The Kapustin-Li formula and the evaluation of (closed) $\mathfrak{sl}(N)$-foams

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Abstract

This notes intended to supplement a talk given by the author at the Institut de Mathématiques de Jussieu - Université Paris 7 in January 2010. The purpose of the talk was to give an explanation of the use of Kapustin-Li formula in the evaluation of the closed foams used in the construction of the $\mathfrak{sl}(N)$-link homology. This notes contains material from the author’s PhD thesis [17] as well as the explanation and the proof of some facts about the Kapustin-Li formula that were omitted from [11].

Contents

1 Introduction 2
2 Matrix factorizations 2
3 Schur polynomials and the cohomology of partial flag varieties 7
4 Foams 10
5 The KL formula and the evaluation of closed foams 12
6 Odds and ends 28
References 30
1 Introduction

This notes intended to supplement a talk given by the author at the Institut de Mathématiques de Jussieu - Université Paris 7 in January 2010 when he was postdoc at the aforementioned institution. The purpose of the talk was to give an explanation of the use of Kapustin-Li formula in the evaluation of the closed foams used in the construction of the $\mathfrak{sl}(N)$-link homology. This notes contains material from my thesis [17] as well as the explanation and the proof of some facts about the Kapustin-Li formula that were omitted from [11].

In the context of categorification foams first appeared in Khovanov’s construction of a topological theory categorifying the $\mathfrak{sl}(3)$-link polynomial [6]. His construction uses cobordisms with singularities, called foams, modulo a finite set of relations. In [9] Khovanov and Rozansky (KR) categorified the $\mathfrak{sl}(N)$-link polynomial for arbitrary $N$, the 1-variable specializations of the 2-variable HOMFLY-PT polynomial. Their construction uses the theory of matrix factorizations, a mathematical tool introduced by Eisenbud in [3] (see also [2, 10, 18]) in the study of maximal Cohen-Macaulay modules over isolated hypersurface singularities and used by Kapustin and Li as boundary conditions for strings in Landau-Ginzburg models [5].

The goal of [11] was to construct a combinatorial topological definition of KR link homology, extending to all $N > 3$ the work of Khovanov [6] for $N = 3$ (see also [12]). Khovanov had to modify considerably his original setting for the construction of $\mathfrak{sl}(2)$ link homology in order to produce his $\mathfrak{sl}(3)$ link homology. It required the introduction of singular cobordisms with a particular type of singularity, which he called foams. The jump from $\mathfrak{sl}(3)$ to $\mathfrak{sl}(N)$, for $N > 3$, requires the introduction of a new type of singularity. The latter is needed for proving invariance under the third Reidemeister move. The introduction of the new singularities makes it much harder to evaluate closed foams and we do not know how to do it combinatorially. Instead we use the Kapustin-Li formula [5], which was introduced by A. Kapustin and Y. Li in [5] in the context of topological Landau-Ginzburg models with boundaries and adapted to foams by Khovanov and Rozansky [7]. The downside is that our construction does not yet allow us to deduce a (fast) algorithm for computing $\mathfrak{sl}(N)$ link homology. A positive side-effect is that it allows us to show that for any link the homology using foams is isomorphic to KR homology. Furthermore the combinatorics involved in establishing certain identities among foams gets much harder for arbitrary $N$. The theory of symmetric polynomials, in particular Schur polynomials, is used to handle that problem.

2 Matrix factorizations

This section contains a brief review of matrix factorizations and the properties that will be used throughout this notes. All the matrix factorizations in this notes are $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$-graded. Let $R$ be a polynomial ring over $\mathbb{Q}$ in a finite number of variables. We take the $\mathbb{Z}$-degree of each polynomial to be twice its total degree. This way $R$ is $\mathbb{Z}$-graded. Let $W$ be a homogeneous element of $R$ of degree $2m$. A matrix factorization of $W$ over $R$ is given by a $\mathbb{Z}/2\mathbb{Z}$-graded free module
We define

\[ M \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_0 \]

such that \( d_1 d_0 = W \text{Id}_{M_0} \) and \( d_0 d_1 = W \text{Id}_{M_1} \). We call \( W \) the potential. The \( \mathbb{Z} \)-grading of \( R \) induces a \( \mathbb{Z} \)-grading on \( M \). The shift functor \( \{ k \} \) acts on \( M \) as

\[ M\{k\} = M_0\{k\} \xrightarrow{d_0} M_1\{k\} \xrightarrow{d_1} M_0\{k\}, \]

where its action on the modules \( M_0, M_1 \) means an upward shift by \( k \) units on the \( \mathbb{Z} \)-grading.

A homomorphism \( f: M \rightarrow M' \) of matrix factorizations of \( W \) is a pair of maps of the same degree \( f_i: M_i \rightarrow M_i' \) \((i = 0, 1)\) such that the diagram

\[
\begin{array}{c}
M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_0 \\
f_0 \downarrow \quad \downarrow f_1 \quad \downarrow f_0 \\
M'_0 \xrightarrow{d'_0} M'_1 \xrightarrow{d'_1} M'_0
\end{array}
\]

commutes. It is an isomorphism of matrix factorizations if \( f_0 \) and \( f_1 \) are isomorphisms of the underlying modules. Denote the set of homomorphisms of matrix factorizations from \( M \) to \( M' \) by

\[ \text{Hom}_{\text{MF}}(M, M'). \]

It has an \( R \)-module structure with the action of \( R \) given by \( r(f_0, f_1) = (r f_0, r f_1) \) for \( r \in R \). Matrix factorizations over \( R \) with homogeneous potential \( W \) and homomorphisms of matrix factorizations form a graded additive category, which we denote by \( \text{MF}_R(W) \). If \( W = 0 \) we simply write \( \text{MF}_R \).

Another description of matrix factorizations is obtained by assembling the differentials \( d_0 \) and \( d_1 \) into an endomorphism \( D \) of the \( \mathbb{Z}/2\mathbb{Z} \)-graded free \( R \)-module \( M = M_0 \oplus M_1 \) such that

\[ D = \begin{pmatrix} 0 & d_1 \\ d_0 & 0 \end{pmatrix}, \quad \deg_{\mathbb{Z}/2\mathbb{Z}} D = 1, \quad D^2 = W \text{Id}_M. \]

In this case we call \( D \) the twisted differential.

