \xi_k^{i+p},$$

where we have used a change of variable $r = \ell - p, s = p$ in the middle sum. Similar computations beginning with the case $1 \otimes_{k-1} \xi_k^j$ give the same result for $(\mathrm{Id} \otimes \epsilon) \circ (\mathrm{Id} \otimes X^+ \otimes \mathrm{Id}) \circ (\eta \otimes \mathrm{Id})$. Finally, injectivity follows again from the fact that $\Omega_{k(k-1)k}$ and $\Omega_{k(k+1)k}$ are free $\mathbb{Q}$-modules. ☐

Thanks to the injection $u$ we see $\Omega_{k(k-1)k}$ as a sub-superbimodule of $\Omega_{k(k+1)k}$ and we define the quotient

$$\frac{\Omega_{k(k+1)k}}{\Omega_{k(k-1)k}} = \frac{\Omega_{k(k+1)k}}{\operatorname{im} u}.$$

A priori this bimodule may not belong to $\mathbf{ExtFlag}_\lambda$. However, as we will see, it is isomorphic to some 1-morphism in $\mathbf{ExtFlag}_\lambda$.

LEMMA 4.15. *The maps $\mu$ and $\pi$ induce surjective morphisms on the quotient*

$$\overline{\mu} : \frac{\Omega_{k(k+1)k}}{\Omega_{k(k-1)k}} \twoheadrightarrow \Pi\Omega_k^\xi, \quad \overline{\pi} : \frac{\Omega_{k(k+1)k}}{\Omega_{k(k-1)k}} \twoheadrightarrow \Omega_k^\xi,$$

*of degrees, respectively,* $(2k+2,-2)$ *and* $(-2k,0)$.

*Proof.* We have to show that $\operatorname{im} u \subset \ker \mu$ and $\operatorname{im} u \subset \ker \pi$. By Lemma 4.3, it is sufficient to show that the maps $\mu$ and $\pi$ are zero on $1 \otimes_{k+1} 1, \xi_k \otimes_{k+1} 1, \dots, \xi_k^{k-1} \otimes_{k+1} 1$, which is immediate from the definition of these maps. ☐

LEMMA 4.16. *The morphism*

$$\bar{\iota} \colon \Omega_k^\xi \to \frac{\Omega_{k(k+1)k}}{\Omega_{k(k-1)k}},$$

*defined as the composite of $\iota$ with the projection on the quotient, is still injective and the inverse of $\overline{\pi}$.*

*Proof.* To show injectivity, we only have to prove that $\operatorname{im}\iota \cap \operatorname{im} u = \{0\}$ which is straightforward, since by Lemma 4.3 there are no occurrences of $\xi_k^{\geqslant k} \otimes_{k+1} 1$ in $\operatorname{im} u$. The invertibility property is immediate. ☐

LEMMA 4.17. *The induced morphism $\overline{\mu}$ is right invertible, with inverse given by*

$$\overline{\mu}^{-1} : \Pi\Omega_k^{\xi} \hookrightarrow \frac{\Omega_{k(k+1)k}}{\Omega_{k(k-1)k}}, \qquad \xi^i \mapsto \sum_{\ell=0}^{i} (-1)^{\ell} x_{\ell,k} \otimes_{k+1} \xi_{k+1}^{i-\ell} s_{k+1,k+1} + \operatorname{im} u$$
$$= \sum_{\ell=0}^{i} (-1)^{\ell} \xi_{k+1}^{i-\ell} \otimes_{k+1} x_{\ell,k} s_{k+1,k+1} + \operatorname{im} u.$$

*Proof.* It suffices to prove that $\xi_{k+1}$ slides over the tensor product, as $s_{k+1,k+1}$ already does. For $i \geqslant k$, this comes from Proposition 4.4. For $i < k$ it follows from the fact that for all $X, Y \in \Omega_k$ and $r + s < k$ we have

$$\begin{aligned}
&u(s_{k,k}\xi_k^r X \otimes_{k-1} \xi_k^s Y - (-1)^{p(X)+p(Y)} \xi_k^r X \otimes_{k-1} \xi_k^s Y s_{k,k}) \\
&\overset{(14)}{=} \xi_k^s X \otimes_{k+1} Y \xi_{k+1}^r (s_{k,k+1} + \xi_{k+1} s_{k+1,k+1}) \\
&\quad - (-1)^{p(X)+p(Y)} (s_{k,k+1} + \xi_{k+1} s_{k+1,k+1}) \xi^s X \otimes_{k+1} Y \xi_{k+1}^r \\
&= \xi_{k+1}^s X \otimes_{k+1} Y \xi_{k+1}^{r+1} s_{k+1,k+1} - (-1)^{p(X)+p(Y)} \xi_{k+1}^{s+1} s_{k+1,k+1} X \otimes_{k+1} Y \xi_{k+1}^r,
\end{aligned}$$

since $s_{k,k+1}$ commutes. In the quotient, this is zero and thus $\xi_{k+1} s_{k+1,k+1}$ commutes. The invertibility is showed by the same computations as in the proof of Proposition 4.6. ☐

LEMMA 4.18. *There is an equality of graded dimensions*

$$\operatorname{gdim} \frac{\Omega_{k(k+1)k}}{\Omega_{k(k-1)k}} = \left(q^{2k} + \pi\lambda^2 q^{-2k-2}\right) \operatorname{gdim} \Omega_k^{\xi}.$$

*Proof.* Let $Q = \operatorname{gdim} \mathbb{Q}[\xi] = \frac{1}{1-q^2}$. We compute

$$\begin{aligned}
\operatorname{gdim} \Omega_{k(k-1)k} &= \frac{(\operatorname{gdim} \Omega_{k,k-1})^2}{\operatorname{gdim} \Omega_{k-1}} = \frac{\left(Q(1+q^{-2k}\pi\lambda^2) \operatorname{gdim} \Omega_{k-1}\right)^2}{\operatorname{gdim} \Omega_{k-1}} \\
&= Q^2(1-q^{2k})(1+q^{-2k}\pi\lambda^2) \operatorname{gdim} \Omega_k,
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{gdim} \Omega_{k(k+1)k} &= \frac{(\operatorname{gdim} \Omega_{k,k+1})^2}{\operatorname{gdim} \Omega_{k+1}} = \frac{\left(Q(1+q^{-2k-2}\pi\lambda^2) \operatorname{gdim} \Omega_{k}\right)^2}{\frac{1+q^{-2k-2}\pi\lambda^2}{1-q^{2k+2}} \operatorname{gdim} \Omega_k} \\
&= Q^2(1-q^{2k+2}) \left(1+q^{-2k-2}\pi\lambda^2\right) \operatorname{gdim} \Omega_k.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{gdim} \frac{\Omega_{k(k+1)k}}{\Omega_{k(k-1)k}} &= \operatorname{gdim} \Omega_{k(k+1)k} - \operatorname{gdim} \Omega_{k(k-1)k} \\
&= Q\left(q^{2k} + \pi\lambda^2 q^{-2k-2}\right) \operatorname{gdim} \Omega_k,
\end{aligned}$$

as stated. ☐

Lemma 4.19. *The equalities*

$$\overline{\mu} \circ \overline{\iota} = 0 \quad \text{and} \quad \overline{\pi} \circ \overline{\mu}^{-1} = 0$$

*hold.*

*Proof.* It suffices to consider the occurrences of $s_{k+1,k+1}$ in the image of $\overline{\iota}$ and $\overline{\mu}^{-1}$, and the claim follows. □

Theorem 4.20. *There is a degree-preserving isomorphism*

$$\frac{\Omega_{k(k+1)k}}{\Omega_{k(k-1)k}} \cong \Pi\Omega_k^\xi\langle -2k-2, 2\rangle \oplus \Omega_k^\xi\langle 2k, 0\rangle,$$

*given by $\overline{\mu} \oplus \overline{\pi}$ with inverse $\overline{\mu}^{-1} \oplus \overline{\iota}$.*

*Proof.* The Lemmas 4.15–4.19 above imply that $\overline{\mu} \oplus \overline{\pi}$ is a split surjection with inverse $\overline{\mu}^{-1} \oplus \overline{\iota}$ and that the dimensions agree, and thus it is an isomorphism. □

Remark 4.21. We do *not* have a direct sum decomposition

$$\Omega_{k(k+1)k} \not\cong \Omega_{k(k-1)k} \oplus \Pi\Omega_k^\xi\langle -2k-2, 2\rangle \oplus \Omega_k^\xi\langle 2k, 0\rangle$$

because there is no surjective morphism $\Omega_{k(k+1)k} \to \Omega_{k(k-1)k}$. As a matter of fact, there is no injective morphism $\Omega_k \to \Omega_{k(k-1)k}$ either.

Definition 4.22. Define the shifted superbimodules

$$\overline{\Omega}_{k+1,k} = \Omega_{k+1,k}\langle -k, 0\rangle, \quad \overline{\Omega}_{k,k+1} = \Omega_{k,k+1}\langle k+2, -1\rangle, \quad \overline{\Omega}_k^\xi = \Pi\Omega_k[\xi]\langle 1, 0\rangle.$$

In terms of these superbimodules the isomorphism in Theorem 4.20 takes the form (the notation $\overline{\Omega}_{k(k+1)k}$ and $\overline{\Omega}_{k(k-1)k}$ should be clear)

$$\frac{\overline{\Omega}_{k(k+1)k}}{\overline{\Omega}_{k(k-1)k}} \cong \overline{\Omega}_k^\xi\langle -2k-1, 1\rangle \oplus \Pi\overline{\Omega}_k^\xi\langle 2k+1, -1\rangle. \tag{22}$$

Note that by (22) we get

$$\operatorname{gdim} \overline{\Omega}_k^\xi = -\frac{\pi}{q-q^{-1}} \operatorname{gdim} \Omega_k,$$

$$\operatorname{gdim} \frac{\overline{\Omega}_{k(k+1)k}}{\overline{\Omega}_{k(k-1)k}} = -\frac{\pi q^{-2k}(\lambda q^{-1}) + q^{2k}(\lambda q^{-1})^{-1}}{q-q^{-1}} \operatorname{gdim} \Omega_k,$$

which agrees with the commutator relation (6) for $\lambda q^{-1}$ and $e_{-2k}$ when specializing $\pi = -1$.

We now arrive to the main result in this section as a corollary of Theorem 4.20.

Corollary 4.23. *There is a short exact sequence*

$$0 \to \overline{\Omega}_{k(k-1)k} \to \overline{\Omega}_{k(k+1)k} \to \overline{\Omega}_k^\xi\langle -2k-1, 1\rangle \oplus \Pi\overline{\Omega}_k^\xi\langle 2k+1, -1\rangle \to 0. \tag{23}$$

The following result will be useful in the sequel.

Proposition 4.24. *The superbimodules $\overline{\Omega}_{k(k+1)k}$ and $\overline{\Omega}_{k(k-1)k}$ are sweet and decompose as left $\Omega_k$-modules as*

$$\overline{\Omega}_{k(k+1)k} \overset{\Omega_k\text{-smod}}{\cong} \oplus_{[k+1]} \left(\Omega_k\langle k+2, -1\rangle \oplus \Pi\Omega_k\langle -k, 1\rangle\right) \otimes \mathbb{Q}[\xi],$$

$$\overline{\Omega}_{k(k-1)k} \overset{\Omega_k\text{-smod}}{\cong} \oplus_{[k]} \left(\Omega_k\langle k+1,-1\rangle \oplus \Pi\Omega_k\langle -k+1,1\rangle\right) \otimes \mathbb{Q}[\xi],$$

*and as right $\Omega_k$-modules, as*

$$\overline{\Omega}_{k(k+1)k} \overset{\text{smod-}\Omega_k}{\cong} \oplus_{[k+1]} \left(\Omega_k\langle k+2,-1\rangle \oplus \Pi\Omega_k\langle -k,1\rangle\right) \otimes \mathbb{Q}[\xi],$$

$$\overline{\Omega}_{k(k-1)k} \overset{\text{smod-}\Omega_k}{\cong} \oplus_{[k]} \left(\Omega_k\langle k+1,-1\rangle \oplus \Pi\Omega_k\langle -k+1,1\rangle\right) \otimes \mathbb{Q}[\xi],$$

*Proof.* By Lemma 4.2, there are decompositions as left supermodules

$$\Omega_{k+1,k} \overset{\Omega_k\text{-smod}}{\cong} \oplus_{\{k+1\}}\Omega_{k+1},$$

$$\Omega_{k,k+1} \overset{\Omega_k\text{-smod}}{\cong} (1+s_{k+1,k+1})\Omega_k \otimes \mathbb{Q}[\xi],$$

such that

$$\overline{\Omega}_{k+1,k} \overset{\text{smod-}\Omega_k}{\cong} \oplus_{[k+1]}\Omega_{k+1},$$

$$\overline{\Omega}_{k,k+1} \overset{\text{smod-}\Omega_k}{\cong} \left(\Omega_k\langle k+2,-1\rangle \oplus \Pi\Omega_k\langle -k,1\rangle\right) \otimes \mathbb{Q}[\xi].$$

We conclude by combining these two decompositions. The proof is similar for the decomposition as right supermodules. □

REMARK 4.25. The decompositions as a left and as a right supermodule are similar but the splitting maps are *not* superbimodule maps (c.f. Remark 4.21).

## 4.2. *nilHecke action*

The *nilHecke algebra* $\mathrm{NH}_n$, which appears in the context of cohomologies of flag varieties and Schubert varieties (see, for example, [**33**, § 4]), is an essential ingredient in the categorification of quantum groups and has become quite ubiquitous in higher representation theory. Recall that it is the unital, associative $\Bbbk$-algebra freely generated by $x_j$ for $1 \leqslant j \leqslant n$ and $\partial_j$ for $1 \leqslant j \leqslant n-1$ with relations

$$\begin{aligned}
&x_i x_j = x_j x_i, && \\
&\partial_i x_j = x_j \partial_i \quad \text{if } |i-j|>1, && \partial_i\partial_j = \partial_j\partial_i \quad \text{if } |i-j|>1,\\
&\partial_i x_i = x_{i+1}\partial_i + 1, && \partial_i^2 = 0,\\
&x_i\partial_i = \partial_i x_{i+1} + 1, && \partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}.
\end{aligned} \tag{24}$$

Here $\Bbbk$ is a ring which, unless stated otherwise, we will take as $\mathbb{Q}$.

PROPOSITION 4.26. *There is an action of the nilHecke algebra* $\mathrm{NH}_n$ *on* $\Omega_{m,m+n}$.

*Proof.* We view $\Omega_{m,m+n}$ as $\Omega_{m,m+1} \otimes \cdots \otimes \Omega_{m+n-1,m+n}$ using Lemma 3.3. Let $\partial_i$ act as the operator $X^- : \Omega_{m+i-1,m+i} \otimes \Omega_{m+i,m+i+1} \to \Omega_{m+i-1,m+i} \otimes \Omega_{m+i,m+i+1}$ and $x_i$ as multiplication by $\xi_{m+i}$ in $\Omega_{m+i-1,m+i}$. We get all the relations in (24) from the superbimodule structure of the morphism $X^-$ together with Lemma 4.12, except for the last two on the second column, which can be checked through computations similar to those in [**34**, Lemma 7.10]. □

As a matter of fact, there is an enlarged version of the nilHecke algebra acting on $\Omega_{m,m+n}$, and therefore on $\Omega_{m+n,m}$. From the proof of Proposition 4.13 we see that the nilHecke algebra $\mathrm{NH}_n$

as defined above acts on the ring $\mathbb{Q}[\underline{x}_n,\underline{\omega}_n]=\mathbb{Q}[\underline{x}_n]\otimes\bigwedge^\bullet(\underline{\omega}_n)$, where $\omega_j$ is odd and has bidegree $\deg_{\lambda,q}(\omega_j)=(-2(j+m),2)$. More precisely, $\omega_j$ is identified with $s_{m+j,m+j}\in\Omega_{m+j-1,m+j}$.

DEFINITION 4.27. We define the bigraded (super)algebra $A_n(m)$ as the quotient of the product algebra of $\mathrm{NH}_n$ with $\bigwedge^\bullet(\underline{\omega}_n)$ by the kernel of the action of $\mathrm{NH}_n$ on $\mathbb{Q}[\underline{x}_n,\underline{\omega}_n]$.

The algebra $A_n$ inherits many of the features of $\mathrm{NH}_n$, like the fact that it is left and right noetherian and is free as a left supermodule over $\mathbb{Q}[\underline{x}_n,\underline{\omega}_n]$ and $\mathbb{Q}[\underline{x}_n]$, of ranks $n!$ and $2^n n!$, respectively. It can be given an explicit presentation as a smash product as follows.

PROPOSITION 4.28. *As an abelian group* $A_n(m)=\mathrm{NH}_n\otimes\bigwedge^\bullet(\underline{\omega}_n)$, *where* $\mathrm{NH}_n$ *and* $\bigwedge^\bullet(\underline{\omega}_n)$ *are subalgebras and*

$$x_i\omega_j=\omega_jx_i,\quad \partial_i\omega_j=\begin{cases}\omega_j\partial_i & \text{if } i\neq j,\\ \omega_i\partial_i+\omega_{i+1}\left(\partial_ix_{i+1}-x_{i+1}\partial_i\right) & \text{if } i=j.\end{cases}$$

*Proof.* Only the last relation calls for a proof. We have

$$\begin{aligned}X^-(xs_{k,k}\otimes_k y)&=X^-(x\otimes_k(s_{k,k+1}+\xi_{k+1}s_{k+1,k+1})y)\\&=X^-(x\otimes_k y)s_{k,k+1}+X^-(x\otimes_k y(\xi_{k+1}s_{k+1,k+1}))\\&=s_{k,k}X^-(x\otimes_k y)-X^-(x\otimes_k y)\xi_{k+1}s_{k+1,k+1}+X^-(x\otimes_k y(\xi_{k+1}s_{k+1,k+1})),\end{aligned}$$

for any $x\in\Omega_{k-1,k}$ and $y\in\Omega_{k,k+1}$ with $p(x)+p(y)=0$. The case with parity one is similar. Take $k=m+i-1$ and we get the relation. Faithfulness comes from the basis constructed in Proposition 8.1. $\square$

## 5. *Categorification of the Verma module* $M(\lambda)$

Following the explanations in the Appendix A.3, in order to define the field of formal Laurent series $\mathbb{Q}(\!(q,\lambda)\!)$, we need to choose an additive order on $\mathbb{Z}\times\mathbb{Z}$. By convention, we use $0\prec q\prec\lambda$, with the abuse of notation explained in Example A.48.

### 5.1. *Categories of modules*

Each of the superrings $\Omega_k$ is a noetherian $\mathbb{Z}\times\mathbb{Z}$-graded local superring whose degree (0,0) part is isomorphic to $\mathbb{Q}$. Then every graded projective supermodule (not necessarily finitely generated) is a free graded supermodule [**7**, Proposition 1.5.15], and $\Omega_k$ has (up to isomorphism and grading shift) a unique graded indecomposable projective supermodule.

Let $\Omega_k\text{-smod}_{\mathrm{lf}}$ be the abelian $\mathbb{Z}\times\mathbb{Z}$-graded category of locally finite-dimensional, cone-bounded $\Omega_k$-supermodules, together with the grading-preserving supermodule maps. These are graded supermodules which are finite-dimensional in each degree. Explicitly a bigraded supermodule $M=\oplus_{i,j}M_{i,j}$ is cone-bounded if there exist a cone $C\subset\mathbb{R}^2$ compatible with the fixed order $\prec$ and $m,n\in\mathbb{Z}$, such that $M_{i+m,j+n}=0$ whenever $(i,j)\notin C$. In other words, it is cone-bounded if its graded dimension is in $\mathbb{Q}_\prec(\!(q,\lambda)\!)$.

Every graded projective supermodule $P$ of $\Omega_k$ is of the form $P\cong\Omega_k\otimes A$ where $A$ is a graded abelian group. The superring $\Omega_k$ has (up to isomorphism and grading shift) a unique simple supermodule $S_k=\Omega_k/(\Omega_k)_+$ (here $(\Omega_k)_+$ denotes the submodule of $\Omega_k$ generated by the elements of non-zero bidegree).

Every cone-bounded graded $\Omega_k$-supermodule has a projective cover [8, Theorem 2]. As a matter of fact, it is not hard to construct such a projective cover. For cone-bounded $\Omega_k$-supermodule $M$ form the non-trivial graded abelian group $M/((\Omega_k)_+M) \cong \mathbb{Q} \otimes_{\Omega_k} M$ and form $P = \Omega_k \otimes \mathbb{Q} \otimes_{\Omega_k} M$. Then $P$ is a projective cover of $M$ with the surjection $p\colon P \to M$ given by $a \otimes b \otimes m \mapsto a\sigma(b \otimes m)$ where $\sigma$ is a section of the canonical projection $M \to M/((\Omega_k)_+M) \cong \mathbb{Q} \otimes_{\Omega_k} M$.

The graded dimension of $\Omega_k$ is contained in the cone in $\mathbb{R}^2$ generated by (2,0) and $(-2k, 2)$. Hence it is a special case of Example A.50 (with some minor adjustments for the 'super'). Therefore, $\Omega_k\text{-smod}_{\mathrm{lf}}$ possesses the local Jordan–Hölder property and we get the following:

Proposition 5.1. *The topological Grothendieck group $\boldsymbol{G}_0(\Omega_k\text{-smod}_{\mathrm{lf}})$ is a one-dimensional module over the ring $\mathbb{Z}_\pi(\!(q,\lambda)\!)$ with the $(q,\lambda)$-adic topology, freely generated by either the class of the simple object $S_k$ or the projective object $\Omega_k$.*

Consider the full subcategory of $\Omega_k\text{-smod}_{\mathrm{lf}}$ generated by modules with $\lambda$-grading bounded above and below. We write it $\Omega_k\text{-smod}_{\mathrm{lf}}^{\lambda}$ and it possesses the local Jordan–Hölder property, with a topological Grothendieck group having $(q)$-adic topology. We get the following proposition.

Proposition 5.2. *The topological Grothendieck group $\boldsymbol{G}_0(\Omega_k\text{-smod}_{\mathrm{lf}}^{\lambda})$ is a one-dimensional topological module over the ring $\mathbb{Z}_\pi[\![q]\!][\lambda^{\pm 1}, q^{-1}]$ with the $(q)$-adic topology, freely generated by the class of the simple object $S_k$.*

Now consider the full subcategory $\Omega_k\text{-psmod}_{\mathrm{lfg}}^{\lambda} \subset \Omega_k\text{-smod}_{\mathrm{lf}}^{\lambda}$ consisting of locally finitely generated, cone-bounded projective modules. For the $q$-grading, it is a cone complete, locally Krull–Schmidt category, and we get the following.

Proposition 5.3. *The topological split Grothendieck group $\boldsymbol{K}_0(\Omega_k\text{-psmod}_{\mathrm{lfg}}^{\lambda})$ is a one-dimensional module over the ring $\mathbb{Z}_\pi[\![q]\!][\lambda^{\pm 1}, q^{-1}]$ with the $(q)$-adic topology, freely generated by the class of the projective object $\Omega_k$.*

### 5.2. *The Verma categorification*

Set $\mathcal{M}_k = \Omega_k\text{-smod}_{\mathrm{lfg}}$ and $\mathcal{M}_{k+1,k} = \Omega_{k+1,k}\text{-smod}_{\mathrm{lfg}}$ and for $k \geqslant 0$ consider the functors

$$\mathrm{Ind}_k^{k+1,k}\colon \mathcal{M}_k \to \mathcal{M}_{k+1,k}, \qquad \mathrm{Res}_k^{k+1,k}\colon \mathcal{M}_{k+1,k} \to \mathcal{M}_k,$$

$$\mathrm{Ind}_{k+1}^{k+1,k}\colon \mathcal{M}_{k+1} \to \mathcal{M}_{k+1,k}, \qquad \mathrm{Res}_{k+1}^{k+1,k}\colon \mathcal{M}_{k+1,k} \to \mathcal{M}_{k+1}.$$

It is sometimes useful to arrange them using a diagram as follows.

$$\begin{tikzcd}
\cdots \arrow[dr, shift left] & & \mathcal{M}_{k+1,k} \arrow[dl, shift left, "\mathrm{Res}_{k+1}^{k+1,k}"] \arrow[dr, shift left, "\mathrm{Res}_k^{k+1,k}"'] & & \cdots \arrow[dl, shift left] \\
& \mathcal{M}_{k+1} \arrow[ul, shift left] \arrow[ur, shift left, "\mathrm{Ind}_{k+1}^{k+1,k}"] & & \mathcal{M}_k \arrow[ul, shift left, "\mathrm{Ind}_k^{k+1,k}"'] \arrow[ur, shift left] &
\end{tikzcd}$$

Since $\Omega_{k+1,k}$ is sweet the functors $\mathrm{Ind}_k^{k+1,k}$ and $\mathrm{Ind}_{k+1}^{k+1,k}$ are exact.

For each $k \geqslant 0$ define exact functors $\mathsf{F}_k\colon \mathcal{M}_k \to \mathcal{M}_{k+1}$ and $\mathsf{E}_k\colon \mathcal{M}_{k+1} \to \mathcal{M}_k$ by

$$\mathsf{F}_k = \mathrm{Res}_{k+1}^{k,k+1} \circ \mathrm{Ind}_k^{k,k+1}\langle -k, 0\rangle \quad \text{and} \quad \mathsf{E}_k = \mathrm{Res}_k^{k,k+1} \circ \mathrm{Ind}_{k+1}^{k,k+1}\langle k+2, -1\rangle.$$

Using the language of bimodules, $\mathsf{F}_k$ and $\mathsf{E}_k$ can also be written as

$$\mathsf{F}_k(-) = (\Omega_{k+1} \otimes_{k+1} \Omega_{k+1,k} \otimes_k (-))\,\langle -k, 0\rangle,$$

and

$$\mathsf{E}_k(-) = (\Omega_k \otimes_k \Omega_{k,k+1} \otimes_{k+1} (-)) \langle k+2, -1\rangle,$$

where $\Omega_{k+1,k}$ is seen as a $(\Omega_{k+1,k}, \Omega_k)$-superbimodule and $\Omega_{k,k+1}$ as a $(\Omega_k, \Omega_{k,k+1})$-superbimodule.

PROPOSITION 5.4. *Up to a grading shift the functors $(\mathsf{F}_k, \mathsf{E}_k)$ form an adjoint pair of functors.*

*Proof.* The superbimodule maps $\eta$ and $\epsilon$ from Definitions 4.5 and 4.8 induce, respectively, natural transformations $\mathbb{1}_k \to \mathsf{E}_k\mathsf{F}_k$ and $\mathsf{F}_k\mathsf{E}_k \to \mathbb{1}_{k+1}$ which are the unit and counit of the adjunction $\mathsf{F}_k \dashv \mathsf{E}_k$ by Remark 4.9. □

The functor $\mathsf{F}$ does not admit $\mathsf{E}$ as a left adjoint. As explained in § 1.1.3 this is necessary to categorify infinite-dimensional highest weight $\mathfrak{sl}_2$-modules.

We denote by $\mathsf{Q}_k$ the functor $\mathcal{M}_k \to \mathcal{M}_k$ of tensoring on the left with the shifted $(\Omega_k, \Omega_k)$-superbimodule $\overline{\Omega}_k^{\xi}$. In this context , Corollary 4.23 reads as follows.

PROPOSITION 5.5. *For each $k \in \mathbb{N}_0$ we have an exact sequence*

$$0 \longrightarrow \mathsf{F}_{k-1} \circ \mathsf{E}_{k-1} \longrightarrow \mathsf{E}_k \circ \mathsf{F}_k \longrightarrow \mathsf{Q}_k\langle -2k-1, 1\rangle \ \oplus \Pi\mathsf{Q}_k\langle 2k+1, -1\rangle \longrightarrow 0$$

*of endofunctors on $\mathcal{M}_k$.*

Since the superbimodules used to construct $\mathsf{F}_k$ and $\mathsf{E}_k$ are sweet, see Proposition 4.24, we have the following.

COROLLARY 5.6. *For every $M \in \mathcal{M}_k$ we have an isomorphism*

$$\mathsf{E}_k \circ \mathsf{F}_k(M) \cong \mathsf{F}_{k-1} \circ \mathsf{E}_{k-1}(M) \oplus \mathsf{Q}_k(M)\langle -2k-1, 1\rangle \oplus \Pi\mathsf{Q}_k(M)\langle 2k+1, -1\rangle.$$

Define the functor $\mathsf{K}_k$ as the endofunctor of $\mathcal{M}_k$ which is the auto-equivalence that shifts the bidegree by $(-2k-1, 1)$

$$\mathsf{K}_k\colon \mathcal{M}_k \to \mathcal{M}_k, \quad \mathsf{K}_k(-) = (-)\langle -2k-1, 1\rangle.$$

We have isomorphisms $\mathsf{K}_k \circ \mathsf{E}_k \cong \mathsf{E}_k \circ \mathsf{K}_{k+1}\langle 2, 0\rangle$ and $\mathsf{K}_{k+1} \circ \mathsf{F}_k \cong \mathsf{F}_k \circ \mathsf{K}_k\langle -2, 0\rangle$. Moreover, since $\Pi \circ \mathsf{Q}_k \cong \mathsf{Q}_k \circ \Pi$ we have $\Pi\mathsf{Q}_k \circ \mathsf{K}_k \cong \mathsf{Q}_k \circ \Pi\mathsf{K}_k$.

DEFINITION 5.7. Define the category $\mathcal{M}$ and the endofunctors $\mathsf{F}$, $\mathsf{E}$, $\mathsf{K}$ and $\mathsf{Q}$, as

$$\mathcal{M} = \bigoplus_{k\geqslant 0} \mathcal{M}_k, \quad \mathsf{E} = \bigoplus_{k\geqslant 0} \mathsf{E}_k, \quad \mathsf{F} = \bigoplus_{k\geqslant 0} \mathsf{F}_k, \quad \mathsf{K} = \bigoplus_{k\geqslant 0} \mathsf{K}_k, \quad \mathsf{Q} = \bigoplus_{k\geqslant 0} \mathsf{Q}_k.$$

All the above add up to the following.

THEOREM 5.8. *We have natural isomorphisms of functors*

$$\mathsf{K} \circ \mathsf{K}^{-1} \cong \mathrm{Id}_{\mathcal{M}} \cong \mathsf{K}^{-1} \circ \mathsf{K},$$

$$\mathsf{K} \circ \mathsf{E} \cong \mathsf{E} \circ \mathsf{K}\langle 2, 0\rangle, \quad \mathsf{K} \circ \mathsf{F} \cong \mathsf{F} \circ \mathsf{K}\langle -2, 0\rangle,$$

*and an exact sequence*

$$0 \longrightarrow \mathsf{F} \circ \mathsf{E} \longrightarrow \mathsf{E} \circ \mathsf{F} \longrightarrow \mathsf{Q} \circ \left(\mathsf{K} \oplus \ \Pi\mathsf{K}^{-1}\right) \longrightarrow 0.$$

Remark 5.9. Theorem 5.8 is suggestive from the point of view of categorification of the deformed version $\dot{U}_\lambda$ of quantum $\mathfrak{sl}_2$ from § 2.1.1 if we identify $\mathcal{M}_k$ with the $(\lambda q^{-1-2k})$-weight space.

5.2.1. *NilHecke action.* In § 4.2 we have constructed an action of the nilHecke algebra $\mathrm{NH}_n$ on the superbimodules $\Omega_{k,k+n}$ and $\Omega_{k+n,k}$. The definition of $\mathsf{F}_k$ and $\mathsf{E}_k$ imply the following.

Proposition 5.10. *The nilHecke algebra action on $\Omega_{k,k+n}$ descends to an action on $\mathsf{F}^n$ and on $\mathsf{E}^n$.*

5.2.2. *Grothendieck groups.* For the sake of simplicity we write $K_0(\mathcal{M}_k) = \boldsymbol{K}_0(\Omega_k\text{-psmod}^\lambda_{\mathrm{lfg}})$, $G_0(\mathcal{M}_k) = \boldsymbol{G}_0(\Omega_k\text{-smod}^\lambda_{\mathrm{lf}})$ and $\widehat{G}_0(\mathcal{M}_k) = \boldsymbol{G}_0(\Omega_k\text{-smod}_{\mathrm{lf}})$. We also write

$$K_0(\mathcal{M}) = \bigoplus_{k\geqslant 0} K_0(\mathcal{M}_k)\otimes_{\mathbb{Z}}\mathbb{Q},\quad G_0(\mathcal{M}) = \bigoplus_{k\geqslant 0} G_0(\mathcal{M}_k)\otimes_{\mathbb{Z}}\mathbb{Q},\quad \widehat{G}_0(\mathcal{M}) = \bigoplus_{k\geqslant 0} \widehat{G}_0(\mathcal{M}_k)\otimes_{\mathbb{Z}}\mathbb{Q}.$$

Regarding the behavior of tensoring with $\mathbb{Q}[\xi]$ we have the following.

Lemma 5.11. *The functor of tensoring with $\Omega_k[\xi]$ descends to multiplication by $1/(1-q^2)$ on the different Grothendieck groups $K_0(\mathcal{M})$, $G_0(\mathcal{M})$ and $\widehat{G}_0(\mathcal{M})$.*

*Proof.* This follows from the fact that, since the variable $\xi$ commutes with the variables used to construct $\Omega_k$, we get $\Omega_k[\xi]\cong\bigoplus_{i\geqslant 0}\Omega_k\{2i\}$ and thus $[\Omega_k[\xi]] = (1+q^2+\cdots)[\Omega_k] = 1/(1-q^2)[\Omega_k]$. ☐

The categorical $\mathfrak{sl}_2$-action on projective supermodules is very nice, the functors $\mathsf{F}_k$, $\mathsf{E}_k$ and $\mathsf{Q}_k$ satisfy

$$\mathsf{F}_k(\Omega_k) = \oplus_{[k+1]}\Omega_{k+1},\quad \mathsf{E}_k(\Omega_{k+1}) = \mathsf{Q}_k\Omega_k\langle -k-1,1\rangle \oplus \Pi\mathsf{Q}_k\Omega_k\langle k+1,-1\rangle.$$

On the Grothendieck groups we have

$$[\mathsf{F}_k(\Omega_k)] = [k+1]_q[\Omega_{k+1}], \tag{25}$$

and

$$[\mathsf{E}_k(\Omega_{k+1})] = -\frac{\pi(\lambda q^{-1})q^{-k} + (\lambda q^{-1})^{-1}q^k}{q-q^{-1}}[\Omega_k]. \tag{26}$$

Here we have used the notation $[-]_q$ for quantum integers to avoid confusion with the notation for equivalence classes in the Grothendieck groups. The action on simples can also be computed to be

$$\mathsf{F}_k(S_k) = \mathbb{Q}[x_{1,k+1}, s_{k+1,k+1}]\langle -k,0\rangle,\quad \mathsf{E}_k(S_{k+1}) = \oplus_{\{k+1\}}S_k\langle k+2,-1\rangle.$$

On the Grothendieck group $G_0(\mathcal{M})$ (and thus on $\widehat{G}_0(\mathcal{M})$) we have

$$[\mathsf{F}_k(S_k)] = [\mathbb{Q}[x_{1,k+1}, s_{k+1,k+1}]\langle -k,0\rangle] = -\frac{\pi(\lambda q^{-1})q^{-k} + (\lambda q^{-1})^{-1}q^k}{q-q^{-1}}\lambda q^{-2k-2}[S_{k+1}], \tag{27}$$

and

$$[\mathsf{E}_k(S_{k+1})] = [\oplus_{\{k+1\}}S_k\langle k+2,-1\rangle] = [k+1]_q\lambda^{-1}q^{2k+2}[S_k]. \tag{28}$$

We can now state the main result of this section.

THEOREM 5.12. *The functors* $\mathsf{F}$ *and* $\mathsf{E}$ *induce an action of quantum* $\mathfrak{sl}_2$ *on the Grothendieck groups* $K_0(\mathcal{M})$, $G_0(\mathcal{M})$ *and* $\widehat{G}_0(\mathcal{M})$, *after specializing* $\pi = -1$. *With this action there are* $\mathbb{Q}[\![q]\!][q^{-1}, \lambda^{\pm 1}]$*-linear isomorphisms*

$$K_0(\mathcal{M}) \cong M_A(\lambda), \quad G_0(\mathcal{M}) \cong M_A^*(\lambda),$$

*of* $\dot{U}_\lambda$*-modules, with* $A = \mathbb{Q}[\![q]\!][q^{-1}, \lambda^{\pm 1}]$, *and a* $\mathbb{Q}(\!(q, \lambda)\!)$*-linear isomorphism*

$$\widehat{G}_0(\mathcal{M}) \cong M(\lambda),$$

*of* $U_q(\mathfrak{sl}_2)$*-representations. Moreover, these isomorphisms send classes of projective indecomposables to canonical basis elements and classes of simples to dual canonical elements, whenever this makes sense.*

*Proof.* By exactness and Theorem 5.8, the action of the functors $\mathsf{F}$, $\mathsf{E}$ and $\mathsf{K}$ descend to an action on the Grothendieck groups that satisfies the $\mathfrak{sl}_2$-relations.

Propositions 5.2 and 5.3 yield two isomorphisms $K_0(\mathcal{M}) \cong M_A(\lambda)$ and $G_0(\mathcal{M}) \cong M_A^*(\lambda)$ as $\mathbb{Q}[\![q]\!][q^{-1}, \lambda^{\pm 1}]$-modules by sending, respectively, $[\Omega_k]$ to $m_k$ and $[S_k]$ to $m^k$.

Proposition 5.1 gives an isomorphism $\widehat{G}_0(\mathcal{M}) \cong M(\lambda)$ of $\mathbb{Q}(\!(q, \lambda)\!)$-vector spaces. Comparing the action of $E$ and $F$ on the canonical basis (7) with (25) and (26), and on the dual canonical basis (9) with (27) and (28), concludes the proof. ☐

We finish this section with a categorification of the Shapovalov forms defined in § 2.3. For $M, N \in \mathcal{M}$ denote by $M^{\mathrm{op}}$ the right module given by acting with the opposite algebra. Then we consider the bigraded (super)vector space

$$M^{\mathrm{op}} \otimes_{(\oplus_k \Omega_k)} N.$$

Since both $M$ and $N$ are cone-bounded, locally finite-dimensional, we get

$$\operatorname{sdim} M^{\mathrm{op}} \otimes_{(\oplus_k \Omega_k)} N \in \mathbb{Q}(\!(q, \lambda)\!).$$

For the sake of keeping the notations short, we will write $\otimes_{\mathcal{M}}$ for $\otimes_{(\oplus_k \Omega_k)}$.

THEOREM 5.13. *In the Grothendieck groups,*

$$([M], [N])_\lambda = \operatorname{sdim} M^{\mathrm{op}} \otimes_{\mathcal{M}} N,$$

*where* $(-,-)_\lambda$ *is the universal Shapovalov from* § 2.3.

*Proof.* Let $\mathbb{Q}$ be the unique projective indecomposable in $\Omega_0\text{-smod}_{\mathrm{lfg}}$. We have $([\mathbb{Q}], [\mathbb{Q}])_\lambda = \operatorname{sdim} \mathbb{Q} \otimes_{\Omega_0} \mathbb{Q} = 1$. Moreover, by construction

$$\begin{aligned}
(\mathsf{F}X)^{\mathrm{op}} \otimes_{\mathcal{M}} Y &\cong ((\oplus_k \Omega_{k+1,k}\langle -k, 0\rangle) \otimes_{\mathcal{M}} X)^{\mathrm{op}} \otimes_{\mathcal{M}} Y \\
&\cong X^{\mathrm{op}} \otimes_{\mathcal{M}} (\oplus_k \Omega_{k,k+1}\langle -k, 0\rangle) \otimes_{\mathcal{M}} Y \\
&\cong X^{\mathrm{op}} \otimes_{\mathcal{M}} (\oplus_k (\Omega_{k,k+1}\langle k+2, -1\rangle)\langle -2k-1, 1\rangle)\langle -1, 0\rangle \otimes_{\mathcal{M}} Y \\
&\cong X^{\mathrm{op}} \otimes_{\mathcal{M}} (\mathsf{KE}Y\langle -1, 0\rangle) \\
&\cong X^{\mathrm{op}} \otimes_{\mathcal{M}} (\rho(\mathsf{F})Y),
\end{aligned}$$

for any $X, Y \in \mathcal{M}$. Finally, the bilinearity is obvious from the behavior of the dimension with respect to direct sum and tensor product. ☐

For, $M, N \in \mathcal{M}$, denote by

$$\mathrm{HOM}_{\mathcal{M}}(M,N) = \bigoplus_{i,j,k\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}} \mathrm{Hom}_{\mathcal{M}}(M, \Pi^k N\langle i,j\rangle)$$

the enriched Hom-spaces. They consist of $\mathbb{Z}\times\mathbb{Z}$-graded $\mathbb{Q}$-(super)vector spaces of morphisms.

THEOREM 5.14. *In the Grothendieck groups, whenever* $\mathrm{HOM}_{\mathcal{M}}(M,N)$ *has a locally finite cone-bounded dimension, we have*

$$\langle [M], [N]\rangle_\lambda = \operatorname{sdim} \mathrm{HOM}_{\mathcal{M}}(M,N),$$

*where* $\langle - \rangle_\lambda$ *is the twisted Shapovalov from* § 2.3.