The free \( R \)-module \( \text{Hom}_R(M, M') \) of graded \( R \)-module homomorphisms from \( M \) to \( M' \) is a 2-complex

\[ \text{Hom}_R^0(M, M') \xrightarrow{d} \text{Hom}_R^1(M, M') \xrightarrow{d} \text{Hom}_R^0(M, M') \]

where

\[ \text{Hom}_R^0(M, M') = \text{Hom}_R(M_0, M_0') \oplus \text{Hom}_R(M_1, M_1') \]

\[ \text{Hom}_R^1(M, M') = \text{Hom}_R(M_0, M_1') \oplus \text{Hom}_R(M_1, M_0') \]

and for \( f \) in \( \text{Hom}_R^I(M, M') \) the differential acts as

\[ df = d_{M'} f - (-1)^i f d_M. \]

We define

\[ \text{Ext}(M, M') = \text{Ext}^0(M, M') \oplus \text{Ext}^1(M, M') = \text{Ker} d / \text{Im} d, \]

3
and write Ext$_{(m)}(M, M')$ for the elements of Ext$(M, M')$ with $\mathbb{Z}$-degree $m$. Note that for $f \in \text{Hom}_{MF}(M, M')$ we have $df = 0$. We say that two homomorphisms $f, g \in \text{Hom}_{MF}(M, M')$ are homotopic if there is an element $h \in \text{Hom}_{R}^{1}(M, M')$ such that $f - g = dh$.

Denote by Hom$_{HMF}(M, M')$ the $R$-module of homotopy classes of homomorphisms of matrix factorizations from $M$ to $M'$ and by HMF$_{R}(W)$ the homotopy category of MF$_{R}(W)$.

We denote by $M(1)$ and $M_{\bullet}$ the factorizations

\[
\begin{array}{ccc}
M_{1} & \overset{d_{1}}{\longrightarrow} & M_{0} \\
\downarrow & & \downarrow \\
M_{1} & \overset{d_{0}}{\longrightarrow} & M_{0}
\end{array}
\]

and

\[
(M_{0})^{\ast} \overset{-((d_{1})^{\ast})}{\longrightarrow} (M_{1})^{\ast} \overset{(d_{0})^{\ast}}{\longrightarrow} (M_{0})^{\ast}
\]

respectively. Factorization $M(1)$ has potential $W$ while factorization $M_{\bullet}$ has potential $-W$. We call $M_{\bullet}$ the dual factorization of $M$.

We have

\[
\begin{align*}
\text{Ext}^{0}(M, M') & \cong \text{Hom}_{HMF}(M, M') \\
\text{Ext}^{1}(M, M') & \cong \text{Hom}_{HMF}(M, M'(1))
\end{align*}
\]

The tensor product $M \otimes_{R} M_{\bullet}$ has potential zero and is therefore a 2-complex. Denoting by HMF the homology of matrix factorizations with potential zero we have

\[
\text{Ext}(M, M') \cong \text{HMF}(M' \otimes_{R} M_{\bullet})
\]

and, if $M$ is a matrix factorization with $W = 0$,

\[
\text{Ext}(R, M) \cong \text{HMF}(M).
\]

Let $R = \mathbb{Q}[x_{1}, \ldots, x_{k}]$ and $W \in R$. The Jacobi algebra of $W$ is defined as

\[
J_{W} = R/(\partial_{1}W, \ldots, \partial_{k}W),
\]

where $\partial_{i}$ means the partial derivative with respect to $x_{i}$. Writing the differential as a matrix and differentiating both sides of the equation $D^{2} = W$ with respect to $x_{i}$ we get $D(\partial_{i}D) + (\partial_{i}D)D = \partial_{i}W$. We thus see that multiplication by $\partial_{i}W$ is homotopic to the zero endomorphism and that the homomorphism

\[
\begin{array}{ccc}
R & \rightarrow & \text{End}_{HMF}(M), \\
r & \mapsto & m(r)
\end{array}
\]

factors through the Jacobi algebra of $W$.

Let $f, g \in \text{End}(M)$. We define the supercommutator of $f$ and $g$ as

\[
[f, g]_{s} = fg - (-1)^{\deg_{R/\mathbb{Z}}(f) \deg_{R/\mathbb{Z}}(g)}gf.
\]

The supertrace of $f$ is defined as

\[
\text{STr}(f) = \text{Tr}((-1)^{gr}f)
\]

where the grading operator $(-1)^{gr} \in \text{End}(M_{0} \oplus M_{1})$ is given by

$(m_{0}, m_{1}) \mapsto (m_{0}, -m_{1}), \quad m_{0} \in M_{0}, \ m_{1} \in M_{1}$. 

4
If \( f \) and \( g \) are homogeneous with respect to the \( \mathbb{Z}/2\mathbb{Z} \)-grading we have that
\[
\text{STr}(fg) = (-1)^{\deg_{\mathbb{Z}/2\mathbb{Z}}(f) \deg_{\mathbb{Z}/2\mathbb{Z}}(g)} \text{STr}(gf),
\]
and
\[
\text{STr}([f; g]_s) = 0.
\]
There is a canonical isomorphism of \( \mathbb{Z}/2\mathbb{Z} \)-graded \( \mathbb{R} \)-modules
\[
\text{End}(M) \cong M \otimes_R M_\bullet.
\]
Choose a basis \( \{ i \} \) of \( M \) and define a dual basis \( \{ j \} \) of \( M_\bullet \) by \( \langle j | i \rangle = \delta_{i,j} \), where \( \delta \) is the Kronecker symbol. There is a natural pairing map \( M \otimes M_\bullet \to R \) called the super-contraction that is given on basis elements \( i \langle j \rangle \) by
\[
| i \rangle \langle j | \mapsto (-1)^{\deg_{\mathbb{Z}/2\mathbb{Z}}(|i\rangle) \deg_{\mathbb{Z}/2\mathbb{Z}}(|j\rangle)} \langle j | i \rangle = \delta_{i,j}.
\]
The super-contraction induces a map \( \text{End}(M) \to R \) which coincides with the supertrace. When \( M \) and \( M_\bullet \) are factors in a tensor product \( (M \otimes M_\bullet) \otimes (M_\bullet \otimes M_\bullet) \) the super-contraction of \( M \) with \( M_\bullet \) induces a map \( \text{STr}_M : \text{End}(M \otimes M_\bullet) \to \text{End}(N) \) called the partial super-trace (w.r.t. \( M \)).

### 2.1 Koszul Factorizations

For \( a, b \) homogeneous elements of \( R \), an elementary Koszul factorization \( \{a, b\} \) over \( R \) with potential \( ab \) is a factorization of the form
\[
R \xrightarrow{a} R \{ \frac{1}{2} (\deg_{\mathbb{Z}} b - \deg_{\mathbb{Z}} a) \} \xrightarrow{b} R.
\]
When we need to emphasize the ring \( R \) we write this factorization as \( \{a, b\}_R \). The tensor product of matrix factorizations \( M_i \) with potentials \( W_i \) is a matrix factorization with potential \( \sum W_i \). We restrict to the case where all the \( W_i \) are homogeneous of the same degree. Throughout this notes we use tensor products of elementary Koszul factorizations \( \{a_i, b_i\} \) to build bigger matrix factorizations, which we write in the form of a Koszul matrix as
\[
\begin{pmatrix}
  a_1 & b_1 \\
  \vdots & \vdots \\
  a_k & b_k
\end{pmatrix}
\]
We denote by \( \{a, b\} \) the Koszul matrix which has columns \( \{a_1, \ldots, a_k\} \) and \( \{b_1, \ldots, b_k\} \). If \( \sum_{i=1}^k a_i b_i = 0 \) then \( \{a, b\} \) is a 2-complex whose homology is an \( R/(a_1, \ldots, a_k, b_1, \ldots, b_k) \)-module, since multiplication by \( a_i \) and \( b_i \) are null-homotopic endomorphisms of \( \{a, b\} \).