*Proof.* We have $\langle [\mathbb{Q}], [\mathbb{Q}]\rangle_\lambda = \operatorname{sdim}\mathrm{HOM}_{\mathcal{M}_0}(\mathbb{Q},\mathbb{Q}) = 1$ by construction. It follows from Proposition 5.4 that for any $X, Y \in \mathcal{M}$,

$$\mathrm{HOM}_{\mathcal{M}}(\mathsf{F}X, Y) \cong \mathrm{HOM}_{\mathcal{M}}(X, \mathsf{KE}Y\langle -1,0\rangle) \cong \mathrm{HOM}_{\mathcal{M}}(X, \tau(\mathsf{F})Y).$$

Finally, $q^m\lambda^n\langle \mathsf{F}^i[\mathbb{Q}], \mathsf{F}^j[\mathbb{Q}]\rangle_\lambda = \langle q^{-m}\lambda^{-n}\mathsf{F}^i[\mathbb{Q}], \mathsf{F}^j[\mathbb{Q}]\rangle_\lambda = \langle \mathsf{F}^i[\mathbb{Q}], q^m\lambda^n\mathsf{F}^j[\mathbb{Q}]\rangle_\lambda$, is a consequence of the definition of the (enriched) Hom spaces in a bigraded category. □

### 5.3. 2-*Verma modules*

As explained in §§ 1.1.3 and 5.2, the functors $(\mathsf{F}, \mathsf{E})$ are adjoint (up to grading shifts) but not biadjoint. In order to accommodate our construction to the concept of strong 2-representations and $Q$-strong 2-representations (in the sense of Rouquier [**46**] and Cautis-Lauda [**10**], respectively) we adjust their definitions into the notion of a 2-Verma module for $\mathfrak{sl}_2$.

Since we need to work with short exact sequences of 1-morphisms, we require the Hom-categories of a 2-Verma module to be Quillen exact [**44**, § 2] (see also [**25**, Appendix A]). Recall that a full subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is closed under extensions if for all short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ with $A$ and $C$ in $\mathcal{C}$, then $B$ is also in $\mathcal{C}$. An additive full subcategory of an abelian category, closed under extensions, is called *Quillen exact*.

All results from § A.2 apply for Quillen exact categories if the category admits unions and intersections of admissible objects. That is whenever there are short exact sequences

$$0 \to A_1 \to B \to C_1 \to 0, \quad 0 \to A_2 \to B \to C_2 \to 0$$

in $\mathcal{C}$ then there are also short exact sequences

$$0 \to A_1 \cap A_2 \to B \to X \to 0, \quad 0 \to A_1 \cup A_2 \to B \to Y \to 0$$

in $\mathcal{C}$.

DEFINITION 5.15. Let $c$ be either an integer or a formal parameter and define $\varepsilon_c$ to be zero if $c \in \mathbb{Z}$ and to be 1 otherwise. Let $\Lambda_c = c - 2\mathbb{N}_0$ be the support. A 2-*Verma module* for $\mathfrak{sl}_2$ with the highest weight $q^c$ consists of a bigraded $\Bbbk$-linear idempotent complete, 2-category $\mathfrak{M}$ admitting a parity 2-functor $\Pi : \mathfrak{M} \to \mathfrak{M}$, where

- the objects of $\mathfrak{M}$ are indexed by weights $\mu \in \Lambda_c$;
- there are identity 1-morphisms $\mathbb{1}_\mu$ for each $\mu$, as well as 1-morphisms $\mathsf{F}\mathbb{1}_\mu \colon \mu \to \mu - 2$ in $\mathfrak{M}$ and their grading shift. We also assume that $\mathsf{F}\mathbb{1}_\mu$ has a right adjoint and define the 1-morphism $\mathsf{E}\mathbb{1}_{\mu-2} \colon \mu - 2 \to \mu$ as a grading shift of a right adjoint of $\mathsf{F}\mathbb{1}_\mu$,

$$\mathsf{E}\mathbb{1}_{\mu-2} = (\mathsf{F}\mathbb{1}_\mu)_R\langle \mu + 2 - c, -\varepsilon_c\rangle.$$

- the Hom-spaces between objects are locally additive, cone complete, Quillen exact categories.

On this data we impose the following conditions:

(1) The identity 1-morphism $\mathbb{1}_\mu$ of the object $\mu$ is isomorphic to the zero 1-morphism if $\mu \notin \Lambda_c$.
(2) The enriched $\mathrm{HOM}_{\mathfrak{M}}(\mathbb{1}_\mu, \mathbb{1}_\mu)$ is cone-bounded for all $\mu$.
(3) There is an exact sequence
$$0 \longrightarrow \mathsf{FE}\mathbb{1}_\mu \longrightarrow \mathsf{EF}\mathbb{1}_\mu \longrightarrow \mathsf{Q}_\mu\langle -c+\mu, \epsilon_c\rangle \oplus \ \Pi\mathsf{Q}_\mu\langle c-\mu, -\epsilon_c\rangle \longrightarrow 0,$$
where $\mathsf{Q}_\mu := \bigoplus_{k\geqslant 0} \Pi\mathbb{1}_\mu\langle 2k+1, 0\rangle$.
(4) For each $k \in \mathbb{N}_0$, $\mathsf{F}^k\mathbb{1}_\mu$ carries a faithful action of the enlarged nilHecke algebra.

Let $\mathcal{C}_0$ be a full subcategory of an abelian category $\mathcal{A}$. For all $i \geqslant 0$, define recursively $\mathcal{C}_{i+1}$ as the full subcategory of $\mathcal{A}$ containing all the objects of $\mathcal{C}_i$ and all $B$ for all short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ with $A$ and $C$ in $\mathcal{C}_i$. We call $\bigcup_i \mathcal{C}_i$ the *completion under extensions of* $\mathcal{C}_0$ *in* $\mathcal{A}$. It is clear that if $\mathcal{C}_0$ is also additive, then its completion under extension is Quillen exact.

Form the 2-category $\mathfrak{M}'(\lambda q^{-1})$ whose objects are the categories $\mathcal{M}_k$, the 1-morphisms are locally finite, cone-bounded direct sums of shifts of functors from $\{E_k, F_k, Q_k, \mathrm{Id}_k\}$ and the 2-morphisms are (grading-preserving) natural transformations of functors. We define $\mathfrak{M}(\lambda q^{-1})$ as the completion under extensions of $\mathfrak{M}'(\lambda q^{-1})$ in the abelian category of all functors. In this case $\mathfrak{M}(\lambda q^{-1})$ is a 2-Verma module for $\mathfrak{sl}_2$. Now take the cone completion of $\mathbf{ExtFlag}_\lambda$ from § 4.1, namely add the cone-bounded, locally finite coproduct of $\Omega_k$. Then the completion under extensions of this 2-category in $\mathbf{Bim}^s$ is also a 2-Verma module, equivalent to $\mathfrak{M}(\lambda q^{-1})$.

## 6. *Categorification of the Verma modules with integral highest weight*

In this section we give a categorical interpretation of the evaluation map $\mathrm{ev}_n \colon M(\lambda q^{-1}) \to M(n)$ for $n \in \mathbb{Z}$.

### 6.1. *Categorification of* $M(-1)$

Forgetting the $\lambda$-gradings of the superrings $\Omega_k$ and $\Omega_{k,k+1}$ from § 3.1 results in single graded superrings that we denote $\Omega_k(-1)$ and $\Omega_{k,k+1}(-1)$, respectively. We write $\langle s\rangle$ for the shift of the $q$-grading up by $s$ units.

Define $\mathcal{M}_k(-1) = \Omega_k(-1)\text{-smod}_{\mathrm{lf}}$ and $\mathcal{M}_{k+1,k}(-1) = \Omega_{k+1,k}(-1)\text{-smod}_{\mathrm{lf}}$ with the functors
$$\mathsf{F}_k \colon \mathcal{M}_k(-1) \to \mathcal{M}_{k+1}(-1),$$
$$\mathsf{E}_k \colon \mathcal{M}_{k+1}(-1) \to \mathcal{M}_k(-1),$$
$$\mathsf{Q}_k \colon \mathcal{M}_k(-1) \to \mathcal{M}_k(-1),$$
$$\mathsf{K}_k \colon \mathcal{M}_k(-1) \to \mathcal{M}_k(-1),$$
as in § 5. Denote also $\mathcal{M}(-1) = \oplus_{k\geqslant 0}\mathcal{M}_k(-1)$.

Since the $q$-grading in $\Omega_k(-1)$ and $\Omega_{k+1,k}(-1)$ is bounded from below and both superrings have one-dimensional lowest degree part, all the results in § 5 can be transported to the singly graded case. Note that either $\Omega_k(-1)$ and $\Omega_{k+1,k}(-1)$ are the product of a graded local ring with degree zero part isomorphic to $\mathbb{Q}$ with a finite-dimensional superring.

The (topological) Grothendieck group $G_0(\mathcal{M}_k(-1))$ is one-dimensional and generated either by the class of the projective indecomposable, either by the class of the simple object, both unique up to isomorphism and grading shift. Also note that $K_0(\mathcal{M}_k(-1))$ is generated by the unique projective and it is homeomorphic to $G_0(\mathcal{M}_k(-1))$. However, we prefer to use $G_0(\mathcal{M}_k(-1))$ which seems to be a more natural choice since the dual canonical basis in $K_0$ only exists as a formal power series.

Define the 2-category $\mathfrak{M}(-1)$ like $\mathfrak{M}(\lambda q^{-1})$ but with the $\mathcal{M}_k(-1)$s as objects. Collapsing the $\lambda$-grading defines a forgetful 2-functor $U : \mathfrak{M}(\lambda q^{-1}) \to \mathfrak{M}(-1)$. It is clear that $\mathfrak{M}(-1)$ is a 2-Verma module.

Having the forgetful 2-functor $U$ at hand, our strategy is to first categorify the shifted universal Verma module $M(\lambda q^n)$ for arbitrary $n$ and then to apply $U$ to get a categorification of the Vermas with integral highest weight.

Remark 6.1. While this approach yields a categorification of $M(n)$, it is interesting challenge to construct one where the $\mathfrak{sl}_2$-commutator relation is given in the form of a finite direct sum.

### 6.2. *Categorification of the shifted Verma module $M(\lambda q^n)$ for $n \in \mathbb{Z}$*

Let $n \in \mathbb{Z}, n < 0$ be fixed and let $G_{-n-1,k}$ and $G_{-n-1,k,k+1}$ the varieties of partial flags in $\mathbb{C}^\infty$

$$G_{-n-1,k} = \{(U_{-n-1}, U_k) | \dim_{\mathbb{C}} U_{-n-1} = -n-1, \dim_{\mathbb{C}} U_k = k,\ 0 \subset U_{-n-1} \subset U_k \subset \mathbb{C}^\infty\},$$

and

$$G_{-n-1,k,k+1} = \{(U_{-n-1}, U_k, U_{k+1}) | \dim_{\mathbb{C}} U_{-n-1} = -n-1, \dim_{\mathbb{C}} U_k = k, \dim_{\mathbb{C}} U_{k+1} = k+1,$$
$$0 \subset U_{-n-1} \subset U_k \subset U_{k+1} \subset \mathbb{C}^\infty\}.$$

Their cohomologies are generated by the Chern classes

$$H(G_{-n-1,k}) \cong \mathbb{Q}[x_{1,k}, \dots, x_{-n-1,k}, z_{1,k}, \dots, z_{k,k}],$$

with $\deg_q(x_{i,k}) = 2i$, $\deg_q(z_{i,k}) = 2i$, and

$$H(G_{-n-1,k,k+1}) \cong \mathbb{Q}[x_{1,k+1}, \dots, x_{-n-1,k+1}, z_{1,k+1}, \dots, z_{k,k+1}, \xi_{k+1}],$$

with $\deg_q(x_{i,k+1}) = 2i$, $\deg_q(z_{i,k+1}) = 2i$ and $\deg_q(\xi_{k+1}) = 2$.

The forgetful map $G_{-n-1,k} \to G_k$ gives $H(G_{-n-1,k})$ the structure of a $(H(G_k), H(G_{-n-1,k}))$-superbimodule and similarly for $H(G_{-n-1,k,k+1})$, which becomes a $(H(G_{k,k+1}), H(G_{-n-1,k,k+1}))$-bimodule under the forgetful map $G_{-n-1,k,k+1} \to G_{k,k+1}$. Tensoring on the left with $H(G_{-n-1,k})$ (over $H(G_k)$) and with $H(G_{-n-1,k,k+1})$ (over $H(G_{k,k+1})$) gives exact functors from $H(G_k)\text{-smod}_{\mathrm{lfg}}$ to $H(G_{-n-1,k})\text{-smod}_{\mathrm{lfg}}$ and from $H(G_{k,k+1})\text{-smod}_{\mathrm{lfg}}$ to $H(G_{-n-1,k,k+1})\text{-smod}_{\mathrm{lfg}}$, respectively.

For each $j \in \mathbb{N}_0$ put

$$X_{-n-1,j} = \begin{cases} H(G_{-n-1,j}) \otimes \mathrm{Ext}_{H(G_j)}(S_j, S_j) & \text{if } j \geqslant -n-1 \\ 0 & \text{else,} \end{cases}$$

and

$$X_{-n-1,j,j+1} = \begin{cases} H(G_{-n-1,j,j+1}) \otimes \mathrm{Ext}_{H(G_{j+1})}(S_{j+1}, S_{j+1}) & \text{if } j \geqslant -n-1 \\ 0 & \text{else,} \end{cases}$$

and for all $k \in \mathbb{N}_0$ define the superrings

$$\Omega_k^n = X_{-n-1,-n-1+k} \otimes_k \Omega_{-n-1+k},$$

$$\Omega_{k,k+1}^n = X_{-n-1,-n-1+k,-n+k} \otimes_{k,k+1} \Omega_{-n-1+k,-n+k}.$$

Now take $n \in \mathbb{N}_0$. Let also $\Omega_k^n \subset \Omega_k$ and $\Omega_{k,k+1}^n \subset \Omega_{k,k+1}$ be the sub-superrings

$$\Omega_k^n = \mathbb{Q}[x_{1,k}, \dots, x_{k,k}, s_{-n,k}, \dots, s_{k-1-n,k}],$$

and

$$\Omega^n_{k,k+1} = \mathbb{Q}[w_{1,k},\dots,w_{k,k},\xi_{k+1},\sigma_{-n,k+1},\dots,\sigma_{k-n,k+1}],$$

where we compute $s_{i,k}$ and $\sigma_{i,k+1}$ for $i \leqslant 0$ recursively with the formulas

$$s_{i,k} = -\sum_{\ell=1}^{k} x_{\ell,k}s_{i+\ell,k}, \quad \sigma_{i,k+1} = -\sum_{\ell=1}^{k}(x_{\ell,k}+\xi_{k+1}x_{\ell-1,k})\sigma_{i+\ell,k+1}.$$

After a suitable change of variables we can write

$$\begin{aligned}\Omega^n_k &= \mathbb{Q}[x_{1,k},\dots,x_{k,k},\tilde{s}_{1,k},\dots,\tilde{s}_{k,k}]\\ \Omega^n_{k,k+1} &= \mathbb{Q}[x_{1,k},\dots,x_{k,k},\xi_{k+1},\tilde{s}_{1,k+1},\dots,\tilde{s}_{k+1,k+1}],\end{aligned}\tag{29}$$

with $\deg_{q,\lambda,q}(\tilde{s}_{i,k}) = \deg_{q,\lambda}(\tilde{s}_{i,k+1}) = (2n-2i,2)$.

In order to define the analogous of the category $\mathcal{M}$ from § 5 note that we get

$$\phi^*_k(x_{i,k}) = x_{i,k+1}, \quad \phi^*_k(\tilde{s}_{i,k}) = \tilde{s}_{i,k+1}+\xi_{k+1}\tilde{s}_{i+1,k+1},\tag{30}$$

$$\psi^*_k(x_{i,k+1}) = x_{i,k+1}+\xi_{k+1}x_{i-1,k+1}, \quad \psi^*_{k+1}(\tilde{s}_{i,k+1}) = \tilde{s}_{i,k+1}.\tag{31}$$

As in Definition 4.22, we define the shifted superbimodules.

Definition 6.2. For $n \in \mathbb{Z}$ and $k \in \mathbb{N}_0$ we put

$$\overline{\Omega}^n_{k+1,k} = \Omega^n_{k+1,k}\langle -k,0\rangle, \quad \overline{\Omega}^n_{k,k+1} = \Omega^n_{k,k+1}\langle k-n+1,-1\rangle,$$

and define

$$\overline{\Omega}^n_{k(k\pm1)k} = \overline{\Omega}^n_{k,k\pm1}\otimes_{k\pm1}\overline{\Omega}^n_{k\pm1,k}, \quad \overline{\Omega}^{\xi,n}_k = \Pi\Omega^n_k[\xi]\langle 1,0\rangle.$$

The analogue of Corollary 4.23 reads as below.

Lemma 6.3. *There are short exact sequences of* $(\Omega^n_k,\Omega^n_k)$*-superbimodules*

$$0 \longrightarrow \overline{\Omega}^n_{k(k-1)k} \longrightarrow \overline{\Omega}^n_{k(k+1)k} \longrightarrow \overline{\Omega}^{\xi,n}_k\langle n-2k,1\rangle \oplus \Pi\overline{\Omega}^{\xi,n}_k\langle 2k-n,-1\rangle \longrightarrow 0.$$

*Proof.* For $n<0$ we regard $\Omega^n_k$ and $\Omega^n_{k,k+1}$ as algebras over $\Omega_{-n-1}$ and apply the analysis leading to Corollary 4.23. For $n \geqslant 0$ we use the presentations of $\Omega^n_k$ and $\Omega^n_{k,k+1}$ in equation (29) and apply again the analysis leading to Corollary 4.23. In both cases the claim follows by tracing carefully the bidegrees in the homomorphisms in § 4.1. □

We now define categories $\mathcal{M}^n_k = \Omega^n_k\text{-smod}_{\mathrm{lfg}}$ and $\mathcal{M}^n = \oplus_{k\geqslant 0}\mathcal{M}^n_k$ as we did $\mathcal{M}$ in § 5. We also define $\mathfrak{M}(\lambda q^n)$ as the 2-category with objects $\mathcal{M}^n_k$, for $k \in \mathbb{N}_0$. The 1-morphisms of $\mathfrak{M}(\lambda q^n)$ are direct sums of grading shifts of the functors

$$\begin{aligned}
&\mathsf{F}_k\colon \mathcal{M}^n_k \to \mathcal{M}^n_{k+1} && \mathsf{F}_k(-) = \mathrm{Res}^{k,k+1}_{k+1}\circ\left(\Omega^n_{k+1,k}\otimes_k(-)\right)\langle -k,0\rangle,\\
&\mathsf{E}_k\colon \mathcal{M}^n_{k+1} \to \mathcal{M}^n_k && \mathsf{E}_k(-) = \mathrm{Res}^{k,k+1}_{k}\circ\left(\Omega^n_{k,k+1})\otimes_{k+1}(-)\right)\langle k-n+1,-1\rangle,\\
&\mathsf{Q}_k\colon \mathcal{M}^n_k \to \mathcal{M}^n_k && \mathsf{Q}_k(-) = \Omega^n_k[\xi]\otimes_k(-)\langle 1,0\rangle,\\
&\mathsf{K}_k\colon \mathcal{M}^n_k \to \mathcal{M}^n_k && \mathsf{K}_k(-) = (-)\langle n-2k,1\rangle,
\end{aligned}$$

and the 2-morphisms are (grading-preserving) natural transformations between these functors. As before, we put

$$\mathsf{E} = \bigoplus_{k\geqslant 0} \mathsf{E}_k, \quad \mathsf{F} = \bigoplus_{k\geqslant 0} \mathsf{F}_k, \quad \mathsf{K} = \bigoplus_{k\geqslant 0} \mathsf{K}_k, \quad \text{and} \quad \mathsf{Q} = \bigoplus_{k\geqslant 0} \mathsf{Q}_k.$$

The analogs of Theorems 5.8 and 5.12 follow as in § 5 using Lemma 6.3. We state them below for the record.

THEOREM 6.4. *We have natural isomorphisms of exact endofunctors of $\mathcal{M}^n$*

$$\mathsf{K}\circ\mathsf{K}^{-1} \cong \mathrm{Id}_{\mathcal{M}} \cong \mathsf{K}^{-1}\circ\mathsf{K},$$

$$\mathsf{K}\circ\mathsf{E} \cong \mathsf{E}\circ\mathsf{K}\langle 2,0\rangle, \quad \mathsf{K}\circ\mathsf{F} \cong \mathsf{F}\circ\mathsf{K}\langle -2,0\rangle,$$

*and an exact sequence*

$$0 \longrightarrow \mathsf{F}\circ\mathsf{E} \longrightarrow \mathsf{E}\circ\mathsf{F} \longrightarrow \mathsf{Q}\circ\big(\mathsf{K}\oplus\ \Pi\mathsf{K}^{-1}\big) \longrightarrow 0.$$

All the above imply that $\mathfrak{M}(\lambda q^n)$ is a 2-Verma module for $\mathfrak{sl}_2$.

THEOREM 6.5. *The three Grothendieck groups $G_0(\mathcal{M}^n)$, $K_0(\mathcal{M}^n)$ and $\widehat{G}_0(\mathcal{M}^n)$, together with the action induced by functors $\mathsf{F}$ and $\mathsf{E}$, are, respectively, isomorphic with the $\dot{U}_\lambda$-modules $M_A^*(\lambda q^n)$ and $M_A(\lambda q^n)$, and with the $U_q(\mathfrak{sl}_2)$-module $\widehat{M}(\lambda q^n)$.*

### 6.3. *Categorification of the Verma module $M(n)$ for $n \in \mathbb{Z}$*

As before, we apply the map that forgets the $\lambda$-grading on the superrings $\Omega_k^n$ and $\Omega_{k,k+1}^n$ to obtain singly graded superrings $\Omega_k(n)$ and $\Omega_{k,k+1}(n)$. In this case we still have that the $q$-grading in $\Omega_k(n)$ and $\Omega_{k+1,k}(n)$ is bounded from below and both superrings have one-dimensional lowest degree part, and we can use all the results in § 5. Note that again both $\Omega_k(n)$ and $\Omega_{k+1,k}(n)$ are the product of a graded local ring with degree zero part isomorphic to $\mathbb{Q}$ with a finite-dimensional superring.

Denote by $\mathcal{M}(n)$ the image of $\mathcal{M}^n$ under the forgetful functor. We keep the notation $\mathsf{F}_k$, $\mathsf{E}_k$, $\mathsf{Q}_k$ and $\mathsf{K}_k$ for $U(\mathsf{F}_k)$, $U(\mathsf{E}_k)$, $U(\mathsf{Q}_k)$ and $U(\mathsf{K}_k)$. The 2-categories $\mathfrak{M}(n)$ constructed in the obvious way, yield 2-Verma modules.

It is easy to see that

$$\mathsf{F}_k\Omega_k^n \cong \oplus_{[k+1]}\Omega_{k+1}^n,$$

and

$$\mathsf{E}_k\Omega_{k+1}^n \cong \mathsf{Q}_k\Omega_k^n\langle n-k\rangle \oplus \Pi\mathsf{Q}_k\Omega_k^n\langle k-n\rangle,$$

which means that in the Grothendieck group we have

$$[\mathsf{F}][\Omega_k^n] = [k+1]_q[\Omega_{k+1}^n], \quad [\mathsf{E}][\Omega_{k+1}^n] = [n-k+1]_q[\Omega_k^n] \tag{32}$$

after specializing $\pi = -1$.

We now proceed to analyze the cases $n < 0$ and $n \geqslant 0$ separately.

6.3.1. *The case $n < 0$.* The arguments in the proof of Theorem 5.12 can be applied almost unchanged to get the following.

THEOREM 6.6. *For $n < 0$ the Grothendieck groups $G_0(\mathcal{M}(n))$ and $K_0(\mathcal{M}(n))$, together with the action induced by functors $\mathsf{F}$ and $\mathsf{E}$, are isomorphic with the Verma module $M(n)$.*

The composite of the functor on $\mathcal{M}_k$ obtained by tensoring with the appropriate superbimodules $X_{n-1,k}$ and $X_{n-1,k,k+1}$ with the forgetful functor $U$ defines a 2-functor $\mathrm{EV}_{-|n|}\colon \mathfrak{M} \to \mathfrak{M}(n)$ which is exact, takes projectives to projectives, and categorifies the evaluation map $\mathrm{ev}_{-|n|}\colon M(\lambda q^{-1}) \to M(n)$.

The categorification of $M(n)$ using $\mathcal{M}(n)$ for $n < 0$ is not minimal, in the sense that there is a smaller category with the same properties we now describe. Note that $\Omega_k(-|n|)$ and $\Omega_{k,k+1}(-|n|)$ have presentations

$$\Omega_k(-|n|) = \mathbb{Q}[\underline{x}_{n-1}, \underline{s}_{n-1}][z_1, \dots, z_k, s_n, \dots, s_{n+k}],$$

and

$$\Omega_{k,k+1}(-|n|) = \mathbb{Q}[\underline{x}_{n-1}, \underline{s}_{n-1}][z_1, \dots, z_{k+1}, \xi_{k+1}, s_n, \dots, s_{n+k+1}],$$

where $z_i$ and $\xi_{k+1}$ are even with $\deg_q(z_i) = 2i$, $\deg_q(\xi_{k+1}) = 2$ and $s_i$ is odd with $\deg_q(s_i) = -2i$ (recall that $\mathbb{Q}[\underline{x}_{n-1}, \underline{s}_{n-1}] = \Omega_k$).

Let $J_k \subset \Omega_k(-|n|)$ and $J_{k,k+1} \subset \Omega_{k,k+1}(-|n|)$ be the two-sided ideals generated by $(\underline{x}_{n-1}, \underline{s}_{n-1})$ and define the 2-category $\mathcal{M}^{min}(-|n|)$ as before but using the quotient superrings

$$\Omega_k^{min}(-|n|) = \Omega_k(-|n|)/J_k,$$

and

$$\Omega_{k,k+1}^{min}(-|n|) = \Omega_{k,k+1}(-|n|)/J_{k,k+1},$$

instead. We get functors $\mathsf{F}^{min}$, $\mathsf{E}^{min}$, $\mathsf{Q}^{min}$ and $\mathsf{K}^{min}$ with the same properties as in Theorem 6.4 while the Grothendieck group of $\mathcal{M}^{min}(n)$ is still isomorphic to $M(n)$. Using the surjection from $\Omega_k(-|n|)$ to $\Omega_k^{min}(-|n|)$ we can construct an obvious functor $\Psi$ from $\mathcal{M}(n)$ to $\mathcal{M}^{min}(n)$ that sends $\mathcal{M}_k(n)$ to $\mathcal{M}_k^{min}(-|n|)$ and $(\mathsf{F}_k, \mathsf{E}_k, \mathsf{Q}_k, \mathsf{K}_k)$ to $(\mathsf{F}_k^{min}, \mathsf{E}_k^{min}, \mathsf{Q}_k^{min}, \mathsf{K}_k^{min})$. Moreover, $\Psi$ sends projectives to projectives, simples to simples, is exact, full and bijective on objects.

6.3.2. *The case $n \geqslant 0$.* We have to be a bit careful for this case, as the dual canonical basis does not exist for $M(n)$. Indeed, in $G_0(\mathcal{M}(n))$ we get the equality

$$[\Omega_{n+1}^n] = \prod_{i=1}^{n+1} \frac{1 + \pi q^{2n-2i+2}}{1 - q^{2i}} [S_{n+1}^n]$$

and thus after specializing to $\pi = -1$, it gives $[\Omega_{n+1}^n] = 0$. Of course, the element $(1 + \pi)$ is not invertible in $\mathbb{Z}_\pi[\![q]\!][q^{-1}]$ and thus we cannot use the projective resolution of $S_{n+1}^n$ to generate it with the projective $\Omega_{n+1}^n$ neither. In fact, we have $[F_n] = 0$, and $G_0(\mathcal{M}(n))$ contains $V(n)$ as a submodule. The action of $U_q(\mathfrak{sl}_2)$ on $G_0(\mathcal{M}(n))/V(n)$ is trivial with $E = F = 0$.

Therefore, the Grothendieck group $G_0(\mathcal{M}(n))$ is not the way we want to decategorify $\mathcal{M}(n)$ and we will work only with $K_0(\mathcal{M}(n))$, which is freely generated by the classes or projectives $[\Omega_k^n]$. Again comparing the action on the canonical basis and (32) gives the following theorem.

THEOREM 6.7. *For each $n \geqslant 0$ the Grothendieck group $K_0(\mathcal{M}(n))$, together with the action induced by functors $\mathsf{F}$ and $\mathsf{E}$, is isomorphic to the Verma module $M(n)$.*

## 7. *Categorification of the finite-dimensional irreducibles from the Verma categorification*

### 7.1. *The ABC of the DG world*

We start by recollecting some basic facts about DG-algebras and their (derived) categories of modules following closely the exposition in [**5**, § 10].

A *differential graded algebra* (DG-algebra for short) $(A, d)$ is a $\mathbb{Z}$-graded associative unital algebra $A$ with 1 in degree zero, equipped with an additive endomorphism $d$ of degree $-1$ satisfying

$$d^2 = 0, \quad d(ab) = d(a)b + (-1)^{\deg a} ad(b), \quad d(1) = 0.$$

A *homomorphism* between DG-algebras $(A, d)$ and $(A', d')$ is a homomorphism $\phi\colon A \to B$ of algebras intertwining the differentials, $\phi \circ d = d \circ \phi$.

A left *DG-module* $M$ over $A$ is a $\mathbb{Z}$-graded left $A$-module with a differential $d_M : M_i \to M_{i-1}$ such that, for all $a \in A$ and all $m \in M$,

$$d_M(am) = d(a)m + (-1)^{\deg(a)} ad_M(m).$$

We have the analogous notion for right $A$-modules and bimodules. Denote by $(A, d)$-mod the abelian category of (left) DG-modules over $A$. We say $P \in (A, d)$-mod is *projective* if for every acyclic $M$ in $(A, d)$-mod the complex $\mathrm{Hom}_A(P, M)$ is also acyclic. The homology $H(M)$ of a DG-module over $A$ is the usual homology of the chain complex $M$. It is a graded module over the graded ring $H(A)$. We say that $A$ is *formal* if it is quasi-isomorphic to $H(A)$, that is if there exists a map $A \to H(A)$ or $H(A) \to A$ inducing an isomorphism on the homology.

Two morphisms $f, g\colon M \to N$ in $(A, d)$-mod are *homotopic* if there is a degree one map $s\colon N \to M$ such that $f - g = sd_M + d_N s$. The *homotopy category* $\mathcal{K}_A$, which is a triangulated category, is obtained from $(A, d)$-mod by modding out by null-homotopic maps. Inverting quasi-isomorphisms results in the *derived category* $\mathcal{D}(A)$, which is triangulated and idempotent complete.

The localization functor gives an isomorphism from the full subcategory of $\mathcal{K}_A$ consisting of projective objects to the derived category $\mathcal{D}(A)$. We say $M$ is compact if the canonical map

$$\oplus_{i\in I} \mathrm{Hom}_{\mathcal{D}(A)}(M, N_i) \to \mathrm{Hom}_{\mathcal{D}(A)}(M, \oplus_{i\in I} N_i)$$

is an isomorphism for every arbitrary direct sum of DG-modules. Note that $A$ is compact projective. Let $\mathcal{D}^c(A)$ be the full subcategory of $\mathcal{D}(A)$ consisting of compact modules. It is also idempotent complete.

For a homomorphism of DG-algebras $\phi\colon A \to B$ the *derived induction* functor is the derived functor associated with the bimodule ${}_B B_A$,

$$\mathrm{Ind}_A^B = B \otimes_A^{\mathbf{L}} (-)\colon \mathcal{D}(A) \to \mathcal{D}(B).$$

The *derived restriction* functor is given by taking the derived Hom-functor:

$$\mathrm{Res}_A^B = \mathbf{R}\mathrm{Hom}_B(B_A, -)\colon \mathcal{D}(B) \to \mathcal{D}(A).$$

The forgetful functor via the map $\phi$ is exact and therefore, lifts trivially to the derived setting. This lift coincides with the derived restriction functor. The above functors are adjoint:

$$\mathrm{Hom}_{\mathcal{D}(B)}(\mathrm{Ind}_A^B(M), N) \cong \mathrm{Hom}_{\mathcal{D}(A)}(M, \mathrm{Res}_A^B(N)).$$

If $\phi$ is a quasi-isomorphism then $\mathrm{Ind}_A^B$ and $\mathrm{Res}_A^B$ are mutually inverse equivalences of categories.

We define the Grothendieck group of $A$ as the Grothendieck group of the triangulated category $\mathcal{D}^c(A)$. As Khovanov pointed out in [27], if $A$ is formal and (graded) noetherian we can describe the Grothendieck group of $A$ via finitely generated $H(A)$-modules.

Of course, all the above generalize easily to the case where $A$ has additional gradings and where the differential is graded over $\mathbb{Z}/2\mathbb{Z}$. In this case we speak of graded (or bigraded) DG-algebras, graded (or bigraded) homomorphisms and graded versions of all the categories above.

7.2. *The differentials $d_n$*

We next introduce differentials on $\Omega_k$ and $\Omega_{k,k+1}$ turning them into DG-algebras for the parity degree. Recall from § 3.1 that the rings $\Omega_k$ and $\Omega_{k,k+1}$ have presentations

$$\Omega_k = \mathbb{Q}[x_{1,k}, \dots, x_{k,k}, s_1, \dots, s_k] \quad \text{and} \quad \Omega_{k,k+1} = \mathbb{Q}[x_{1,k}, \dots, x_{k,k}, \xi_{k+1}, s_{1,k+1}, \dots, s_{k+1,k+1}].$$

DEFINITION 7.1. Define maps $d_n^k \colon \Omega_k \to \Omega_k$ and $d_n^{k,k+1} \colon \Omega_{k,k+1} \to \Omega_{k,k+1}$ of bidegree $\langle 2n+2, -2\rangle$ and parity $-1$ by

$$d_n^k(x_{r,k}) = 0, \quad d_n^k(s_{r,k}) = Y_{n-r+1,k},$$

and

$$d_n^{k,k+1}(x_{r,k}) = 0, \quad d_n^{k,k+1}(\xi_{k+1}) = 0, \quad d_n^{k,k+1}(s_{r,k+1}) = Y_{n-r+1,k+1},$$

respecting the Leibniz rule

$$d_n^k(ab) = d_n^k(a)b + (-1)^{p(a)} a d_n^k(b) \quad d_n^{k,k+1}(ab) = d_n^{k,k+1}(a)b + (-1)^{p(a)} a d_n^{k,k+1}(b).$$

From now on we use $d_n$ to denote either $d_n^k$ and $d_n^{k,k+1}$ whenever the $k$ or the $k, k+1$ are clear from the context. The maps $d_n$ satisfy $d_n \circ d_n = 0$ and therefore $\Omega_k$ and $\Omega_{k,k+1}$ become DG-algebras with $d_n$ of bidegree $\langle 2n+2, -2\rangle$ which we denote $(\Omega_k, d_n)$ and $(\Omega_{k,k+1}, d_n)$. These algebras are bigraded and *differential graded* with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading (also known as the parity).

LEMMA 7.2. *For $k > n$, the DG-algebras $(\Omega_k, d_n)$ and $(\Omega_{k,k+1}, d_n)$ are acyclic.*

*Proof.* Let $k > n$. Then $d_n^k(s_{n+1,k}) = Y_{0,k} = 1$ and $d_n^{k,k+1}(s_{n+1,k+1}) = 1$. □

REMARK 7.3. For $n = 0$ the DG-algebra $(\Omega_{k,k+1}, d_0)$ is acyclic for all $k$ and the DG-algebra $(\Omega_k, d_0)$ is acyclic unless $k = 0$ and in this case $(\Omega_0, d_0) \cong \mathbb{Q}$.

The DG-rings $(\Omega_k, d_n)$ and $(\Omega_{k,k+1}, d_n)$ have a nice geometric interpretation.

PROPOSITION 7.4. *The DG-rings $(\Omega_k, d_n)$ and $(\Omega_{k,k+1}, d_n)$ are formal.* (1) *The DG-ring $(\Omega_k, d_n)$ is quasi-isomorphic to the cohomology of the Grassmannian variety $H(G_{k;n})$ of $k$-planes in $\mathbb{C}^n$.* (2) *The DG-ring $(\Omega_{k,k+1}, d_n)$ is quasi-isomorphic to the cohomology of the partial flag variety $H(G_{k,k+1;n})$ of $k, k+1$-planes in $\mathbb{C}^n$.*

*Proof.* Let $(d_n(\underline{s}_k))$ denote the two-sided ideal of $\mathbb{Q}[\underline{x}_k]$ generated by $d_n(s_{1,k}), \dots, d_n(s_{k,k})$ and let $(d_n(\underline{\sigma}_{k+1}))$ denote the two-sided ideal of $\mathbb{Q}[\underline{x}_k, \xi_{k+1}]$ generated by $d_n(\sigma_{1,k+1}), \dots, d_n(\sigma_{k+1,k+1})$. If we equip the quotient rings

$$\mathbb{Q}[\underline{x}_k]/(d_n(\underline{s}_k)) \quad \text{and} \quad \mathbb{Q}[\underline{x}_k, \xi_{k+1}]/(d_n(\underline{\sigma}_{k+1})),$$

with the zero differential, an easy exercise shows that the obvious surjections $\Omega_k \twoheadrightarrow \mathbb{Q}[\underline{x}_k]/(d_n(\underline{s}_k))$ and $\Omega_{k,k+1} \twoheadrightarrow \mathbb{Q}[\underline{x}_k, \xi_{k+1}]/(d_n(\underline{\sigma}_{k+1}))$, are quasi-isomorphisms.

Formula (11) together with the definition of the differential show that $Y_{n-k+r,k} \in (d_n(\underline{s}_k))$ for all $r \geqslant 1$ and thus give a presentation

$$\mathbb{Q}[\underline{x}_k]/(d_n(\underline{s}_k)) \cong \mathbb{Q}[\underline{x}_k, \underline{Y}_{(n-k)}]/I_{k,n},$$

with $I_{k,n}$ the ideal generated by the homogeneous terms in the equation

$$(1 + x_{1,k}t + \dots + x_{k,k}t^k)(1 + Y_{1,k}t + \dots + Y_{(n-k),k}t^{n-k}) = 1,$$

which is a presentation for the cohomology ring $H(G_{k;n})$ and thus proves part (1). The second claim is proved in the same way. □

### 7.3. *A category of DG-bimodules*

PROPOSITION 7.5. *The maps $\phi_k^*$ and $\psi_{k+1}^*$ from § 3.4 commute with the differentials $d_n$.*

*Proof.* From the definitions and the Leibniz rule we have that the diagrams

$$\begin{tikzcd} s_{r,k} \arrow[r, mapsto, "d_n"] \arrow[d, mapsto, "\phi^*"'] & Y_{n-r+1,k} \arrow[d, mapsto, "\phi^*"] \\ s_{r,k} + \xi_{k+1} s_{r-1,k} \arrow[r, mapsto, "d_n"'] & Y_{n-r+1,k+1} + \xi_{k+1} Y_{n-r,k+1} \end{tikzcd}$$

and

$$\begin{tikzcd} s_{r,k+1} \arrow[r, mapsto, "d_n"] \arrow[d, mapsto, "\psi^*"'] & Y_{n-r+1,k+1} \arrow[d, mapsto, "\psi^*"] \\ s_{r,k+1} \arrow[r, mapsto, "d_n"'] & Y_{n-r+1,k+1} \end{tikzcd}$$

commute. The statement now follows using the Leibniz rule recursively. □

In the following we write DG $(k,s)_n$-bimodule for a $((\Omega_k, d_n), (\Omega_s, d_n))$-bimodule. From the proposition, $(\Omega_{k,k+1}, d_n)$ is a DG $(k+1,k)_n$-bimodule. The $(\Omega_k, \Omega_k)$-bimodules $\Omega_{k,k+1} \otimes_{k+1} \Omega_{k+1,k}$ and $\Omega_{k,k-1} \otimes_{k-1} \Omega_{k-1,k}$ get structures of DG $(k,k)_n$-bimodules with $d_n$ satisfying the Leibniz rule $d_n(a \otimes b) = d_n(a) \otimes b + (-1)^{p(a)} a \otimes d_n(b)$.