Note that the action of the shift \( \langle 1 \rangle \) on \( \{a, b\} \) is equivalent to switching terms in one line of \( \{a, b\} \):
\[
\{a, b\} \langle 1 \rangle \cong \begin{pmatrix}
  \vdots & \vdots \\
  a_{i-1} & b_{i-1} \\
  -b_i & -a_i \\
  a_{i+1} & b_{i+1} \\
  \vdots & \vdots
\end{pmatrix}
\begin{pmatrix}
  \frac{1}{2} (\deg_{\mathbb{Z}} b_i - \deg_{\mathbb{Z}} a_i) \\
  \frac{1}{2} (\deg_{\mathbb{Z}} a_i - \deg_{\mathbb{Z}} b_i)
\end{pmatrix}.
\]
If we choose a different row to switch terms we get a factorization which is isomorphic to this one. We also have that
\[ \{a, b\} \ast \cong \{a - b, k\} \{s_k\}, \]
where
\[ s_k = \sum_{i=1}^{k} \deg_z a_i - \frac{k}{2} \deg_z W. \]

Let \( R = \mathbb{Q}[x_1, \ldots, x_k] \) and \( R' = \mathbb{Q}[x_2, \ldots, x_k] \). Suppose that \( W = \sum_i a_i b_i \in R' \) and \( x_1 - b_i \in R' \), for a certain \( 1 \leq i \leq k \). Let \( c = x_1 - b_i \) and \( \{\hat{a}, \hat{b}\} \) be the matrix factorization obtained from \( \{a, b\} \) by deleting the \( i \)-th row and substituting \( x_1 \) by \( c \).

**Lemma 2.1** (excluding variables). The matrix factorizations \( \{a, b\} \) and \( \{\hat{a}, \hat{b}\} \) are homotopy equivalent.

In [9] one can find the proof of this lemma and its generalization with several variables.

The following lemma contains three particular cases of Proposition 3 in [9]:

**Lemma 2.2** (Row operations). We have the following isomorphisms of matrix factorizations
\[
\begin{align*}
\{a_i, b_j\} & \cong \{a_i - \lambda a_j, b_j\} \\
\{a_j, b_i\} & \cong \{a_j + \lambda b_j, b_i\}
\end{align*}
\]

for \( \lambda \in R \). If \( \lambda \) is invertible in \( R \), we also have
\[
\{a_i, b_j\} \overset{[i,j]_\lambda}{\cong} \{\lambda a_i, \lambda^{-1} b_j\}.
\]

**Proof.** It is straightforward to check that the pairs of matrices
\[
[i,j]_\lambda = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \quad [i,j]^{\prime}_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

define isomorphisms of matrix factorizations. \( \square \)

Recall that a sequence \( (a_1, a_2, \ldots, a_k) \) is called regular in \( R \) if \( a_j \) is not a zero divisor in \( R/(a_1, a_2, \ldots, a_{j-1}) \), for \( j = 1, \ldots, k \). The proof of the following lemma can be found in [8].

**Lemma 2.3.** Let \( \mathbf{b} = (b_1, b_2, \ldots, b_k) \), \( \mathbf{a} = (a_1, a_2, \ldots, a_k) \) and \( \mathbf{a}' = (a'_1, a'_2, \ldots, a'_k) \) be sequences in \( R \). If \( \mathbf{b} \) is regular and \( \sum_i a_i b_i = \sum_i a'_i b_i \) then the factorizations
\[
\{\mathbf{a}, \mathbf{b}\} \quad \text{and} \quad \{\mathbf{a}', \mathbf{b}\}
\]
are isomorphic.

A factorization \( M \) with potential \( W \) is said to be contractible if it is isomorphic to a direct sum of factorizations of the form
\[
R \xrightarrow{\frac{1}{2} \deg_z W} R \quad \text{and} \quad R \xrightarrow{-\frac{1}{2} \deg_z W} R.
\]
3 Schur polynomials and the cohomology of partial flag varieties

In this section we recall some basic facts about Schur polynomials and the cohomology of partial flag varieties.

3.1 Schur polynomials

A nice basis for homogeneous symmetric polynomials is given by the Schur polynomials. If \( \lambda = (\lambda_1, \ldots, \lambda_k) \) is a partition such that \( \lambda_1 \geq \ldots \geq \lambda_k \geq 0 \), then the Schur polynomial \( \pi_\lambda(x_1, \ldots, x_k) \) is given by the following expression:

\[
\pi_\lambda(x_1, \ldots, x_k) = \frac{\det(x_i^{\lambda_j+k-j})}{\Delta},
\]

where \( \Delta = \prod_{i<j} (x_i - x_j) \), and by \( \det(x_i^{\lambda_j+k-j}) \), we have denoted the determinant of the \( k \times k \) matrix whose \((i, j)\) entry is equal to \( x_i^{\lambda_j+k-j} \). Note that the elementary symmetric polynomials are given by \( \pi_{1,0,0,\ldots,0}, \pi_{1,1,0,\ldots,0}, \pi_{1,1,1,\ldots,1} \). There are multiplication rules for the Schur polynomials which show that any \( \pi_{\lambda_1,\lambda_2,\ldots,\lambda_k} \) can be expressed in terms of the elementary symmetric polynomials.

If we do not specify the variables of the Schur polynomial \( \pi_\lambda \), we will assume that these are exactly \( x_1, \ldots, x_k \), with \( k \) being the length of \( \lambda \), i.e.

\[
\pi_{\lambda_1,\ldots,\lambda_k} := \pi_{\lambda_1,\ldots,\lambda_k}(x_1, \ldots, x_k).
\]

In this notes we only use Schur polynomials of two and three variables. In the case of two variables, the Schur polynomials are indexed by pairs of nonnegative integers \((i, j)\), such that \( i \geq j \), and (2) becomes

\[
\pi_{i,j} = \sum_{\ell=j}^{i} x_1^{i-j} x_2^{j-\ell}.
\]

Directly from Pieri’s formula we obtain the following multiplication rule for the Schur polynomials in two variables:

\[
\pi_{i,j} \pi_{a,b} = \sum \pi_{x,y},
\]

where the sum on the r.h.s. is over all indices \( x \) and \( y \) such that \( x+y = i+j+a+b \) and \( a+i \geq x \geq \max(a+j,b+i) \). Note that this implies \( \min(a+j,b+i) \geq y \geq b+j \). Also, we shall write \( \pi_{x,y} \in \pi_{i,j} \pi_{a,b} \) if \( \pi_{x,y} \) belongs to the sum on the r.h.s. of (3). Hence, we have that \( \pi_{x,x} \in \pi_{i,j} \pi_{a,b} \) iff \( a+j = b+i = x \) and \( \pi_{x+1,x} \in \pi_{i,j} \pi_{a,b} \) iff \( a+j = x+1, b+i = x \) or \( a+j = x, b+i = x+1 \).

We shall need the following combinatorial result which expresses the Schur polynomial in three variables as a combination of Schur polynomials of two variables. For \( i \geq j \geq k \geq 0 \), and the triple \((a,b,c)\) of nonnegative integers, we define

\[
(a,b,c) \sqsubset (i,j,k),
\]

if \( a+b+c = i+j+k, i \geq a \geq j \) and \( j \geq b \geq k \). We note that this implies that \( i \geq c \geq k \), and hence \( \max\{a,b,c\} \leq i \).
Lemma 3.1.

\[ \pi_{i,j,k}(x_1,x_2,x_3) = \sum_{(a,b,c) \in (i,j,k)} \pi_{a,b}(x_1,x_2)x_3^c, \]

Proof. From the definition of the Schur polynomial, we have

\[ \pi_{i,j,k}(x_1,x_2,x_3) = \frac{(x_1x_2x_3)^k}{(x_1-x_2)(x_1-x_3)(x_2-x_3)} \det \begin{pmatrix} x_1^{i-k+1} & x_1^{j-k+1} & 1 \\ x_2^{i-k+1} & x_2^{j-k+1} & 1 \\ x_3^{i-k+1} & x_3^{j-k+1} & 1 \end{pmatrix}. \]

After subtracting the last row from the first and the second one of the last determinant, we obtain

\[ \pi_{i,j,k} = \frac{(x_1x_2x_3)^k}{(x_1-x_2)(x_1-x_3)(x_2-x_3)} \det \begin{pmatrix} x_1^{i-k+2} - x_3^{i-k+1} & x_1^{j-k+1} - x_3^{j-k+1} \\ x_2^{i-k+2} - x_3^{i-k+1} & x_2^{j-k+1} - x_3^{j-k+1} \end{pmatrix}, \]

and so

\[ \pi_{i,j,k} = \frac{(x_1x_2x_3)^k}{x_1-x_2} \sum_{m=0}^{i-k+1} \sum_{n=0}^{j-k+1-m} (x_1^{m}x_2^{n} - x_1^{n}x_2^{m}) x_3^{i+j-2k+1-m-n}. \]

Finally, after expanding the last determinant we obtain

(4)

\[ \pi_{i,j,k} = \frac{(x_1x_2x_3)^k}{x_1-x_2} \sum_{m=0}^{i-k+1} \sum_{n=0}^{j-k+1-m} (x_1^{m}x_2^{n} - x_1^{n}x_2^{m}) x_3^{i+j-2k+1-m-n}. \]

We split the last double sum into two: the first one when \( m \) goes from 0 to \( j-k \), denoted by \( S_1 \), and the other one when \( m \) goes from \( j-k+1 \) to \( i-k+1 \), denoted by \( S_2 \). To show that \( S_1 = 0 \), we split the double sum further into three parts: when \( m < n, m = n \) and \( m > n \). Obviously, each summand with \( m = n \) is equal to 0, while the summands of the sum for \( m < n \) are exactly the opposite of the summands of the sum for \( m > n \). Thus, by replacing only \( S_2 \) instead of the double sum in (4) and after rescaling the indices \( a = m + k - 1, b = n + k \), we get

\[ \pi_{a,b} = \frac{(x_1x_2x_3)^k}{x_1-x_2} \sum_{m=0}^{i-k+1} \sum_{n=0}^{j-k+1-m} (x_1^{m}x_2^{n} - x_1^{n}x_2^{m}) x_3^{i+j-2k+1-m-n} \]

\[ = \sum_{a=j}^{i} \sum_{b=k}^{j} \pi_{a,b} x_3^{i+j+k+a-b} = \sum_{(a,b,c) \in (i,j,k)} \pi_{a,b} x_3^c, \]

as wanted. \( \square \)

Of course there is a multiplication rule for three-variable Schur polynomials which is compatible with (3) and the lemma above, but we do not want to discuss it here. For details see [4].

3.2 The cohomology of partial flag varieties

In this notes the rational cohomology rings of partial flag varieties play an essential role. The partial flag variety \( Fl_{d_1,d_2,\ldots,d_l} \), for \( 1 \leq d_1 < d_2 < \ldots < d_l = N \), is defined by

\[ Fl_{d_1,d_2,\ldots,d_l} = \{ V_{d_1} \subset V_{d_2} \subset \ldots \subset V_{d_l} = \mathbb{C}^N | \dim(V_i) = i \}. \]

8
A special case is $Fl_{k,N}$, the Grassmannian variety of all $k$-planes in $\mathbb{C}^N$, also denoted $\mathcal{G}_{k,N}$. The dimension of the partial flag variety is given by

$$\dim Fl_{d_1,d_2,...,d_l} = N^2 - \sum_{i=1}^{l-1} (d_i+1 - d_i)^2 - d_1^2.$$ 

The rational cohomology rings of the partial flag varieties are well known and we only recall those facts that we need in this notes.

**Lemma 3.2.** $H(\mathcal{G}_{k,N})$ is isomorphic to the vector space generated by all $\pi_{i_1,i_2,\ldots}k$ modulo the relations

$$\pi_{N-k+1,0,\ldots,0} = 0, \quad \pi_{N-k+2,0,\ldots,0} = 0, \quad \ldots, \quad \pi_{N,0,\ldots,0} = 0,$$

where there are exactly $k-1$ zeros in the multi-indices of the Schur polynomials.

A consequence of the multiplication rules for Schur polynomials is that

**Corollary 3.3.** The Schur polynomials $\pi_{i_1,i_2,\ldots,i_k}$, for $N - k \geq i_1 \geq i_2 \geq \ldots \geq i_k \geq 0$, form a basis of $H(\mathcal{G}_{k,N})$.

Thus, the dimension of $H(\mathcal{G}_{k,N})$ is $\binom{N}{k}$, and up to a degree shift, its graded dimension is $\binom{N}{k}$.

Another consequence of the multiplication rules is that

**Corollary 3.4.** The Schur polynomials $\pi_{1,0,0,\ldots,0}, \pi_{1,1,0,\ldots,0}, \ldots, \pi_{1,1,1,\ldots,1}$ (the elementary symmetric polynomials) generate $H(\mathcal{G}_{k,N})$ as a ring.