Define the DG $(k+1,k)_n$-bimodule

$$(\check{\Omega}_{k+1,k}, d_n) = (\overline{\Omega}_{k+1,k}, d_n)\langle 0,0\rangle = (\Omega_{k+1,k}, d_n)\langle -k, 0\rangle,$$

and the DG $(k,k+1)_n$-bimodule

$$(\check{\Omega}_{k,k+1}, d_n) = (\overline{\Omega}_{k,k+1}, d_n)\langle -n,1\rangle = (\Omega_{k,k+1}, d_n)\langle k+1-n, 0\rangle.$$

Note that some of the maps in § 4.1 do not extend to the various DG-bimodules above, for example, $\pi$ from Proposition 4.6. However, we can equip $\Omega_k^\xi \oplus \Pi\Omega_k^\xi\langle -2k-2\rangle$ with a differential given by $d_n(\xi^i \oplus 0) = 0$ and

$$d_n(0 \oplus Y_{j,k}^\xi) = \pi(Y_{n-k,k+1} \otimes \xi_{k+1}^j) \oplus 0 = \sum_{p=k-j}^{n-k} Y_{p-k+j,k}^\xi Y_{n-k-p,k} \oplus 0,$$

for all $i,j \geqslant 0$, such that it becomes a DG-bimodule over $(\Omega_k, d_n^k)$. This differential commutes with $\pi + \mu$, as $d_n^{k,k+1}$ does with $u$, and by consequence we get a short exact sequence of $(k,k)_n$-bimodules

$$0 \to \left(\Omega_{k(k-1)k}, d_n\right) \to \left(\Omega_{k(k+1)k}, d_n\right) \to \left(\Omega_k^\xi\langle 2k,0\rangle \oplus \Pi\Omega_k^\xi\langle -2k-2,2\rangle, d_n\right) \to 0.$$

By the snake lemma, it descends to a long exact sequence of $H(\Omega_k, d_n) \cong H(G_{k;n})$-bimodules

$$\begin{tikzcd} & \cdots \arrow[r] & H^1\left(\Omega_{k(k+1)k}, d_n\right) \arrow[r] & H^1\left(\Omega_k^\xi\langle 2k,0\rangle \oplus \Pi\Omega_k^\xi\langle -2k-2,2\rangle, d_n\right) \arrow[dll] \\ H^0\left(\Omega_{k(k-1)k}, d_n\right) \arrow[r] & H^0\left(\Omega_{k(k+1)k}, d_n\right) \arrow[r] & H^0\left(\Omega_k^\xi\langle 2k,0\rangle \oplus \Pi\Omega_k^\xi\langle -2k-2,2\rangle, d_n\right) \arrow[dll] & \\ H^1\left(\Omega_{k(k-1)k}, d_n\right) \arrow[r] & \cdots & & \end{tikzcd}$$

Proposition 7.4 tells us the homology of $(\Omega_{k(k+1)k}, d_n)$ is concentrated in parity 0 and thus we have a long exact sequence

$$\begin{tikzcd} & 0 \arrow[r] & H^1\left(\Omega_k^\xi\langle 2k,0\rangle \oplus \Pi\Omega_k^\xi\langle -2k-2,2\rangle, d_n\right) \arrow[dll] \\ H^0\left(\Omega_{k(k-1)k}, d_n\right) \arrow[r] & H^0\left(\Omega_{k(k+1)k}, d_n\right) \arrow[r] & H^0\left(\Omega_k^\xi\langle 2k,0\rangle \oplus \Pi\Omega_k^\xi\langle -2k-2,2\rangle, d_n\right) \arrow[dll] \\ 0. & & \end{tikzcd}$$

For $n-2k \geqslant 0$, we get that $d_n(0 \oplus 1)$ is a polynomial with a dominant monomial $\xi_{k+1}^{n-2k}$ and $d_n(0 \oplus Y_{j,k}^\xi) \neq 0$. It means the homology of $(\Omega_k^\xi\langle 2k,0\rangle \oplus \Pi\Omega_k^\xi\langle -2k-2,2\rangle, d_n)$ is concentrated in parity 0 and given by $\bigoplus_{\{n-2k\}} q^{2k} H(G_{k;n})$. Therefore we get the following short exact sequence:

$$\begin{aligned} 0 \to H(G_{k,k-1;n}) &\otimes_{H(G_{k-1;n})} H(G_{k-1,k;n}) \\ &\hookrightarrow H(G_{k,k+1;n}) \otimes_{H(G_{k+1;n})} H(G_{k+1,k;n}) \twoheadrightarrow \bigoplus_{\{n-2k\}} q^{2k} H(G_{k;n}) \to 0. \end{aligned}$$

For $n-2k \leqslant 0$, we get $d_n(0 \oplus Y_{j,k}^\xi) = 0$ for $j < 2k-n$ and $d_n(0 \oplus Y_{2k-n,k}^\xi) = 1 \oplus 0$. Thus the homology is concentrated in parity 1 and isomorphic to $\bigoplus_{\{2k-n\}} q^{-2k-2}\lambda^2 \Pi H(G_{k;n})$. After shifting by the degree of the connecting homomorphism, it yields the short exact sequence

$$\begin{aligned} 0 \to \bigoplus_{-\{2k-n\}} q^{2k} H(G_{k;n}) &\hookrightarrow H(G_{k,k-1;n}) \otimes_{H(G_{k-1;n})} H(G_{k-1,k;n}) \\ &\twoheadrightarrow H(G_{k,k+1;n}) \otimes_{H(G_{k+1;n})} H(G_{k+1,k;n}) \to 0. \end{aligned}$$

It is not hard to see that both short exact sequences split, with an obvious splitting morphism for the projection in the first one and an obvious left inverse for the injection in the second one, obtained by the same kind of expressions as $\iota$ and $\pi$.

In conclusion, we recover the well-known commutator of the categorical $\mathfrak{sl}_2$ action using cohomology of the finite Grassmannians and two-step flag varieties, which is employed in the categorification of the irreducible finite-dimensional $\mathfrak{sl}_2$-modules.

Proposition 7.6. *We have quasi-isomorphisms of bigraded DG $(k,k)_n$-bimodules*

$$(\check{\Omega}_{k,k+1} \otimes_{k+1} \check{\Omega}_{k+1,k}, d_n) \cong (\check{\Omega}_{k,k-1} \otimes_{k-1} \check{\Omega}_{k-1,k}, d_n) \oplus_{[n-2k]} \Omega_k^{d_n}, \quad \textit{if } n-2k \geqslant 0,$$

$$(\check{\Omega}_{k,k-1} \otimes_{k-1} \check{\Omega}_{k-1,k}, d_n) \cong (\check{\Omega}_{k,k+1} \otimes_{k+1} \check{\Omega}_{k+1,k}, d_n) \oplus_{[2k-n]} \Omega_k^{d_n}, \quad \textit{if } n-2k \leqslant 0.$$

### 7.4. *A categorification of $V(n)$*

Let $\mathcal{V}_k(n)$ (respectively, $\mathcal{V}_{k,k+1}(n)$) be the derived category of bigraded, left, compact $(\Omega_k, d_n)$ modules (respectively, bigraded, left, compact $(\Omega_{k,k+1}, d_n)$-modules) and define the functors

$$\mathrm{Ind}_k^{k+1,k}\colon \mathcal{V}_k(n) \to \mathcal{V}_{k+1,k}(n), \qquad \mathrm{Res}_k^{k+1,k}\colon \mathcal{V}_{k+1,k}(n) \to \mathcal{V}_k(n),$$
$$\mathrm{Ind}_{k+1}^{k+1,k}\colon \mathcal{V}_{k+1}(n) \to \mathcal{V}_{k+1,k}(n), \quad \mathrm{Res}_{k+1}^{k+1,k}\colon \mathcal{V}_{k+1,k}(n) \to \mathcal{V}_{k+1}(n).$$

For each $k \geqslant 0$ define the functors

$$\mathsf{F}_k(-) = \mathrm{Res}_{k+1}^{k,k+1} \circ \left(\Omega_{k+1,k}^{d_n} \otimes_k^{\mathbf{L}} (-)\right) \langle -k, 0\rangle,$$

and

$$\mathsf{E}_k(-) = \mathrm{Res}_{k}^{k,k+1} \circ \left(\Omega_{k,k+1}^{d_n} \otimes_{k+1}^{\mathbf{L}} (-)\right) \langle k+1-n, 0\rangle,$$

where $\Omega_{k+1,k}^{d_n}$ is seen as a DG $(\Omega_{k+1,k}, \Omega_k)_n$-bimodule and $\Omega_{k,k+1}^{d_n}$ as a DG $(\Omega_k, \Omega_{k,k+1})_n$-bimodule.

Proposition 7.7 and Theorem 7.8 are a direct consequence of Propositions 7.4 and 7.5, together with well-known results: see, for example, [**13**, § 6.2; **36**, §§ 3.4 and 5.3] (see also [**11**, § 5.3] for the ungraded case). There is an equivalence of triangulated categories between $\mathcal{V}_k(n)$ and the bounded derived category $\mathcal{D}^b(H(G_{k;n})\text{-mod})$.

Proposition 7.7. *The functors $\mathsf{F}_k$ and $\mathsf{E}_k$ are biadjoint up to a shift. Moreover we have natural isomorphisms of functors*

$$\mathsf{E}_k \circ \mathsf{F}_k \cong \mathsf{F}_{k-1} \circ \mathsf{E}_{k-1} \oplus_{[n-2k]} \mathrm{Id}_k, \quad \textit{if } \ n-2k \geqslant 0,$$

*and*

$$\mathsf{F}_{k-1} \circ \mathsf{E}_{k-1} \cong \mathsf{E}_k \circ \mathsf{F}_k \oplus_{[2k-n]} \mathrm{Id}_k, \quad \textit{if } \ n-2k \leqslant 0.$$

Theorem 7.8. *Define the category $\mathcal{V}(n) = \bigoplus_{k\geqslant 0} \mathcal{V}_k(n)$. We have a $\mathbb{Z}[q,q^{-1}]$-linear isomorphism of $U_q(\mathfrak{sl}_2)$-modules, $K_0(\mathcal{V}(n)) \cong V(n)$, for all $n \geqslant 0$.*

All the above can be applied without difficulty to $\Omega_k^m$ and $\Omega_{k,k+1}^m$ in $\mathcal{M}(\lambda q^m)$, with $m \geqslant 0$. After passing to the derived category we get a category isomorphic to $\mathcal{V}(n+m+1)$. In particular if we take $m = N$ and $d_0$, this yields a differential on $\mathcal{M}(N)$ with $q$-grading 2.

### 7.5. *nilHecke action*

Recall the ring $\Omega_{k,k+1,\dots,k+s} = \mathbb{Q}[\underline{x}_k, \underline{\xi}_s, \underline{s}_{k+s}]$ from § 3.4. It has the structure of a DG-algebra with differential $d_n$ given by

$$d_n(x_r) = 0, \quad d_n(\xi_r) = 0, \quad d_n(s_r) = Y_{n-r+1}.$$

and $Y_i$ such that

$$(1 + x_1 t + \cdots + x_k t^k)(1 + \xi_1 t + \cdots + \xi_s t^s)(1 + Y_1 t + \cdots + Y_i t^i + \cdots) = 1.$$

It is a DG $(k, k+s)_n$-bimodule quasi-isomorphic to

$$(\Omega_{k,k+1} \otimes_{k+1} \Omega_{k+1,k+2} \otimes_{k+2} \cdots \otimes_{k+n-1} \Omega_{k+n-1,k+n}, d_n).$$

As explained in § 4.2, the nilHecke algebra $\mathrm{NH}_m$ acts on the ring $\Omega_{k,\dots,k+m}$ as endomorphisms of $(\Omega_{k+m}, \Omega_k)$-bimodules. We now show this action extends to the DG context.

PROPOSITION 7.9. *The nilHecke algebra* $\mathrm{NH}_m$ *acts as endomorphisms of the DG* $(k, k+m)_n$*-bimodule* $(\Omega_{k,\dots,k+m}, d_n)$ *and of the DG* $(k+m, k)_n$*-bimodule* $(\Omega_{k+m,\dots,k}, d_n)$*.*

*Proof.* It is sufficient to verify that the action of the $\partial_i$s from $\mathrm{NH}_m$ commute with the differential $d_n$ on $(\Omega_{k,\dots,k+m}, d_n)$ since the action of $x_i$, which comes to multiplying by $\xi_i$, obviously commutes from the definition $d_n(\xi_i) = 0$. The commutation with the action of $\partial_i$ follows from the fact that $X^-$ is a bimodule morphism with parity 0 and that the differential is a bimodule endomorphism. Indeed, suppose $f$ is a polynomial in $x_{i,k}$ and $\underline{\xi}_m$, then $X^-(d_n(f)) = 0 = d_n(X^-(f))$. Suppose now recursively that $f \in \Omega_{k,k+m}$ respects such a relation. From the bimodule structure, we get

$$
\begin{aligned}
d_n(X^-(fs_{i,k+m})) &= d_n(X^-(f)s_{i,k+m}) = d_n(X^-(f))s_{i,k+m} + (-1)^{p(f)}X^-(f)d_n(s_{i,k+m}) \\
&= X^-(d_n(f))s_{i,k+m} + (-1)^{p(f)}X^-(f)Y_{n-i+1}, \\
X^-(d_n(fs_{i,k+m})) &= X^-(d_n(f)s_{i,k+m}) + (-1)^{p(f)}X^-(fd_n(s_{i,k+m})) \\
&= X^-(d_n(f))s_{i,k+m} + (-1)^{p(f)}X^-(f)Y_{n-i+1},
\end{aligned}
$$

which is enough to conclude the proof since we can express every $s_{i,k+j}$ as a combination of $s_{i,k+m}$ and $\xi_{k+i}$. ☐

COROLLARY 7.10. *The nilHecke algebra* $\mathrm{NH}_s$ *acts as endomorphisms of* $\mathsf{E}^s$ *and of* $\mathsf{F}^s$*.*

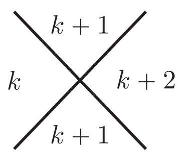
This action coincides with the one from Lauda [**34**] and Chuang–Rouquier [**11**].

## 8. *Verma categorification and a diagrammatic algebra*

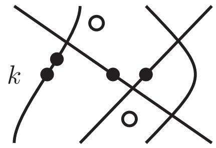
In § 4.2 we have constructed an extended version of the nilHecke algebra which acts on $\Omega_{k,k+n}$. In this section we study this algebra more closely and give it a diagrammatic version in the same spirit as the KLR algebras [**28**, **34**, **46**], as isotopy classes of diagrams modulo relations.

### 8.1. *The superalgebra* $A_n$

Consider the collection of braid-like diagrams on the plane connecting $n$ points on the horizontal axis $\mathbb{R} \times \{0\}$ to $n$ points on the horizontal line $\mathbb{R} \times \{1\}$ with no critical point when projected onto the $y$-axis, such that the strands can never turn around. We allow the strands to intersect each other without triple intersection points. We can decorate the strands with (black) dots and the regions with (white) dots, called floating dots, except the leftmost one. Moreover, we equip the diagram with a height function such that we cannot have two floating dots on the same height. Also, the regions are labeled by integers such that each time we cross a strand from left to right the label increases by1:

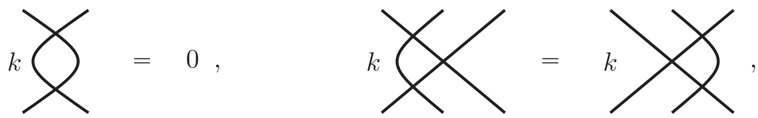


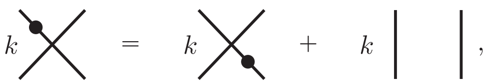

From this rule, it is enough to write a label for the leftmost region of a diagram. Furthermore, these diagrams are taken up to isotopy which does not create any critical point and preserves the relative height of the floating dots as well as the labeling of the regions. An example of such diagram is

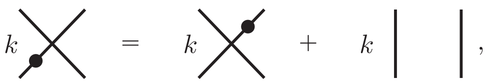


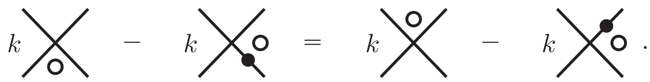
Fix a field $\Bbbk$ and denote by $A_n$ the $\Bbbk$-superalgebra obtained by the linear combinations of these diagrams together with multiplication given by gluing diagrams on top of each other whenever the labels of the regions agree and zero otherwise. In our conventions $ab$ means stacking the diagram for $a$ atop the one for $b$, whenever they are composable. Our diagrams are subjected to the following local relations:

 (33)

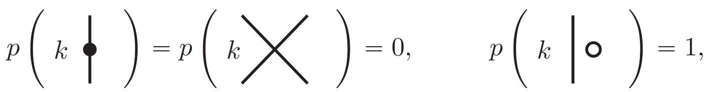
 (34)

 (35)

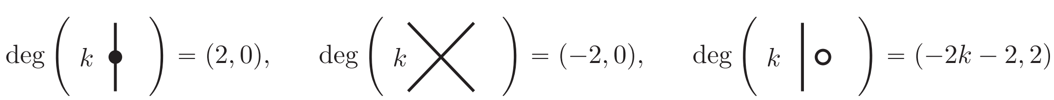
 (36)

 (37)

We turn $A_n$ into bigraded superalgebras by setting the *parity*

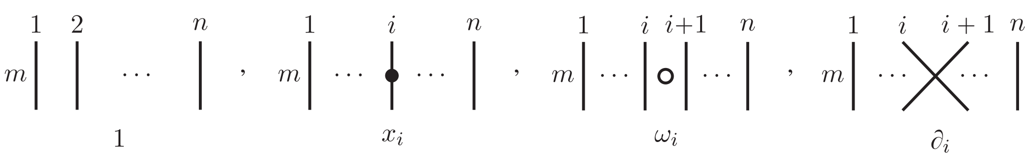


and the $\mathbb{Z} \times \mathbb{Z}$-degrees as



One can check easily that all relations preserve the bidegree and the parity.

We write $A_n(m)$ for the sub-superalgebra consisting of diagrams with a label $m$ on the leftmost region. This superalgebra is generated by the diagrams



There is an obvious canonical inclusion of the nilHecke algebra $\mathrm{NH}_n$ into $A_n(m)$, the former seen as a superalgebra concentrated in parity zero. Note that $A_n(m)$ coincides with the superalgebra introduced in § 4.2.

PROPOSITION 8.1. *The superalgebra $A_n(m)$ admits a basis given by the elements*

$$x_1^{k_1} \dots x_n^{k_n} \omega_1^{\delta_1} \dots \omega_n^{\delta_n} \partial_\vartheta$$

*for all reduced word $\vartheta \in S_n$, $k_i \in \mathbb{N}$ and $\delta_i \in \{0,1\}$, with*

$$\partial_\vartheta = \partial_{i_1} \dots \partial_{i_r}$$

*for $\vartheta = \tau_{i_1} \dots \tau_{i_r}$, $\tau_i$ being the transposition exchanging $i$ with $i+1$.*

*Proof.* Using the relations (35)–(37) we can push all dots and floating dots to the top of the diagrams. By the relations (33), we can have a maximum of one floating dot on the immediate right of a strand, on the top of each diagram. By relations (34) we get the decomposition in $\partial_w$ as in the nilHecke algebra (see [**28**, § 2.3], for example). Thus the family above is generating. The action described in the section below shows easily they act as linearly independent operators, concluding the proof. □

## 8.2. *The action of $A_n(m)$ on polynomial rings*

The superalgebra $A_n(m)$ acts on $\mathbb{Q}[\underline{x}_n] \otimes \bigwedge^\bullet(\underline{\omega}_n)$ with $x_i$ and $\omega_i$ acting by left multiplication while the action of $\partial_i$ is defined by

$$\partial_i(1) = 0, \quad \partial_i(x_j) = \begin{cases} 1 & \text{if } j = i, \\ -1 & \text{if } j = i+1, \\ 0 & \text{otherwise}, \end{cases} \quad \partial_i(\omega_j) = \begin{cases} -\omega_{j+1} & \text{if } j = i, \\ 0 & \text{otherwise}, \end{cases}$$

together with the rule

$$\partial_i(fg) = \partial_i(f)g + f\partial_i(g) - (x_i - x_{i+1})\partial_i(f)\partial_i(g), \tag{38}$$

for all $f, g \in \mathbb{Q}[\underline{x}_n] \otimes \bigwedge^\bullet(\underline{\omega}_n)$.