Furthermore, we can introduce a non-degenerate trace form on $H(\mathcal{G}_{k,N})$ by giving its values on the basis elements

$$\varepsilon(\pi_\lambda) = \begin{cases} (-1)^{\frac{n-\lambda}{2}}, & \lambda = (N-k,\ldots,N-k) \\ 0, & \text{else} \end{cases}$$

This makes $H(\mathcal{G}_{k,N})$ into a commutative Frobenius algebra. One can compute the basis dual to $\{\pi_\lambda\}$ in $H(\mathcal{G}_{k,N})$, with respect to $\varepsilon$. It is given by

$$\hat{\pi}_{\lambda_1,\ldots,\lambda_k} = (-1)^{\frac{n-\lambda}{2}} \pi_{N-k,\lambda_k,\ldots,N-k,\lambda_1}.$$ 

We can also express the cohomology rings of the partial flag varieties $Fl_{1,2,N}$ and $Fl_{2,3,N}$ in terms of Schur polynomials. Indeed, we have

$$H(Fl_{1,2,N}) = \mathbb{Q}[x_1,x_2]/(\pi_{N-1,0},\pi_{N,0}),$$

$$H(Fl_{2,3,N}) = \mathbb{Q}[x_1+x_2,x_1x_2]/(\pi_{N-2,0,0},\pi_{N-1,0,0},\pi_{N,0,0}).$$

The natural projection map $p_1: Fl_{1,2,N} \to \mathcal{G}_{2,N}$ induces

$$p_1^*: H(\mathcal{G}_{2,N}) \to H(Fl_{1,2,N}),$$

which is just the inclusion of the polynomial rings. Analogously, the natural projection map $p_2: Fl_{2,3,N} \to \mathcal{G}_{3,N}$, induces

$$p_2^*: H(\mathcal{G}_{3,N}) \to H(Fl_{2,3,N}),$$

which is also given by the inclusion of the polynomial rings.
4 Foams

In this section we begin to define the foams we will work with (foams were called pre-foams in [11] and in [17]. This distinction is irrelevant for the purposes of this notes). The philosophy behind these foams will be explained in Section 5. The basic examples of foams are given in Figure 1. These foams are composed of three types of facets: simple, double and triple facets. The double facets are coloured and the triple facets are marked to show the difference. Intersecting such a foam with a plane results in a web, as long as the plane avoids the singularities where six facets meet, such as on the right in Figure 1. Recall that a web is a planar trivalent graph with three types of edges: simple, double and triple which contain closed loops (simple, double, triple) and that only the simple edges are equipped with an orientation.

We adapt the definition of a world-sheet foam given in [15] to our setting.

**Definition 4.1.** Let \( s_γ \) be a finite closed oriented 4-valent graph, which may contain disjoint circles. We assume that all edges of \( s_γ \) are oriented. A cycle in \( s_γ \) is defined to be a circle or a closed sequence of edges which form a piece-wise linear circle. Let \( \Sigma \) be a compact orientable possibly disconnected surface, whose connected components are white, coloured or marked, also denoted by simple, double or triple. Each component can have a boundary consisting of several disjoint circles and can have additional decorations which we discuss below. A closed foam \( u \) is the identification space \( \Sigma/s_γ \) obtained by gluing boundary circles of \( \Sigma \) to cycles in \( s_γ \) such that every edge and circle in \( s_γ \) is glued to exactly three boundary circles of \( \Sigma \) and such that for any point \( p \in s_γ \):

1. if \( p \) is an interior point of an edge, then \( p \) has a neighborhood homeomorphic to the letter \( Y \) times an interval with exactly one of the facets being double, and at most one of them being triple. For an example see Figure 1;

2. if \( p \) is a vertex of \( s_γ \), then it has a neighborhood as shown on the r.h.s. in Figure 1.

We call \( s_γ \) the singular graph, its edges and vertices singular arcs and singular vertices, and the connected components of \( u−s_γ \) the facets.

Furthermore the facets can be decorated with dots. A simple facet can only have black dots (\( \bullet \)), a double facet can also have white dots (\( \bigcirc \)), and a triple facet besides black and white dots can have double dots (\( \varnothing \)). Dots can move freely on a facet but are not allowed to cross singular arcs. See Figure 2 for examples of foams.
Figure 2: a) A foam. b) An open foam

Note that the cycles to which the boundaries of the simple and the triple facets are glued are always oriented, whereas the ones to which the boundaries of the double facets are glued are not. Note also that there are two types of singular vertices. Given a singular vertex $v$, there are precisely two singular edges which meet at $v$ and bound a triple facet: one oriented toward $v$, denoted $e_1$, and one oriented away from $v$, denoted $e_2$. If we use the “left hand rule”, then the cyclic ordering of the facets incident to $e_1$ and $e_2$ is either $(3, 2, 1)$ and $(3, 1, 2)$ respectively, or the other way around. We say that $v$ is of type I in the first case and of type II in the second case (see Figure 3). When we go around a triple facet we see that there have to be as many singular vertices of type I as there are of type II for the cyclic orderings of the facets to match up. This shows that for a closed foam the number of singular vertices of type I is equal to the number of singular vertices of type II.

We can intersect a foam $u$ generically by a plane $W$ in order to get a web, as long as the plane avoids the vertices of $s_\gamma$. The orientation of $s_\gamma$ determines the orientation of the simple edges of the web according to the convention in Figure 4.

Suppose that for all but a finite number of values $i\in]0, 1[$, the plane $W \times i$ intersects $u$ generically. Suppose also that $W \times 0$ and $W \times 1$ intersect $u$ generically and outside the vertices of $s_\gamma$. We call $W \times I \cap u$ an open foam. Interpreted as morphisms we read open foams from bottom to top, and their composition consists of placing one foam on top of the other, as long as their boundaries are isotopic and the orientations of the simple edges coincide.

We now define the $q$-degree of a foam. Let $u$ be a foam, $u_1$, $u_2$ and $u_3$ the disjoint union of its simple and double and marked facets respectively and $s_\gamma(u)$ its singular graph. Define the
Figure 4: Orientations near a singular arc

Partial $q$-gradings of $u$ as

$$q_i(u) = \chi(u_i) - \frac{1}{2}\chi(\partial u_i \cap \partial u), \quad i = 1, 2, 3$$

$$q_s(y(u)) = \chi(s_y(u)) - \frac{1}{2}\chi(\partial s_y(u)).$$

where $\chi$ is the Euler characteristic and $\partial$ denotes the boundary.

**Definition 4.2.** Let $u$ be a foam with $d_\bullet$ dots of type $\bullet$, $d_\circ$ dots of type $\circ$ and $d_\bigcirc$ dots of type $\bigcirc$. The $q$-grading of $u$ is given by

$$q(u) = -\sum_{i=1}^{3} i(N - i)q_i(u) - 2(N - 2)q_s(y(u)) + 2d_\bullet + 4d_\circ + 6d_\bigcirc.$$

The following result is a direct consequence of the definitions.

**Lemma 4.3.** $q(u)$ is additive under the glueing of foams.

## 5 The KL formula and the evaluation of closed foams

Let us briefly recall the philosophy behind the foams. Loosely speaking, to each closed foam should correspond an element in the cohomology ring of a configuration space of planes in some big $\mathbb{C}^M$. The singular graph imposes certain conditions on those planes. The evaluation of a foam should correspond to the evaluation of the corresponding element in the cohomology ring. Of course one would need to find a consistent way of choosing the volume forms on all of those configuration spaces for this to work. However, one encounters a difficult technical problem when working out the details of this philosophy. Without explaining all the details, we can say that the problem can only be solved by figuring out what to associate to the singular vertices. Ideally we would like to find a combinatorial solution to this problem, but so far it has eluded us. That is the reason why we are forced to use the KL formula.

We denote a simple facet with $i$ dots by $i$. 
Recall that \( \pi_{k,m} \) can be expressed in terms of \( \pi_{1,0} \) and \( \pi_{1,1} \). In the philosophy explained above, the latter should correspond to \( \bullet \) and \( \circ \) on a double facet respectively. We can then define

\[
\left\{ (k,m) \right\}
\]

as being the linear combination of dotted double facets corresponding to the expression of \( \pi_{k,m} \) in terms of \( \pi_{1,0} \) and \( \pi_{1,1} \). Analogously we expressed \( \pi_{p,q,r} \) in terms of \( \pi_{1,0,0} \), \( \pi_{1,1,0} \) and \( \pi_{1,1,1} \) (see Section 3). The latter correspond to \( \bullet, \circ \) and \( \circ \) on a triple facet respectively, so we can make sense of

\[
\left[ (p,q,r) \right].
\]

In the sequel, we shall give a definition of the KL formula for the evaluation of foams and state some of its basic properties. The KL formula was introduced by A. Kapustin and Y. Li [5] to generalize Vafa’s work [16] in the context of the evaluation of 2-dimensional TQFTs to the case of smooth surfaces with boundary. It was later extended to the case of foams by M. Khovanov and L. Rozansky in [7], who interpreted singular arcs as boundary conditions as in [5]. Khovanov and Rozansky adapted the KL formula to a general sort of foam. In this notes we have to specify the input data which allows us to use it for the evaluation of our foams. The normalization is ours and is used to obtain integral relations.

### 5.1 The general framework

Let \( u = \Sigma / s \gamma \) be a closed foam with singular graph \( s \gamma \) and without any dots on it. Let \( F \) denote an arbitrary \( i \)-facet, \( i \in \{1,2,3\} \), with a 1-facet being a simple facet, a 2-facet being a double facet and a 3-facet being a triple facet.

Each \( i \)-facet can be decorated with dots, which correspond to generators of the rational cohomology ring of the Grassmannian \( G^i_N, \) i.e. \( H(G^i_N, \mathbb{Q}) \). Alternatively, we can associate to every \( i \)-facet \( F, i \) variables \( x^F_1, \ldots, x^F_i \), with \( \deg x^F_i = 2i \), and the potential \( W(x^F_1, \ldots, x^F_i) \), which is the polynomial defined such that

\[
W(\sigma_1, \ldots, \sigma_i) = y_1^{N+1} + \ldots + y_i^{N+1},
\]

where \( \sigma_j \) is the \( j \)-th elementary symmetric polynomial in the variables \( y_1, \ldots, y_i \). The Jacobi algebra \( J_W \)

\[
J_W = \mathbb{Q}[x^F_1, \ldots, x^F_i] / (\partial_W),
\]

where \( \partial_W \) denote the ideal generated by the partial derivatives of \( W \), is isomorphic to the rational cohomology ring of the Grassmannian \( G^i_N, \). Note that up to a multiple the top degree nonvanishing element in this Jacobi algebra is \( \pi_{N-i, \ldots, N-i} \) (multiindex of length \( i \), i.e. the polynomial in variables \( x^F_1, \ldots, x^F_i \) which gives \( \pi_{N-i, \ldots, N-i} \) after replacing the variable \( x^F_j \) by \( 1 \)'s, \( 1 \leq j \leq i \) (see also Subsection 3.1). We define the trace (volume) form \( \varepsilon \) on \( H(G^i_N, \mathbb{Q}) \) by giving it on the basis of the Schur polynomials:

\[
\varepsilon(\pi_{j_1, \ldots, j_i}) = \begin{cases} (-1)^{\frac{i}{2}} & \text{if } (j_1, \ldots, j_i) = (N-i, \ldots, N-i) \\ 0 & \text{else} \end{cases}.
\]

The KL formula associates to \( u \) an element in the product of the cohomology rings of the Jacobi algebras \( J \), over all the facets in the foam. Alternatively, we can see this element as a
polynomial, $KL_u \in J$, in all the variables associated to the facets. Now, let us put some dots on $u$. Recall that a dot corresponds to an elementary symmetric polynomial. So a linear combination of dots on $u$ is equivalent to a polynomial, $f$, in the variables of the dotted facets. Let $\varepsilon$ denote the product of the trace forms $\varepsilon_F$ over all facets of $u$. The value of this dotted foam we define to be

$$\langle u \rangle_{KL} := \varepsilon \left( \prod_F \frac{\det(\partial_i \partial_j W_F)^{g(F)}}{(N + 1)^{g'(F)}} KL_u f \right).$$

The product is over all facets $F$ and $W_F$ is the potential associated to $F$. For any $i$-facet $F$, $i = 1, 2, 3$, the symbol $g(F)$ denotes the genus of $F$ and $g'(F) = ig(F)$. If $u$ is a closed surface without singularities we define $KL_u = 1$ and $\langle \rangle_{KL}$ reduces to an extension to colored closed surfaces of the formula introduced by Vafa in [16]. The Vafa factor

$$\prod_F \frac{\det(\partial_i \partial_j W_F)^{g(F)}}{(N + 1)^{g'(F)}}$$

computes the contribution of the handles in the facets of $u$.

Having explained the general idea, we are left with defining the element $KL_u$ for a dotless foam. For that we have to explain Khovanov and Rozansky’s extension of the KL formula to foams [7], which uses the theory of matrix factorizations.