PROPOSITION 8.2. *Formulas above define an action of $A_n(m)$ on $\mathbb{Q}[\underline{x}_n] \otimes \bigwedge^\bullet(\underline{\omega}_n)$.*

*Proof.* The commutation relations produced by isotopies are immediate from $\partial_i$ being zero on $x_j$ and $\omega_j$ for $j \notin \{i, i+1\}$ together with the rule (38). The relation $\partial_i^2 = 0$ can easily be proved by induction. It is straightforward on 1 from the definition. Let $f, g \in \mathbb{Q}[\underline{x}_n] \otimes \bigwedge^\bullet(\underline{\omega}_n)$ be such that $\partial_i^2(f) = \partial_i^2(g) = 0$ and compute

$$\partial_i^2(fg) \overset{(38)}{=} \partial_i\left(\partial_i(f)g + f\partial_i(g) - (x_i - x_{i+1})\partial_i(f)\partial_i(g)\right) \overset{(38)}{=} 0.$$

The Reidemeister III relation (34) is proved in a same way, we leave the details for the reader. The nilHecke relations (35) and (36) are easy computations. Indeed we have

$$\begin{aligned} \partial_i(x_{i+1}f) + f &\overset{(38)}{=} -f + x_{i+1}\partial_i(f) + (x_i - x_{i+1})\partial_i(f) \\ &= x_i\partial_i(f), \\ \partial_i(x_i f) &\overset{(38)}{=} f + x_i\partial_i(f) - (x_i - x_{i+1})\partial_i(f) \\ &= x_{i+1}\partial_i(f) + f, \end{aligned}$$

for all $f \in \mathbb{Q}[\underline{x}_n] \otimes \bigwedge^\bullet(\underline{\omega}_n)$. The second to last relation is direct from $\partial_i$ acting as zero on $\omega_{i+1}$. Finally, computing

$$\begin{aligned}\partial_i(\omega_i f) &\overset{(38)}{=} -\omega_{i+1} f + \omega_i \partial_i(f) + (x_i - x_{i+1})\omega_{i+1}\partial_i(f) \\ &\overset{(35)}{=} \omega_i \partial_i(f) + \partial_i(x_{i+1}\omega_{i+1} f) - \omega_{i+1} x_{i+1} \partial_i(f)\end{aligned}$$

gives (37). □

### 8.3. *Symmetric group action*

The action of $\mathrm{NH}_n$ on $\mathbb{Q}[\underline{x}_n]$ with $x_i$ acting by multiplication and $\partial_i$ by divided difference operators

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}},$$

induces an action of $\mathrm{NH}_n$ on $\mathbb{Q}[\underline{x}_n] \otimes \bigwedge^\bullet(\underline{\omega}_n)$. This action goes through an action of the symmetric group $S_n$ on $\mathbb{Q}[\underline{x}_n] \otimes \bigwedge^\bullet(\underline{\omega}_n)$, given by

$$s_i(x_j) = \begin{cases} x_{i+1} & \text{if } j = i, \\ x_i & \text{if } j = i+1, \\ x_j & \text{otherwise,} \end{cases} \tag{39}$$

and

$$s_i(\omega_j) = \begin{cases} \omega_i + (x_i - x_{i+1})\omega_{i+1} & \text{if } j = i, \\ \omega_j & \text{otherwise,} \end{cases} \tag{40}$$

and $s_i(fg) = s_i(f)s_i(g)$.

With this $S_n$-action, the action of $\partial_i$ satisfies the usual Leibnitz rule for the Demazure operator, $\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g)$. Moreover, this action coincides with the one defined before since

$$\begin{aligned}\partial_i(f)g + f\partial_i(g) - (x_i - x_{i+1})\partial_i(f)\partial_i(g) &= \frac{(f - s_i(f))g + f(g - s_i(g)) - (f - s_i(f))(g - s_i(g))}{x_i - x_{i+1}} \\ &= \frac{fg - s_i(fg)}{x_i - x_{i+1}}.\end{aligned}$$

### 8.4. *The action of $A_n(k+m)$ on $\Omega^m_{k,k+n}$.*

We have already observed in § 6.2 that the ring $\Omega_{m,m+n}$ can be written as

$$\Omega_{m,m+n} \cong \Omega_{m,m+1} \otimes_{m+1} \dots \otimes_{m+n-1} \Omega_{m+n-1,m+n}.$$

We therefore get an action of $A_n(m)$ on $\Omega_{m,m+n}$ given by

$m+i$  $\mapsto \xi_{i+1}$, $m+i$  $\mapsto s_{i+1,i+1}$,

$m+i$  $\mapsto \left(X^- : \Omega_{i,i+1} \otimes \Omega_{i+1,i+2} \to \Omega_{i,i+1} \otimes \Omega_{i+1,i+2}\right).$

This action agrees with the one defined in § 8.2 and this can be generalized to get an action of $A_n(k+m)$ on $\Omega^m_{k,k+n}$.

8.5. *Categorification*

We define

$$A(m) = \bigoplus_{n\geqslant 0} A_n(m).$$

The usual inclusion $A_n(m) \hookrightarrow A_{n+1}(m)$ that adds a strand at the right of a diagram from $A_n(m)$ gives rise to induction and restriction functors $\mathsf{F}$ and $\mathsf{E}$ on $A(m)\text{-smod}_{\text{lfg}}$ that satisfy the $\mathfrak{sl}_2$-relations. Our results in §§ 4–6 imply that $A_n(m)\text{-smod}_{\text{lfg}}$ categorifies the $(\lambda q^{m-1-2n})$-weight space of $M(\lambda q^{m-1})$, and that $A(m)\text{-smod}_{\text{lfg}}$ categorifies the Verma module $\mathcal{M}(\lambda q^{m-1})$. This is explained in details in [**41**]. The categorification of the Verma modules with integral highest weight using specializations of the superalgebras $A_n(m)$ follows as a consequence of our results in § 6.

8.6. *Cyclotomic quotients*

We can turn $A_n(N+1)$ into a DG-algebra, equipping it with a differential of degree $(0,-2)$ defined by

$$d_N(x_i) = 0, \quad d_N(\partial_i) = 0, \quad d_N(\omega_i) = (-1)^i h_{N-i+1}(\underline{x}_i),$$

together with the parity graded Leibniz rule (as before, the parity is the cohomological degree of $d_n$). We stress that we take the complete homogeneous symmetric polynomial on only the first $i$ variables, and therefore it commutes with $\partial_j$ for all $j \neq i$ and respects (37) for $\partial_i$ (recall that $Y_{i,k} = (-1)^i h_i(\xi_k)$).

PROPOSITION 8.3. *The DG-algebra $(A_n(N+1), d_N)$ is quasi-isomorphic to the cyclotomic quotient of the nilHecke algebra $\mathrm{NH}_n^N = \mathrm{NH}_n / (x_1^N)$,*

$$(A_n(N+1), d_N) \cong \mathrm{NH}_n^N .$$

*Proof.* It suffices to see that $d_N(\omega_1) = h_N(x_1) = x_1^N$ since $d_N(\omega_i)$ lies in the ideal generated by $x_1^N$ (see, for example, [**17**, Proposition 2.8]). □

## Appendix. *Topological Grothendieck groups*

Recall that the Grothendieck group (respectively, split Grothendieck group), denoted $G_0$ (respectively, $K_0$), is defined in general for abelian (respectively, additive) categories as the free group generated by the classes of objects up to isomorphism quotiented by

$$[B] = [A] + [C],$$

whenever there exists a short exact sequence $0 \to A \to B \to C \to 0$ (respectively, an isomorphism $B \cong A \oplus C$). We call *distinguished triplet* $(A, B, C)$ these short exact sequences and decompositions into direct sums. In the case of abelian categories with finite length objects (respectively, Krull–Schmidt categories), they are given by the free $\mathbb{Z}$-module generated by the classes of simple objects (respectively, indecomposable objects).

If $\mathcal{C}$ has a (not necessarily strong) $\mathbb{Z}/2\mathbb{Z}$-action $\Pi : \mathcal{C} \to \mathcal{C}$, then $G_0(\mathcal{C})$ and $K_0(\mathcal{C})$ become modules over $\mathbb{Z}_\pi = \mathbb{Z}[\pi]/(\pi^2 - 1)$ with $\pi[M] = [\Pi M]$. When the category is *strictly $\mathbb{Z}$-graded*, namely $A\langle k\rangle \ncong A$ for all $A \in \mathcal{C}$ and $k \in \mathbb{Z}_0$ , then $K_0(\mathcal{C})$ (respectively, $G_0(\mathcal{C})$) becomes as $\mathbb{Z}[q, q^{-1}]$-module freely generated by the classes of indecomposable objects (respectively, simple objects), up to shift. The action is given by a shift in the degree

$$q[M] = [M\langle 1\rangle], \quad q^{-1}[M] = [M\langle -1\rangle].$$

This means the action of a polynomial $p \in \mathbb{Z}[q, q^{-1}]$ can be viewed as (cf. § 3.3)

$$p(q, q^{-1})[M] = \left[M^{\oplus p}\right].$$

This story generalizes in the obvious way to the case of multigraded categories.

We look for similar results when working with objects admitting infinite filtrations or that decompose into infinitely many indecomposables, such that the Grothendieck groups become modules over $\mathbb{Z}[\![q]\!][q^{-1}]$. This will allow us making sense of expressions like

$$[X] = (1 + q^2 + q^4 + \cdots + q^{2i} + \cdots)[S] = \frac{1}{1-q^2}[S],$$

for some objects $X$ and $S$ such that $(X\langle 2\rangle, X, S)$ is a distinguished triplet. In general this procedure fails and one can see easily that the Grothendieck group collapses using Eilenberg swindle arguments. To avoid this outcome in our construction, we work with categories where these decompositions and filtrations are controlled and essentially unique.

Most of the arguments are sensibly similar to the ones used in the finite case, which can be found, for example, in [**39**, Appendix].

A.1. *Topological split Grothendieck group $K_0$.* The aim of this section is to define a notion of Krull–Schmidt categories admitting infinite decompositions. For the split Grothendieck group not to collapse, we need to control the occurrences of the indecomposables in these decompositions and they should be essentially unique. That is for every other possible decomposition, the indecomposables are in bijection and have the same grading. Since we are working in a graded context, we require that in each decomposition the indecomposables are in a finite number in each degree and the degrees are bounded from below.

Definition A.1. We say that a coproduct in a strictly $\mathbb{Z}$-graded category $\mathcal{C}$ is *locally finite* if it is finite in each degree. By this we mean the coproduct is of the form

$$\coprod_{i\in\mathbb{Z}} (A_1^{\oplus k_{1,i}} \oplus \cdots \oplus A_n^{\oplus k_{n,i}})\langle i\rangle,$$

for some $A_1, \dots, A_n \in \mathcal{C}$ and $k_{j,i} \in \mathbb{N}$. Moreover, we say that it is *left-bounded*, or *bounded from below*, if there is some $m \in \mathbb{Z}$ such that $k_{j,i} = 0$ for all $i < m$.

An additive category admits all finite products and coproducts, and those are equivalent and called biproducts. In the same spirit, we define the stronger notion of right complete locally additive category.

Definition A.2. We say that an additive, strictly $\mathbb{Z}$-graded category $\mathcal{C}$ is *locally additive* if all its locally finite coproducts are biproducts, that is, they are isomorphic to their product counterparts. We write them with a $\bigoplus$ sign and sometimes call them direct sums. Moreover, we say that $\mathcal{C}$ is *right complete* if it admits all left-bounded locally finite coproducts.

We illustrate this notions in the working example below, that will be developed further throughout this section.

Example A.3. Let $R$ be a unital graded $\Bbbk$-algebra, with $\Bbbk$ being a field. Suppose $R = \bigoplus_{i\geqslant 0} R_i$ is locally finite-dimensional with positive dimension and $R_0 = \Bbbk$. We call *locally finitely generated $R$-module* a graded $R$-module $M$ that can be written as $M = \bigoplus_{i\in I} Rx_i$ with $x_i \in M$ and $X = \{x_i\}_{i\in I}$ is finite in each degree.

It is *left-bounded* if there is some $m \in \mathbb{Z}$ such that $\deg(x_i) > m$ for all $i \in I$. The category $R\text{-mod}_{\mathrm{lfg}}$ of left-bounded locally finitely generated $R$-modules with degree *zero* morphisms is right complete and locally additive.

We clearly have all left-bounded locally finite coproducts. We show that they are biproducts. Let

$$M = \bigoplus_{i>m} (A_1^{\oplus k_{1,i}} \oplus \cdots \oplus A_n^{\oplus k_{n,i}})\langle i \rangle \in R\text{-mod}_{\mathrm{lfg}}$$

be a coproduct and $Z$ be an object in $R\text{-mod}_{\mathrm{lfg}}$ with morphisms $f_{j,i} : Z \to A_j^{k_{j,i}}$ (note that $A_j^{\oplus k_{j,i}}$ are biproducts). If $i$ is bounded, then it is a finite coproduct and thus a biproduct, so we suppose without losing generality that for all $m \in \mathbb{Z}$ there exist $i, j \in \mathbb{Z}$ with $i > m$ such that $k_{j,i} > 0$. By the universal property of the direct product, there is a canonical map (in the category of all $R$-modules, but not in $R\text{-mod}_{\mathrm{lfg}}$)

$$r : M \to \prod_{i\in\mathbb{Z}} (A_1^{\oplus k_{1,i}} \oplus \cdots \oplus A_n^{\oplus k_{n,i}})\langle i \rangle.$$

We want to show that the map of $R$-modules

$$\prod_{i,j} f_{j,i} : Z \to \prod_{i\in\mathbb{Z}} (A_1^{\oplus k_{1,i}} \oplus \cdots \oplus A_n^{\oplus k_{n,i}})\langle i \rangle$$

factors through $r$. This is equivalent to show that for homogeneous $x \in Z$ fixed, $f_{i,j}(x) = 0$ for almost all $i, j$. Suppose this does not hold. Then there is a $j \in \mathbb{Z}$ such that for each $m \in \mathbb{Z}$ there exist some $i \in \mathbb{Z}$ with $i > m$ and $f_{i,j}(x) \neq 0$. Thus $\deg(f_{j,i}(x)) = \deg(x) - i$. But $f_{j,i}(x)$ is an homogeneous element of $A_j$ which is left-bounded and that is absurd. By construction $\prod_{i,j} f_{j,i}$ is the unique morphism satisfying the universal property of the product and thus $M$ is a biproduct.

Remark A.4. In a locally additive category, there is a canonical bijection

$$\mathrm{Hom}\left(X, \bigoplus_{k\in I} Y_k\right) \cong \prod_{k\in I} \mathrm{Hom}(X, Y_k).$$

Definition A.5. An object $A$ in a category $\mathcal{C}$ is *small* if every map $f : A \to \coprod_{i\subset I} B_i$ factors through $\coprod_{j\in J} B_j$ for a finite subset $J \subset I$.

Example A.6. In a category of modules, finitely generated modules are small [**48**, § 2].

Definition A.7. We say that a locally additive category is *locally Krull–Schmidt* if every object decomposes into a locally finite direct sum of small objects having local endomorphisms rings.

Remark A.8. Note that a locally Krull–Schmidt category must be idempotent complete. Moreover, an object with local endomorphism ring must be indecomposable, and has only 0 and 1 as idempotents.

It appears the condition of being small allows us to mimic the classical proof of the Krull–Schmidt property of a Krull–Schmidt category. There exists some other results about Krull–Schmidt properties for infinite decompositions [**48**], where the indecomposables are not necessarily small. However they require the category to admit kernels, which we do not have in our construction. For example, we want to use the category of projective modules which certainly does not admit all kernels.

Lemma A.9 [**3**, Lemma 3.3, p. 18]. *In an additive category $\mathcal{C}$, for all $A, B$ and $C$ in $\mathcal{C}$, if $A$ is indecomposable with local endomorphism ring, then*

$$A \oplus B \xrightarrow[\cong]{f=\begin{pmatrix} f_{AA} & f_{AB} \\ f_{CA} & f_{CB} \end{pmatrix}} A \oplus C$$

*with $f_{AA}$ being an unit, implies that $B \cong C$.*

We now prove that each object in a locally Krull–Schmidt category decomposes into an essentially unique direct sum of indecomposables. The idea of the proof is essentially the same as for the classical Krull–Schmidt theorem (see, for example, [**39**, Theorem A6]), with only the smallness property of the indecomposable objects allowing us to restrict the infinite sums of morphisms into finite ones such that we can extract units from them. Also the locally finiteness of the direct sums allows us to use inductive arguments.

Theorem A.10. *In a locally Krull–Schmidt category, given an isomorphism*

$$\bigoplus_{i\in\mathbb{Z}} (A_1^{\oplus k_{1,i}} \oplus \cdots \oplus A_n^{\oplus k_{n,i}})\langle i\rangle \cong \bigoplus_{i\in\mathbb{Z}} (B_1^{\oplus k'_{1,i}} \oplus \cdots \oplus B_m^{\oplus k'_{m,i}})\langle i\rangle,$$

*where the objects $A_j$ are indecomposables with $A_j \ncong A_{j'}\langle i\rangle$ for all $j \neq j'$ and $i \in \mathbb{Z}$, and the same for the objects $B_j$ , then $m = n$, $A_s \cong B_{j_s}\langle \alpha_s\rangle$ for all $s$, and $k_{s,i} = k'_{j_s,i+\alpha_s}$, with $j_s \neq j_{s'}$ if $s \neq s'$.*

*Proof.* Denote $M = \bigoplus_{i\in\mathbb{Z}} (A_1^{\oplus k_{1,i}} \oplus \cdots \oplus A_n^{\oplus k_{n,i}})\langle i\rangle$. Let

$$f_{j,i,k} : B_j\langle i\rangle \to M, \quad q_{j,i,k} : M \to B_j\langle i\rangle,$$

be the injection and projection morphisms given by the biproduct structure, with $1 \leqslant k \leqslant k'_{j,i}$. Fix $i_0 \in \mathbb{Z}$. We have $\mathrm{Id}_M = \prod_{j,i,k} f_{j,i,k} q_{j,i,k}$. Thus $\mathrm{Id}_{A_1} = \pi(\prod_{j,i,k} f_{j,i,k} q_{j,i,k})\imath \in \mathrm{End}(A_1\langle i_0\rangle)$ for each copy of $A_1\langle i_0\rangle$ in $A_1^{\oplus k_{1,i_0}}\langle i_0\rangle$, with $\pi$ and $\imath$ the projection and inclusion of $A_1\langle i_0\rangle$ in $M$. Since $A_1$ is a small object, we can restrict this sum to a finite one. Thus we can write $\pi(\prod_{j,i,k} f_{j,i,k} q_{j,i,k})\imath$ as a finite sum over $j$ and $k$. By local property of $\mathrm{End}(A_1\langle i_0\rangle)$, there exists $j, i, k \in \mathbb{Z}$ such that $x = \pi f q \imath$ is a unit, with $f = f_{j,i,k}$ and $q = q_{j,i,k}$. Take $q\imath x^{-1}\pi f \in \mathrm{End}(B_j\langle i\rangle)$ which is an idempotent, and thus is 0 or $\mathrm{Id}_{B_j\langle i_1\rangle}$. Since it factors through $\mathrm{Id}_{A_1\langle i_0\rangle}$, it cannot be zero. Thus $q\imath x^{-1}$ is an isomorphism with inverse $\pi f$ such that $A_1\langle i_0\rangle \cong B_j\langle i\rangle$, hence $A_1 \cong B_j\langle i - i_0\rangle$. We apply the same reasoning for each $A_j\langle i\rangle$. Since the argument can be applied for the objects $B_j$ as well, we get $n = m$ and $A_s \cong B_{j_s}\langle \alpha_s\rangle$ for some $\alpha_s \in \mathbb{Z}$. Now using Lemma A.9 to cancel each $A_s\langle i\rangle$ with $B_{j_s}\langle i + \alpha_s\rangle$ we conclude that $k_{s,i} = k'_{j_s,i+\alpha_s}$. ☐

Corollary A.11. *A locally Krull–Schmidt category $\mathcal{C}$ possesses the cancelation property for direct sums, namely for all $A, B$ and $C$ in $\mathcal{C}$, $A \oplus B \cong A \oplus C$ implies $B \cong C$.*

Let $K'_0(\mathcal{C})$ be the free $\mathbb{Z}[\![q]\!][q^{-1}]$-module generated by the classes of indecomposable objects in $\mathcal{C}$, up to shift. We equip it with the $(q)$-adic topology. The (usual) split Grothendieck group of a right complete locally Krull–Schmidt category $\mathcal{C}$ has a canonical structure of $\mathbb{Z}[\![q]\!][q^{-1}]$-module with action of a series $p(q, q^{-1}) = \sum_{i>m} k_i q^i$ given by

$$p(q, q^{-1})[M] = \left[M^{\oplus p}\right].$$

Since each object $M \in \mathcal{C}$ admits an essentially unique decomposition into indecomposable objects $M \cong \bigoplus_{i>m}(A_1^{\oplus k_{1,i}} \oplus \cdots \oplus A_n^{\oplus k_{n,i}})\langle i\rangle$ we get a canonical surjective $\mathbb{Z}[\![q]\!][q^{-1}]$-module map

$$f : K_0(\mathcal{C}) \twoheadrightarrow K_0'(\mathcal{C}).$$

This induces a topology on $K_0(\mathcal{C})$, which in general is not Hausdorff as we can have

$$0 \neq [M] - \sum_{i>m} k_{1,i}q^i[A_1] + \cdots + k_{n,i}q^i[A_n] \in \bigcap_{n\geqslant 0} f^{-1}((q)^n).$$

Definition A.12. We define the *topological split Grothendieck group* as

$$\boldsymbol{K}_0(\mathcal{C}) = K_0(\mathcal{C})/\cap_{n\geqslant 0} f^{-1}((q)^n) = K_0(\mathcal{C})/\ker f.$$

The topological split Grothendieck group possesses a canonical $(q)$-adic topology given by the quotient topology, making it a topological module over the topological ring $\mathbb{Z}[\![q]\!][q^{-1}]$. From the definition we see that we have an homeomorphism $\boldsymbol{K}_0(\mathcal{C}) \cong K_0'(\mathcal{C})$. Therefore, we have the following theorem.