### 5.2 Decoration of foams

As we said, to each facet we associate certain variables (depending on the type of facet), a potential and the corresponding Jacobi algebra. If the variables associated to a facet $F$ are $x_1, \ldots, x_i$, then we define $R_F = \mathbb{Q}[x_1, \ldots, x_i]$. It is immediate that the KL formula gives zero if the argument of $\varepsilon$ in Equation 11 contains an element of $\partial_i W_F$: for any $Q \in \otimes F R_F$ we have that

$$\varepsilon(Q \partial_i W_F) = 0.$$

Now we consider the edges. To each edge we associate a matrix factorization whose potential is equal to the signed sum of the potentials of the facets that are glued along this edge. We define it to be a certain tensor product of Koszul factorizations. In the cases we are interested in there are always three facets glued along an edge, with two possibilities: either two simple facets and one double facet, or one simple, one double and one triple facet. In the first case, we denote the variables of the two simple facets by $x$ and $y$ and take the potentials to be $x^{N+1}$ and $y^{N+1}$ respectively, according to the convention in Figure 5. To the double facet we associate the variables $s$ and $t$ and the potential $W(s, t)$. To the edge we associate the matrix factorization which is the tensor product of Koszul factorizations given by

$$MF = \left\{ A', \begin{array}{c} x + y - s \\ x y - t \end{array} \right\},$$

where $A'$ and $B'$ are given by

$$A' = \frac{W(x + y, xy) - W(s, xy)}{x + y - s},$$

$$B' = \frac{W(s, xy) - W(s, t)}{x y - t}.$$
Note that \((x + y - s)A' + (xy - t)B' = x^{N+1} + y^{N+1} - W(s, t)\).

In the second case, the variable of the simple facet is \(x\) and the potential is \(x^{N+1}\), the variables of the double facet are \(s\) and \(t\) and the potential is \(W(s, t)\), and the variables of the triple face are \(p, q\) and \(r\) and the potential is \(W(p, q, r)\).

Define the polynomials

\[
A = \frac{W(x + s, xs + t, xt) - W(p, xs + t, xt)}{x + s - p},
\]

\[
B = \frac{W(p, xs + t, xt) - W(p, q, xt)}{xs + t - q},
\]

\[
C = \frac{W(p, q, xt) - W(p, q, r)}{xt - r},
\]

so that

\[(x + s - p)A + (xs + t - q)B + (xt - r)C = x^{N+1} + W(s, t) - W(p, q, r).\]

To such an edge we associate the matrix factorization given by the following tensor product of Koszul factorizations:

\[
MF_2 = \begin{cases} 
A, \quad x + s - p \\
B, \quad xs + t - q \\
C, \quad xt - r 
\end{cases}
\]

In both cases, if the edges have the opposite orientation we associate the matrix factorizations \((MF_1)_*\) and \((MF_2)_*\) respectively.
Next we explain what we associate to a singular vertex. First of all, for each vertex \( v \), we define its local graph \( \gamma_v \) to be the intersection of a small sphere centered at \( v \) with the foam. Then the vertices of \( \gamma_v \) correspond to the edges of \( u \) that are incident to \( v \), to which we had associated matrix factorizations. In this notes all local graphs \( \gamma_v \) are in fact tetrahedrons. However, recall that there are two types of vertices (see the remarks below Definition 4.1). Label the six facets that are incident to a vertex \( v \) by the numbers 1, 2, 3, 4, 5 and 6. Furthermore, denote the edge along which are glued the facets \( i, j \) and \( k \) by \( (ijk) \). Denote the matrix factorization associated to the edge \( (ijk) \) by \( M_{ijk} \), if the edge points toward \( v \), and by \( (M_{ijk})^* \), if the edge points away from \( v \). Note that \( M_{ijk} \) and \( (M_{ijk})^* \) are both defined over \( R_i \otimes R_j \otimes R_k \).

Now we can take the tensor product of these four matrix factorizations, over the polynomial rings of the facets of the foam, that correspond to the vertices of \( \gamma_v \). This way we obtain the matrix factorization \( M_v \), whose potential is equal to 0, and so it is a 2-complex and we can take its homology.

To each vertex \( v \) we associate an element \( O_v \in \text{HMF}(M_v) \). More precisely, if \( v \) is of type I, then

\[
\text{HMF}(M_v) \cong \text{Ext}(MF_1(x,y,s_1,t_1) \otimes s_1s_1, MF_2(z,s_1,t_1,p,q,r), MF_1(y,z,s_2,t_2) \otimes s_2s_2, MF_2(x,s_2,t_2,p,q,r)).
\]

(18)

If \( v \) is of type II, then

\[
\text{HMF}(M_v) \cong \text{Ext}(MF_1(y,z,s_2,t_2) \otimes s_2s_2, MF_2(x,s_2,t_2,p,q,r), MF_1(x,y,s_1,t_1) \otimes s_1s_1, MF_2(z,s_1,t_1,p,q,r)).
\]

(19)

Both isomorphisms hold up to a global shift in \( q \). Note that

\[
MF_1(x,y,s_1,t_1) \otimes s_1s_1, MF_2(z,s_1,t_1,p,q,r) \simeq MF_1(y,z,s_2,t_2) \otimes s_2s_2, MF_2(x,s_2,t_2,p,q,r),
\]

because both tensor products are homotopy equivalent to the factorization

\[
\begin{cases}
\ast, & x + y + z - p \\
\ast, & xy + xz + yz - q \\
\ast, & xyz - r
\end{cases}
\]

We have not specified the l.h.s. of the latter Koszul matrix, because of Lemma 2.3. If \( v \) is of type I, we take \( O_v \) to be the cohomology class of a fixed degree 0 homotopy equivalence

\[
w_v: MF_1(x,y,s_1,t_1) \otimes s_1s_1, MF_2(z,s_1,t_1,p,q,r) \rightarrow MF_1(y,z,s_2,t_2) \otimes s_2s_2, MF_2(x,s_2,t_2,p,q,r).
\]

The choice of \( O_v \) is unique up to a scalar, because the graded dimension of the Ext-group in (18) is equal to

\[
q^{3N-6} \text{qdim}(H(M_v)) = q^{3N-6} |N| |N - 1| |N - 2| = 1 + q(\ldots),
\]

where \( (\ldots) \) is a polynomial in \( q \). Note that \( M_v \) is homotopy equivalent to the matrix factorization which corresponds to the closure of \( \gamma \) in [9], which allows one to compute the graded dimension above using the results in the latter paper. If \( v \) is of type II, we take \( O_v \) to be the cohomology class of the homotopy inverse of \( w_v \). Note that a particular choice of \( w_v \) fixes \( O_v \) for both types of vertices and that the value of the KL formula for a closed foam does not depend on that choice because there are as many singular vertices of type I as there are of type II (see the remarks below Definition 4.1). We do not know an explicit formula for \( O_v \). Although such a formula would be very interesting to have, we do not need it for the purposes of this notes.
5.3 The KL derivative and the evaluation of closed foams

From the definition, every boundary component of each facet $F$ is either a circle or a cyclicly ordered finite sequence of edges, such that the beginning of the next edge corresponds to the end of the previous edge. For every boundary component choose an edge $e$ and denote the differential of the matrix factorization associated to this edge by $D_e$. Let $R_F = \mathbb{Q}[x_1, \ldots, x_k]$. The KL derivative of $D_e$ in the variables $x_1, \ldots, x_k$ associated to the facet $F$, is an element from $\text{End}(M) \cong M \otimes M^*$, given by:

\begin{equation}
O_{F,e} = \partial D_e^\ast = \frac{1}{k!} \sum_{\sigma \in S_k} (\text{sgn} \sigma) \partial_{\sigma(1)} D_e \partial_{\sigma(2)} D_e \cdots \partial_{\sigma(k)} D_e,
\end{equation}

where $S_k$ is the symmetric group on $k$ letters, and $\partial D$ is the partial derivative of $D$ with respect to the variable $x_i$. For all the other edges $e'$ in the boundary of $F$ we take $O_{F,e'}$ to be the identity. Denote the set of facets whose boundary contains $e$ by $F(e)$. For every edge define $O_e \in \text{End}(M)$ as the composite

$$O_e = \prod_{F \in F(e)} O_{F,e}.$$ 

The order of the factors in $O_e$ is irrelevant as we will prove it in Lemma 5.2.