Theorem A.13. *The topological split Grothendieck group of a right complete locally Krull–Schmidt category equipped with the* $(q)$*-adic topology is a free* $\mathbb{Z}[\![q]\!][q^{-1}]$*-module generated by the classes of indecomposables* (*up to shifts*).

Example A.14. The category $R\text{-mod}_{\mathrm{lfg}}$ from Example A.3 and its subcategory given by the projective modules are both right complete locally Krull–Schmidt.

A.2. *Grothendieck group* $G_0$. Recall that an object $X$ in an abelian category $\mathcal{C}$ has finite length if there exists a finite filtration, called a composition series,

$$X = X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n = 0,$$

where each $X_i/X_{i+1}$ is a simple (non-zero) object. If it exists, it is unique up to permutation thanks to the Jordan–Hölder theorem.

In general this result does not hold for infinite filtrations. In this section, we present some conditions that are sufficient to have uniqueness of such filtrations in a non-artinian category.

Definition A.15. Let $X$ be an object in an abelian category $\mathcal{C}$. A $\mathbb{Z}$*-filtration* is a sequence of subobjects $X_i \subset X$ indexed by $i \in \mathbb{Z}$ such that $X_{i+1} \subset X_i$ and $X_0 = X$ or $X_0 = 0$. We can write this as

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \quad \text{or} \quad X \leftarrow \cdots \leftarrow X_{-2} \leftarrow X_{-1} \leftarrow X_0 = 0.$$

Such a filtration is called *exhaustive* if the direct limit $\varinjlim_i X_i \cong X$, and *Hausdorff* if the inverse limit $\varprojlim_i X_i \cong 0$. We say that it has *simple quotients* if all quotients $X_i/X_{i+1}$ are either 0 or simple.

As for the coproducts in strictly $\mathbb{Z}$-graded category, we can define a notion of locally finiteness for the $\mathbb{Z}$-filtrations.

Definition A.16. If $\mathcal{C}$ is strictly $\mathbb{Z}$-graded, we say that a $\mathbb{Z}$-filtration is *locally finite* if there is some finite set $\{S_j\}_{j\in J}$ of objects in $\mathcal{C}$ such that for all $i \in \mathbb{Z}$

$$X_i/X_{i+1} \cong S_{j_i}\{p_i\}, \quad \text{or} \quad X_i/X_{i+1} \cong 0,$$

for some $j_i, p_i \in \mathbb{Z}$, and for each $S_j\{p\}$ there is a finite number $k_{j,p}$ of such $i$:

$$k_{j,p} = \#\{i \in \mathbb{Z} | X_i/X_{i+1} \cong S_j\langle p\rangle\} \in \mathbb{N}.$$

We call $k_{j,p}$ the degree $p$ multiplicity of $S_j$ in the filtration. We say the filtration is *left-bounded* if there exists $m \in \mathbb{Z}$ such that $k_{j,p} = 0$ for all $p < m$ and $j \in J$.

The infinite counterpart of a composition series for a $\mathbb{Z}$-filtration we choose is defined as the following.

Definition A.17. We say an object $X \in \mathcal{C}$ has a $\mathbb{Z}$-*composition series* if it admits a locally finite, exhaustive, Hausdorff, $\mathbb{Z}$-filtration with simple quotients.

Example A.18. Let $R = \mathbb{Q}[x]$ with $\deg(x) = 2$ and let $S = R/Rx$ be a simple object in the category of (graded) $R$-modules with degree zero morphisms. Let $M$ be a module isomorphic to $R$. Then $M$ admits a $\mathbb{Z}$-composition series

$$M \cong R \leftarrow Rx \leftarrow Rx^2 \leftarrow \cdots$$

with $R/Rx \cong S$, $Rx/Rx^2 \cong S\langle 2\rangle$, etc. Note that in general, for a module category and the $A_i \subset X$ being submodules, then $\varprojlim_i A_i = \bigcap_i A_i \subset X$ and $\varinjlim_i A_i = \bigcup_i A_i \subset X$, and thus $\varprojlim_i Rx^i = \bigcap_i Rx^i = 0$. Another example is $R = \mathbb{Q}[x, y]$, $\deg(x) = n$, $\deg(y) = m$ which has $\mathbb{Z}$-composition series given by 'aliased diagonals' in $\mathbb{N}^2$.

Example A.19. More generally, for $R$ a positively graded $\Bbbk$-algebra having locally finite graded dimension as a $\Bbbk$-vector space, one can define a $\mathbb{Z}$-composition series for $R$ viewed as a module over itself. Indeed we can write $R = \bigoplus_{i\geqslant 0} R^i$, with each $R^i = \bigoplus_{j=1}^{n_i} \Bbbk v_j^i$ being finite-dimensional vector space in degree $i$ and $R^0 \cong \Bbbk$. Then we get a filtration

$$R \leftarrow \bigoplus_{i\geqslant 1} R^i \leftarrow \bigoplus_{i\geqslant 2} R^i \leftarrow \cdots \leftarrow 0,$$

that can be refined into a filtration with simple quotients if we insert

$$\leftarrow \bigoplus_{i\geqslant k+1} R^i \oplus \bigoplus_{j=1}^{n_i-1} \Bbbk v_j^k \leftarrow \bigoplus_{i\geqslant k+1} R^i \oplus \bigoplus_{j=1}^{n_i-2} \Bbbk v_j^k \leftarrow \cdots \leftarrow \bigoplus_{i\geqslant k+1} R^i \oplus \Bbbk v_1^k \leftarrow$$

between each $\bigoplus_{i\geqslant k} R^i \leftarrow \bigoplus_{i\geqslant k+1} R^i$. The simple quotients are given by $\{S_0\}$, with $S_0 \cong \Bbbk$, and multiplicities $k_{0,i} = n_i$.

As a composition series of a finite-length object is essentially unique, we want to establish that up to some mild hypothesis a $\mathbb{Z}$-composition series is also essentially unique. One possible choice of such hypothesis is given by the following.

Definition A.20. An object $X$ with a $\mathbb{Z}$-composition series is said to be *stable for the filtrations* if

- for all pair of Hausdorff filtrations $X \leftarrow A_1 \leftarrow \cdots \leftarrow 0$ and $X \leftarrow B_1 \leftarrow \cdots \leftarrow 0$, for each $i \geqslant 0$ there exists $k$ such that $B_k \subset A_i$;
- for all pair of exhaustive filtrations $X \leftarrow \cdots \leftarrow A_{-1} \leftarrow 0$ and $X \leftarrow \cdots \leftarrow B_{-1} \leftarrow 0$, for each $i \leqslant 0$ there exists $k$ such that $B_i \subset A_k$.

Example A.21. The rings and modules from the previous examples are stable for the filtrations. In general, a $\Bbbk$-algebra as $R$ in Example A.19, viewed as module over itself, is stable for the filtrations, and also are its left-bounded locally finite-dimensional modules. This comes from the fact that such modules are $\Bbbk$-vector spaces, and therefore a filtration of modules yields a filtration of subspaces. For example, suppose $B_k \not\subset A_i$ for all $k$. Then $A_i \subsetneq A_i \cup B_k$. In addition $X/X_i$ must be a finite-dimensional vector space and we get an infinite filtration

$$X \leftarrow B_1 + A_i \leftarrow B_2 + A_i \leftarrow \cdots \leftarrow A_i.$$

Since, as vector spaces, $B_j = B_{j+1} \bigoplus H_j$ for some $H_j$, and $H_j \not\subset A_i$ for arbitrary large $j$ (if not, we would have a $B_k \subset A_i$ since $B_k \bigoplus_{j \geqslant k} H_k$). It means that $X = A_i \bigoplus_j H_j$ where $j$ runs over $\{j \in \mathbb{Z} | H_j \not\subset A_i\}$, which contradicts the fact that $X/X_i$ is finite-dimensional.

Example A.22. The ring $\mathbb{Z}$ view as a module over itself is not stable for the filtrations. Consider a sequence of non-equal prime numbers $p_0, p_1, p_2, \ldots$ and the following Hausdorff filtrations:

$$\mathbb{Z} \leftarrow p_0\mathbb{Z} \leftarrow p_0p_2\mathbb{Z} \leftarrow \cdots \leftarrow p_0 \ldots p_{2k}\mathbb{Z} \leftarrow \cdots \leftarrow 0,$$

$$\mathbb{Z} \leftarrow p_1\mathbb{Z} \leftarrow p_1p_3\mathbb{Z} \leftarrow \cdots \leftarrow p_1 \ldots p_{2k+1}\mathbb{Z} \leftarrow \cdots \leftarrow 0.$$

It is clear that $p_1 \ldots p_{2k+1}\mathbb{Z} \not\subset p_0\mathbb{Z}$ for all $k$ and thus $\mathbb{Z}$ is not stable for the filtrations. As a matter of fact, the filtrations have simple quotients but are not equivalent. Indeed the quotients are, respectively, given by $\mathbb{Z}/p_{2k}\mathbb{Z}$ and $\mathbb{Z}/p_{2k+1}\mathbb{Z}$.

We now proceed to prove that all $\mathbb{Z}$-composition series of an object $X$ which is stable for the filtrations are essentially the same. Since a $\mathbb{Z}$-filtration can take two forms, reaching 0 or $X$, there are three cases to consider. But first we introduce some useful lemmas.

Lemma A.23. *Suppose there is an object $X$ with two subobjects $M$ and $N$ such that $M \neq N$, and $X/M$ and $X/N$ are simple, then*

$$M/(M \cap N) \cong X/N.$$

*Proof.* $X = M \cup N$ and thus $X/N = (M \cup N)/N \cong M/(M \cap N)$ by the second isomorphism theorem. □

Lemma A.24. *Let $M$ and $A$ be subobjects of $X$ such that $A$ admits a $\mathbb{Z}$-composition series*

$$A = A_0 \leftarrow A_1 \leftarrow \cdots \leftarrow 0, \quad or \quad A \leftarrow \cdots \leftarrow A_{-1} \leftarrow A_0 = 0.$$

*Then we get a $\mathbb{Z}$-composition series*

$$A \cap M \leftarrow A_1 \cap M \leftarrow \cdots \leftarrow 0, \quad or \quad A \cap M \leftarrow \cdots \leftarrow A_{-1} \cap M \leftarrow 0.$$

*Proof.* For all $j$ we have

$$\frac{A_j \cap M}{A_{j+1} \cap M} \cong \frac{A_j \cap M}{A_{j+1} \cap (A_j \cap M)} \cong \frac{A_{j+1} \cup (A_j \cap M)}{A_{j+1}}.$$

If $A_j \cap M \subset A_{j+1}$ then we get 0. If not we have $A_{j+1} \subsetneq A_{j+1} \cup (A_j \cap M) \subset A_j$, thus $A_j \cong A_{j+1} \cup (A_j \cap M)$. Therefore $(A_j \cap M)/(A_{j+1} \cap M) \cong (A_j)/(A_{j+1})$ and this concludes the proof. □

We begin with the case where the two filtrations reach $X$.

Proposition A.25. *Let $X$ be a stable for the filtrations object in a strictly $\mathbb{Z}$-graded abelian category. If $X$ admits two $\mathbb{Z}$-composition series of the following form:*

$$X \leftarrow A_1 \leftarrow A_2 \leftarrow \cdots \leftarrow 0,$$

$$X \leftarrow B_1 \leftarrow B_2 \leftarrow \cdots \leftarrow 0,$$

*with respective multiplicities $k_{j,p}$, $k'_{j,p}$ and simple quotients $\{S_j\}_{j\in J}, \{S'_j\}_{j\in J'}$, then for each $s$, $S_s \cong S'_{j_s}\{\alpha_s\}$ for some $j_s, \alpha_s$ and $k_{s,p} = k'_{j_s,p+\alpha_s}$ for all $p \in \mathbb{Z}$. In other words, the $\mathbb{Z}$-composition series have the same quotients and multiplicities.*

*Proof.* Fix $s$ and $p$. The finiteness condition implies that we can reach all $A_i$ such that $A_i/A_{i+1} \cong S_s\{p\}$ in a finite number of steps. So we can take some minimal subfiltration

$$X \leftarrow A_1 \leftarrow A_2 \leftarrow \cdots \leftarrow A_n$$

containing all simple quotients $S_s\{p\}$. We prove by induction on $n$ that $k'_{j_s,p+\alpha_s} \geqslant k_{s,p}$ for some $j_s, \alpha_s$ such that $S_s \cong S'_{j_s}\{\alpha_s\}$, and by symmetry of the argument we get the equality.

By the hypothesis on $X$ there exists some $r$ such that $B_r \subset A_1$. We can suppose $r$ minimal, such that $B_{r-1} \not\subset A_1$. Consider the filtration

$$A_1 \leftarrow A_1 \cap B_1 \leftarrow A_1 \cap B_2 \leftarrow \cdots \leftarrow A_1 \cap B_{r-1} \leftarrow B_r \leftarrow B_{r+1} \leftarrow \cdots \leftarrow 0.$$

We claim that $B_{r-1}/B_r \cong X/A_1$ and that the filtration is in fact a $\mathbb{Z}$-composition series with quotients given by

$$X/B_1, B_1/B_2, \dots, B_{r-2}/B_{r-1}, 0, B_r/B_{r+1}, \dots,$$

and thus can be rewritten as

$$A_1 \leftarrow A_1 \cap B_1 \leftarrow A_1 \cap B_2 \leftarrow \cdots \leftarrow A_1 \cap B_{r-1} \leftarrow B_{r+1} \leftarrow \cdots \leftarrow 0.$$

Since this filtration has the same quotients as the one given by the $B_i$ at the exception of $B_r/B_{r+1} \cong X/A_1$, we can apply the recursion on $X' = A_1$ with the filtration above together with

$$A_1 \leftarrow \cdots \leftarrow A_n \leftarrow \cdots \leftarrow 0.$$

We now prove our claim. First observe that if $A_1 \cap B_{r-1} \neq B_r$ then $B_r \subset A_1 \cap B_{r-1} \subset B_{r-1}$ are strict inclusions and thus $B_{r-1}/B_r$ would not be simple, which is absurd. So we get

$$\frac{A_1 \cap B_{r-1}}{B_r} = 0.$$

Now by the lemma above, since we can suppose $A_1 \neq B_1$, we have

$$\frac{A_1}{A_1 \cap B_1} \cong \frac{X}{B_1},$$

which is simple. Again, if $A_1 \cap B_1 \neq B_2$ (if not, then $r = 2$ and we are finished) the lemma gives

$$\frac{A_1 \cap B_1}{A_1 \cap B_2} = \frac{A_1 \cap B_1}{(A_1 \cap B_1) \cap B_2} \cong \frac{B_1}{B_2}.$$

Suppose now that $\frac{B_{i-1}}{A_1 \cap B_{i-1}} \cong X/A_1$ is simple. We have for $i < r$

$$\frac{B_i}{A_1 \cap B_i} = \frac{B_i}{(A_1 \cap B_{i-1}) \cap B_i} \cong \frac{B_{i-1}}{A_1 \cap B_{i-1}},$$

and thus it is simple. Then in particular, since $B_r = A_1 \cap B_{r-1}$, we have

$$\frac{B_{r-1}}{B_r} = \frac{B_{r-1}}{A_1 \cap B_{r-1}} \cong \frac{X}{A_1}.$$

Moreover, in general for $i < r - 1$ we have

$$\frac{A_1 \cap B_i}{A_1 \cap B_{i+1}} = \frac{A_1 \cap B_i}{(A_1 \cap B_i) \cap B_{i+1}} \cong \frac{B_i}{B_{i+1}}.$$

This finishes the proof. □

The case with the filtrations reaching 0 is similar.

Lemma A.26. *Suppose there is an object $X$ with two simple subobjects $M$ and $N$ such that $M \neq N$, then*

$$(M \cup N)/M \cong N.$$

*Proof.* $M \cap N = 0$ and thus $M \cup N = M \bigoplus N$. □

Proposition A.27. *Let $X$ be a stable for the filtrations object in a strictly $\mathbb{Z}$-graded abelian category. If $X$ admits two $\mathbb{Z}$-composition series of the following form:*

$$X \leftarrow \cdots \leftarrow A_{-2} \leftarrow A_{-1} \leftarrow 0,$$

$$X \leftarrow \cdots \leftarrow B_{-2} \leftarrow B_{-1} \leftarrow 0,$$

*with respective multiplicities $k_{j,p}$, $k'_{j,p}$ and simple quotients $\{S_j\}, \{S'_j\}$, then the $\mathbb{Z}$-composition series have the same quotients and multiplicities.*

*Proof.* The argument is similar to the one from Proposition A.25. □

Finally, the case with one filtration reaching $X$ and the other 0 follows easily from Lemma A.24.

Proposition A.28. *Let $X$ be a stable for the filtrations object in a strictly $\mathbb{Z}$-graded abelian category. If $X$ admits two $\mathbb{Z}$-composition series*

$$X \leftarrow A_1 \leftarrow A_2 \leftarrow \cdots \leftarrow 0,$$

$$X \leftarrow \cdots \leftarrow B_{-2} \leftarrow B_{-1} \leftarrow 0,$$

*with respective multiplicities $k_{j,p}$, $k'_{j,p}$ and simple quotients $\{S_j\}, \{S'_j\}$, then they have the same quotients and multiplicities.*

*Proof.* We prove by induction that all quotients from the first filtration appear as quotients of the second. A similar reasoning shows the converse.

*Case* 1: Suppose $B_i \subset A_1$ for some $i \in \mathbb{Z}$. Take $i$ minimal such that $B_{i-1} \not\subset A_1$. Then $B_{i-1} \cap A_1 \cong B_i$ and

$$\frac{B_{i-1}}{B_i} \cong \frac{B_{i-1}}{B_{i-1} \cap A_1} \cong \frac{B_{i-1} \cup A_1}{A_1} \cong \frac{X}{A_1}.$$

We now apply the proof on

$$A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \cdots \leftarrow 0,$$

$$X \cap A_1 \leftarrow \cdots \leftarrow B_{i-2} \cap A_1 \leftarrow B_{i-1} \cap A_1 \cong B_i \leftarrow \cdots \leftarrow B_{-2} \leftarrow B_{-1} \leftarrow 0.$$

All quotients but $B_{i-1}/B_i$ appears in this filtration since for all $j < i$ we have $B_{j-1} \cap A_1 \not\subset B_j$.

*Case* 2: Suppose $B_{-1} \not\subset A_1$. Then $X = B_{-1} \bigoplus A_1$ and $X/A_1 \cong B_{-1}$. We then apply the argument recursively on

$$A_1 \leftarrow A_2 \leftarrow \cdots \leftarrow 0,$$

$$A_1 \cong X/B_{-1} \leftarrow \cdots \leftarrow B_{-2}/B_{-1} \leftarrow 0,$$

and this concludes the proof. □

Now we introduce some tools and we prove that given a subobject $M$ of $X$ admitting a $\mathbb{Z}$-composition series, then the quotients in the filtration of $M$ and $X/M$ are essentially the same as the ones in the filtration of $X$.

LEMMA A.29. *Let $M \subset X$. Suppose we have $\mathbb{Z}$-composition series*

$$X \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow 0,$$

$$M \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow 0.$$

*Then we get a $\mathbb{Z}$-composition series*

$$X/M \leftarrow \frac{X_1 \cup M}{M} \leftarrow \frac{X_2 \cup M}{M} \leftarrow \cdots \leftarrow 0.$$

*Proof.* First observe that for all $i$, thanks to the third isomorphism theorem, we have

$$\frac{(X_i \cup M)/M}{(X_{i+1} \cup M)/M} \cong \frac{X_i \cup M}{X_{i+1} \cup M}.$$

Then we get

$$\frac{X_i \cup M}{X_{i+1} \cup M} \cong \frac{X_i \cup (X_{i+1} \cup M)}{X_{i+1} \cup M} \cong \frac{X_i}{X_i \cap (X_{i+1} \cup M).}$$

If $X_i \subset X_{i+1} \cup M$, it is 0. If not, we have $X_{i+1} \subset X_i \cap (X_{i+1} \cup M) \subsetneq X_i$ and thus $(X_i \cup M/X_{i+1} \cup M) \cong (X_i/X_{i+1})$. ☐

REMARK A.30. Also note that if $M$ is a subobject of $X$ such that there are $\mathbb{Z}$-composition series

$$M \leftarrow M_1 \leftarrow \cdots \leftarrow 0,$$

$$X/M \leftarrow Z_1 \leftarrow \cdots \leftarrow 0,$$

then there is a filtration with simple quotients

$$X \leftarrow X_1 \leftarrow \cdots \leftarrow M \leftarrow M_1 \leftarrow \cdots \leftarrow 0.$$

*Proof.* Define $X_i$ as the pull-back

$$\begin{tikzcd} X_i \arrow[r] \arrow[d, "\imath'"'] & Z_i \arrow[d, "\imath"] \\ X \arrow[r] & X/M \end{tikzcd}$$

where $\imath$ is a monomorphism and thus so is $\imath'$. We clearly have $\varprojlim_i X_i = M$. Moreover the following diagram has 3 exact column and 2 exact rows

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\cong"'] & X_{i+1} \arrow[r] \arrow[d] & Z_{i+1} \arrow[d] \\ M \arrow[r] \arrow[d] & X_i \arrow[r] \arrow[d] & Z_i \arrow[d] \\ 0 \arrow[r] & X_i/X_{i+1} \arrow[r] & Z_i/Z_{i+1} \end{tikzcd}$$

and by the $3 \times 3$ lemma the third row must be exact too. This means $\frac{X_i}{X_{i+1}} \cong \frac{Z_i}{Z_{i+1}}$. ☐

Proposition A.31. *Let $M \subset X$. Suppose $X, M$ and $X/M$ are stable for the filtrations. If we have $\mathbb{Z}$-composition series*

$$X \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow 0, \tag{A.1}$$

$$M \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow 0, \tag{A.2}$$

$$X/M \leftarrow \frac{X_1 \cup M}{M} \leftarrow \frac{X_2 \cup M}{M} \leftarrow \cdots \leftarrow 0, \tag{A.3}$$

*then all simple quotients of (A.1) appears has quotients with the same multiplicities in (A.2) plus (A.3).*

*Proof.* Consider the filtrations

$$M = M \cap X \leftarrow M \cap X_1 \leftarrow M \cap X_2 \leftarrow \cdots \leftarrow 0, \tag{A.4}$$

$$X/M \leftarrow \frac{X_1 \cup M}{M} \leftarrow \frac{X_2 \cup M}{M} \leftarrow \cdots \leftarrow 0. \tag{A.5}$$

We claim that together they contain exactly all the quotients of (A.1) with the same multiplicities. Indeed, note that $X_j \cap (X_{j+1} \cup M) \cong X_{j+1} \cup (X_j \cap M)$ such that we must have $X_j \cong X_j \cap (X_{j+1} \cup M)$ or $X_{j+1} \cong X_j \cap (X_{j+1} \cup M)$ and thus

$$\frac{X_j \cap M}{X_{j+1} \cap M} \cong \frac{X_j}{X_{j+1}} \quad \text{or} \quad \frac{X_j \cap M}{X_{j+1} \cap M} = 0$$

$$\frac{X_j \cup M}{X_{j+1} \cup M} \cong 0 \qquad\qquad \frac{X_j \cup M}{X_{j+1} \cup M} = \frac{X_j}{X_{j+1}}.$$

Then we conclude the proof using the Proposition A.25 on (A.2) with (A.4), and on (A.3) with (A.5). ☐

All the above can also be proved for all combinations of the two types of $\mathbb{Z}$-composition series, using similar arguments.

Remark A.32. Restricting the filtrations to only $\mathbb{Z}$-filtrations is too strong for what we want to do. Indeed, suppose we have a non-split short exact sequence

$$A \to B \to \bigoplus_{i \geqslant 0} X\langle 2i \rangle,$$

where $X$ and $A$ are simple, then using Remark A.30 we can construct a filtration

$$B \leftarrow B_1 \leftarrow B_2 \leftarrow \cdots \leftarrow A \leftarrow 0$$

which is not a $\mathbb{Z}$-composition series, despite the fact that $B_i/B_{i+1} \cong X\langle 2i \rangle$ and $\varprojlim_i B_i \cong A$. Thus we want to be able to glue $\mathbb{Z}$-filtrations together.

Write $X \prec - - A$ if there exists a $\mathbb{Z}$-filtration

$$X \leftarrow X_1 \leftarrow \cdots \leftarrow A, \quad \text{or} \quad X \leftarrow \cdots \leftarrow X_{-1} \leftarrow A,$$

inducing a $\mathbb{Z}$-composition series on $X/A$.

Definition A.33. We say that $X$ has locally finitely many composition factors if there is a finite filtration

$$X = X^0 \leftarrow\!\!\!< - - X^1 \leftarrow\!\!\!< - - X^2 \leftarrow\!\!\!< - - - \cdots \leftarrow\!\!\!< - - X^n = 0 ,$$

and each $X^i/X^{i+1}$ admits a $\mathbb{Z}$-composition series. We can consider the set $\{S_j\}_{j\in J}$ of all quotients of the filtrations for all the $X^i/X^{i+1}$, up to isomorphism and grading shift, which we call composition factors of $X$. Each of the composition factor has a finite degree $p$ multiplicity for each $p \in \mathbb{Z}$, which is given by the sum of the multiplicities in the $\mathbb{Z}$-composition series of all the quotients $X^i/X^{i+1}$. Moreover, we say that $X$ is stable for the filtrations if for all such filtrations, the quotients $X^i/X^{i+1}$ are stable for the filtrations.

We have now all the tools to prove the main result in this subsection.

Theorem A.34. *Let $X$ be a stable for the filtrations object having locally finitely many composition factors. If there are two filtrations*

$$X = A^0 \leftarrow\!\!\!< - - A^1 \leftarrow\!\!\!< - - A^2 \leftarrow\!\!\!< - - - \cdots \leftarrow\!\!\!< - - A^n = 0 , \tag{A.6}$$

$$X = B^0 \leftarrow\!\!\!< - - B^1 \leftarrow\!\!\!< - - B^2 \leftarrow\!\!\!< - - - \cdots \leftarrow\!\!\!< - - B^m = 0 , \tag{A.7}$$

*with all $A^i/A^{i+1}$ and $B^i/B^{i+1}$ having $\mathbb{Z}$-composition series, then the composition factors are in bijection and have the same multiplicities.*

*Proof.* We can suppose $m \geqslant n$. The proof follows by a double induction on $n$ and $m$. If $n = 1$ and $m = 1$, then the result is given by Propositions A.25 , A.27 or A.28. If $n = 1$, then by Lemma A.24 we can construct

$$X \leftarrow\!\!\!< - - B^1 \leftarrow\!\!\!< - - 0 ,$$

which has the same composition factors as $X \leftarrow\!\!\!< - - 0$ thanks to Proposition A.31. Then we can split this filtration and (A.7) into two parts at the level of $B^1$, on which we can apply the argument recursively. Suppose now $n > 1$. We can construct

$$X \leftarrow\!\!\!< - - A^1 \leftarrow\!\!\!< - - A^1 \cap B^1 \leftarrow\!\!\!< - - A^2 \cap B^1 \leftarrow\!\!\!< - - - \cdots \leftarrow\!\!\!< - - A^n \cap B^1 = 0 , \tag{A.8}$$

$$X \leftarrow\!\!\!< - - B^1 \leftarrow\!\!\!< - - A^1 \cap B^1 \leftarrow\!\!\!< - - A^2 \cap B^1 \leftarrow\!\!\!< - - - \cdots \leftarrow\!\!\!< - - A^n \cap B^1 = 0 . \tag{A.9}$$

By splitting (A.8) at the level of $A^1$ we can apply the reasoning recursively to prove it has the same composition factors as (A.6). It is important to note that (A.8) and (A.9) have the same tail after $A^1 \cap B^1$ and it is given by $n$ which is smaller than $m$. This allows us to use the induction hypothesis. By the same arguments, (A.9) has the same composition factors as (A.7). Now by splitting (A.8) and (A.9) at the level of $A^1 \cap B^1$ and using the induction hypothesis, these two must have the same composition factors as well. This concludes the proof. □

Definition A.35. We say that a strictly $\mathbb{Z}$-graded abelian category has *local Jordan–Hölder property* if every object has locally finitely many composition factors and is stable for the filtrations.

Let $\mathcal{C}$ be a strictly $\mathbb{Z}$-graded, right complete, locally additive category with the local Jordan–Hölder property and every $\mathbb{Z}$-composition series being left-bounded. Following the same path as in the previous subsection, define $G_0'(\mathcal{C})$ to be the free $\mathbb{Z}[\![q]\!][q^{-1}]$-module generated by the classes of simple objects, up to shift. By Theorem A.34 there is a surjective $\mathbb{Z}[\![q]\!][q^{-1}]$-module map

$$g : G_0(\mathcal{C}) \twoheadrightarrow G_0'(\mathcal{C}),$$

inducing a topology on $G_0(\mathcal{C})$. We make it Hausdorff by the following.

DEFINITION A.36. We define the topological Grothendieck group of $\mathcal{C}$ as

$$\boldsymbol{G}_0(\mathcal{C}) = G_0(\mathcal{C})/\cap_{n\geqslant 0}\, g^{-1}((q)^n).$$

Again, the topological Grothendieck group forms a topological module over $\mathbb{Z}[\![q]\!][q^{-1}]$ with the $(q)$-adic topology and we get the following theorem.

THEOREM A.37. *The topological Grothendieck group $\boldsymbol{G}_0(\mathcal{C})$ of a right complete category $\mathcal{C}$, with the local Jordan–Hölder property and every $\mathbb{Z}$-composition series being left-bounded, is a topological $\mathbb{Z}[\![q]\!][q^{-1}]$-module freely generated by the classes of simple objects (up to degree shift).*

An exact, graded functor $\Phi\colon \mathcal{C} \to \mathcal{C}'$ gives rise to a $\mathbb{Z}[\![q]\!][q^{-1}]$-linear map $G_0(\mathcal{C}) \to G_0(\mathcal{C}')$. Also, there is an obvious $\mathbb{Z}[\![q]\!][q^{-1}]$-linear injection $\boldsymbol{G}_0(\mathcal{C}) \hookrightarrow G_0(\mathcal{C})$. We define $[\Phi] : \boldsymbol{G}_0(\mathcal{C}) \to \boldsymbol{G}_0(\mathcal{C}')$ as the composition $\boldsymbol{G}_0(\mathcal{C}) \hookrightarrow G_0(\mathcal{C}) \to G_0(\mathcal{C}') \twoheadrightarrow \boldsymbol{G}_0(\mathcal{C}')$.

EXAMPLE A.38. Let $R$ be a $\Bbbk$-algebra as in the Example A.3. Consider the category $R\text{-mod}_{\mathrm{lf}}$ of locally finite-dimensional $R$-modules, with the dimensions left-bounded and degree *zero* morphisms. It has the local Jordan–Hölder property and every $\mathbb{Z}$-composition series is left-bounded. Moreover, we have the following inclusions of full subcategories

$$R\text{-pmod}_{\mathrm{lfg}} \subset R\text{-mod}_{\mathrm{lf}} \subset R\text{-mod}_{\mathrm{lfg}}\,.$$

Moreover, if the category has enough projectives, then projective resolutions of the simple objects can give a change of basis, such that the topological Grothendieck group becomes also freely generated by the projective objects.

EXAMPLE A.39. Take $R = \mathbb{Q}[x,y]/(y^2)$, with $\deg(x) = \deg(y) = 2$, and its topological Grothendieck group $\boldsymbol{G}_0(R\text{-mod}_{\mathrm{lf}})$ is generated either by the classes of the simple object $S = \mathbb{Q}$ or the projective object $R$, with change of basis given by

$$[R] = \frac{1+q^2}{1-q^2}[S] = (1+q^2)(1+q^2+q^4+\cdots)[S],$$

$$[S] = \frac{1-q^2}{1+q^2}[R] = (1-q^2)(1-q^2+q^4-\cdots)[R].$$

REMARK A.40. Likely it is possible to adapt the results in [**1**], which allows to define a notion of topological Grothendieck group capable of handling infinite relations coming from infinite projective resolution. Probably one can weaken the artinian assumption to locally Jordan–Hölder and mixing it with our results. The local Jordan–Hölder property gives for each object a unique weight filtration, bounded from above (the weight is the opposite of the $\mathbb{Z}$-grading, thus this a $\mathbb{Z}$-filtration bounded from below), with all quotients being finite. This should be usable to compute the topological Grothendieck group of $\mathcal{D}^{\nabla}(\mathcal{C})$ and get a continuous isomorphism $\boldsymbol{K}(\mathcal{D}^{\nabla}(\mathcal{C})) \cong \boldsymbol{G}_0(\mathcal{C})$.

A.3. *Multigrading and field of formal Laurent series.* We now investigate the case of multigrading. But first we need to choose a construction of the field of formal Laurent series $\mathbb{Q}((x_1, \dots, x_p))$.

A.3.1. *Field of formal Laurent series.* We follow the description given in [2]. We fix a grading by $\mathbb{Z}^p$ with $p \in \mathbb{N}$, and we choose an additive order $\prec$ on it. That is, $a \prec b$ implies $a + c \prec b + c$, for every $a, b, c \in \mathbb{Z}^p$.

DEFINITION A.41. We call *cone* a subset $C \subset \mathbb{R}^p$ such that

$$C = \{\alpha_1 v_1 + \dots + \alpha_p v_p | \alpha_1, \dots, \alpha_p \geqslant 0\},$$

for some generating elements $v_1, \dots, v_p \in \mathbb{Z}^p$. Moreover we say $C$ is *compatible with the order* $\prec$ if $0 \prec v_i$ for all $i \in \{1, \dots, p\}$.

REMARK A.42. Usually the definition of cone is more general, but we are interested only in these ones for our discussion.

EXAMPLE A.43. If $p = 1$, then there are only two possible orders:

- if $0 \prec 1$, then there is only one (non-zero) compatible cone given by $[0, \infty[$,
- if $0 \prec -1$, then the only compatible cone is $]-\infty, 0]$.

Let $C \subset \mathbb{R}^p$ be a cone compatible with $\prec$ and define

$$\mathbb{Q}_C[\![x_1, \dots, x_p]\!] = \left\{ \sum_{\boldsymbol{k}=(k_1,\dots,k_p)\in\mathbb{N}^p} a_{\boldsymbol{k}} x_1^{k_1} \dots x_p^{k_p} | a_{\boldsymbol{k}} = 0 \text{ if } \boldsymbol{k} \notin C \right\}.$$

PROPOSITION A.44 [2, Theorem 10]. *The set $\mathbb{Q}_C[\![x_1, \dots, x_p]\!]$ together with the natural addition and multiplication forms a ring.*

*Proof* (Sketch). The important point in the proof of this proposition is that the restriction to cones compatible with the order ensures we only have to multiply a finite number of elements to define each coefficient in a product. □

The next definition is [2, Definition 14].

DEFINITION A.45. We put

$$\mathbb{Q}_\prec[\![x_1, \dots, x_p]\!] = \bigcup_C \mathbb{Q}_C[\![x_1, \dots, x_p]\!],$$

where the union is over all cones compatibles with $\prec$, and we define the field of formal Laurent series as

$$\mathbb{Q}_\prec((x_1, \dots, x_p)) = \bigcup_{\boldsymbol{e}=(e_1,\dots,e_p)\in\mathbb{Z}^p} x_1^{e_1} \dots x_p^{e_p} \mathbb{Q}_\prec[\![x_1, \dots, x_p]\!].$$

THEOREM A.46 [2, Theorem 15]. *The set $\mathbb{Q}_\prec((x_1, \dots, x_p))$ together with the natural addition, multiplication and division forms a field.*

*Proof* (Sketch). There are three main ideas in the proof of this theorem. First, given any pair of cones $C_1, C_2$ compatible with $\prec$, their sum yields a cone $C_3 = C_1 + C_2$, also compatible with $\prec$. Hence we can define a product $\mathbb{Q}_{C_1}[\![x_1, \dots, x_p]\!] \otimes \mathbb{Q}_{C_2}[\![x_1, \dots, x_p]\!] \to \mathbb{Q}_{C_3}[\![x_1, \dots, x_p]\!]$, which in turns define a product on $\mathbb{Q}_{\prec}[\![x_1,\dots,x_p]\!]$. Second, given $f(\boldsymbol{x}) \in \mathbb{Q}_C[\![x_1,\dots,x_p]\!]$ such that $f(0) \neq 0$, one can define recursively a unique $g(\boldsymbol{x}) \in \mathbb{Q}_C[\![x_1,\dots,x_p]\!]$ such that $g(x)f(x) = 1$. Finally, taking the union of all $\mathbb{Q}_{\prec}[\![x_1,\dots,x_p]\!]$ shifted by a monomial allows to write any $f(\boldsymbol{x}) \in \mathbb{Q}_{\prec}(\!(x_1,\dots,x_p)\!)$ as $\boldsymbol{x}^{\boldsymbol{e}}h(x)$ where $h(\boldsymbol{x}) \in \mathbb{Q}_C[\![x_1,\dots,x_p]\!]$ is such that $h(0) \neq 0$. Therefore, we have $\boldsymbol{x}^{-\boldsymbol{e}}h^{-1}(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{e}}h(\boldsymbol{x}) = 1$, which concludes the proof. □

Example A.47. Again, if $p = 1$, we have two possible ways to construct $\mathbb{Q}(\!(x)\!)$:

- if $0 \prec 1$, then we get $\mathbb{Q}_{\prec}(\!(x)\!) = \mathbb{Q}[\![x]\!][x^{-1}]$ and $\frac{1}{x-x^{-1}} = -x(1 + x^2 + \cdots)$,
- if $0 \prec -1$, then $\mathbb{Q}_{\prec}(\!(x)\!) = \mathbb{Q}[\![x^{-1}]\!][x]$ and $\frac{1}{x-x^{-1}} = x^{-1}(1 + x^{-2} + \cdots)$.

Example A.48. Take $p = 2$, then we have six ways to construct $\mathbb{Q}(\!(x_1, x_2)\!)$. In this case we abuse the notation and say, for example, that we choose the order $0 \prec x_1 \prec x_2$ for the order induced by the choice $(0,0) \prec (1,0) \prec (0,1)$, where we suppose (1,0) corresponds to $x_1$ and (0,1) to $x_2$. Then we get

$$\frac{1}{1 - x_1^{-2}x_2^2} = (1 + x_1^{-2}x_2^2 + x_1^{-4}x_2^4 + \cdots).$$

A.3.2. *Grothendieck groups for multigrading.* We fix a multigrading and an additive order on it. Every definition in §§ A.1 and A.2 extends naturally to the multigraded case, except for left-bounded.

Definition A.49. We say that a locally finite coproduct (or filtration) is *cone-bounded* if all its non-zero coefficients are contained in some cone compatible with $\prec$, up to a shift.

It is then straightforward to adapt all results from §§ A.1 and A.2 to the multigraded case, replacing 'left-bounded' by 'cone-bounded'. In accordance to this denomination, we will also say *cone complete* for a category which admits all cone-bounded, locally finite coproducts.

The next example represents the classical case that will be used later on.

Example A.50. As in Example A.19, we can construct filtrations for some multigraded $\Bbbk$-algebras. Suppose $R$ is a unital $\mathbb{Z}^p$-graded $\Bbbk$-algebra, having locally finite dimension. Also suppose its graded dimension is contained in a cone compatible with the chosen order on $\mathbb{Z}^p$, and suppose it is isomorphic to $\Bbbk$ in degree *zero*. Then the additive order on $\mathbb{Z}^p$ restricts to a total order $0 = i_0 \prec i_1 \prec \cdots$ on the homogeneous components of $R = \bigoplus_k R^{i_k}$, which allows us to construct a filtration of submodules

$$R = \bigoplus_{i \succeq i_0} R^i \leftarrow \bigoplus_{i \succeq i_1} R^i \leftarrow \cdots \leftarrow 0,$$

where each subquotient is an homogeneous component $R^{i_j}$, and thus finite-dimensional. Then we can apply the same arguments as in Example A.21 to show locally finite, cone-bounded (that is, its graded dimension is contained in a cone compatible with $\prec$, up to a shift), left $R$-modules are stable for the filtrations, and thus $R\text{-mod}_{\mathrm{lf}}$ has the local Jordan–Hölder property. Therefore $\boldsymbol{G}_0(R\text{-mod}_{\mathrm{lf}})$ is a free $\mathbb{Z}_{\prec}(\!(x_1,\dots,x_p)\!)$-module generated by the classes of simple modules.

The hypothesis in the example above can be weakened a bit by only requiring $R$ to admits a finite collection of indecomposable projective modules, up to isomorphism and shift, each having their dimension locally finite and contained in a cone compatible with $\prec$.

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