Let $\mathcal{V}$ and $\mathcal{E}$ be the sets of all vertices and all edges of the singular graph $s_\gamma$ of a foam $u$. Denote the matrix factorization associated to an edge $e$ by $M_e$ ($M_e = MF_1$ if $e$ is of type $(1,1,2)$ and $M_e = MF_2$ if $e$ is of type $(1,2,3)$). Recall that the factorization $M_v$ associated to a singular vertex is the tensor product of the matrix factorizations associated to the edges that are incident to $v$. Consider the factorization $M_{s_\gamma}$ given by the tensor product

\begin{equation}
M_{s_\gamma} = \left( \bigotimes_{v \in \mathcal{V}} M_v \right) \otimes \left( \bigotimes_{e \in \mathcal{E}} M_e \otimes (M_e)_{\ast} \right).
\end{equation}

From the definition of $M_v$ we see that we can group all the factorizations involved in pairs of mutually dual factorizations: for every edge $e$ we can pair $M_e$ coming from $M_e \otimes (M_e)_{\ast}$ with $(M_e)_{\ast}$ coming from $M_e$ and $(M_e)_{\ast}$ from $M_e \otimes (M_e)_{\ast}$ can be paired with $M_e$ coming from $M_v$. Using super-contraction on each pair we get a map

$$\phi_\gamma: M_{s_\gamma} \to \mathbb{Q}[x_u],$$

where $x_u$ is the set of variables associated to all the facets of $u$.

**Definition 5.1.** $KL_u = \phi_\gamma \left( \bigotimes_{v \in \mathcal{V}} O_v \otimes \left( \bigotimes_{e \in \mathcal{E}} O_e \right) \right)$.

Note that the $O_e$ and $O_v$ can be seen as tensors with indices associated to the facets that meet at $e$ and $v$ respectively. So we can super-contract all the tensor factors $O_e$ and $O_v$, with respect to a particular facet $F$, along a cycle that bounds $F$. From Definition 5.1 we see that if we do this for all boundary components of all facets we also get $KL_u$.

**Lemma 5.2.** $KL_u$ does not depend on the order of the factors in $O_e$. 

17
Proof. Let $e$ be an edge in the boundary of facets $F$ and $F'$. Since the potential $W_e$ is a sum of the individual potentials associated to the facets that are glued along $e$, each depending on its own set of variables, we have $\partial_i \partial'_j W_e = 0$. Therefore, applying $\partial_i \partial'_j$ to both sides of the relation $D^2_e = W_e$ gives

$$[\partial_i D_e, \partial'_j D_e]_s = -[D_e, \partial_i \partial'_j D_e]_s,$$

and the term on the r.h.s. is annihilated after the super-contraction because it is a coboundary. This means that the KL derivatives of $D$ w.r.t. different facets super-commute. \hfill \square

**Lemma 5.3.** KL$_u$ does not depend on the choice of the preferred edges.

**Proof.** It suffices to prove the claim for only one facet $F$ with one boundary component. Label the edges that bound $F$ by $e_1, \ldots, e_k$ and take $e_1$ as the preferred edge of $F$. Suppose first that $F$ is a simple or a triple facet, so that its boundary consists of an oriented cycle of $s_Y$. Suppose also that $e_1$ is oriented from $v_i$ to $v_{i+1}$. Since $[O_{F,e_1}, O_{F',e_1}]_s = 0$ for every $F' \neq F$ we can assume that $O_{e_1} = O_{F,e_1}$ without loss of generality. The contribution to KL$_u$ of the facet $F$ is given by

$$\text{STr}_{W_e} (\partial D_{e_1}^\gamma O_{v_1} O_{v_2} \ldots O_{v_k}),$$

where STr$_{W_e}$ is the partial supertrace w.r.t. the indices associated to $F$.

The relevant part of a small neighborhood of the vertex $v_1$ is depicted in Figure 7, where only the facet $F$ is shown. From Equation (18) it follows that $O_v$ can be seen as a homomorphism from $M_e(e_1) \otimes M_e(e'_1)$ to $M_e(e_2) \otimes M_e(e'_2)$, where $(e_i)$ denotes the variables associated to the facets that are glued along $e$. Therefore we have that $[D, O_v]_s = 0$, where $D = D_{e_1} + D_{e'_1} + D_{e_2} + D_{e'_2}$ and we are using the convention that the composite of two non-composable homomorphisms is zero. Note that $\partial_i D = \partial_i D_{e_1} + \partial_i D_{e_2}$ since $e'_1$ and $e'_2$ are not variables associated to $F$. Therefore $[D, O_v]_s = 0$ implies

$$[\partial_i D, O_v]_s = -[D, \partial_i O_v]_s$$

by partial differentiation w.r.t. a variable of $F$. This implies

$$\text{STr}_{W_e} (\partial D_{e_1}^\gamma O_{v_1} O_{v_2} \ldots O_{v_k}) = \text{STr}_{W_e} (O_{v_1} \partial D_{e_2}^\gamma O_{v_2} \ldots O_{v_k}),$$

since terms involving the r.h.s. of Equation (22) get killed by STr.

Now suppose that $F$ is a double facet. The boundary of $F$ is not an oriented cycle in $s_Y$. Suppose a small neighborhood of $v$ has a part as depicted in Figure 8. In this case $O_v$ can be seen as a homomorphism from $M_e(e_1) \otimes M_e(e'_1)_\bullet$ to $M_e(e_2) \otimes M_e(e'_2)_\bullet$, so that $D$ and $O_v$
super-commute, where \( D = D_{e_1} + (D_{e_1})_+ + (D_{e_2})_+ + D_{e_2} \). Taking a partial derivative of both sides of the relation \([D, O_v]_s\) relative to a variable associated to \( F \) we obtain that

\[
\text{STr}_{\mathcal{W}_F}(\partial D_{e_1}^O v_1 O_{v_2} \ldots O_{v_k}) = \text{STr}_{\mathcal{W}_F}(O_{v_1} \partial(D_{e_2})_+ O_{v_2} \ldots O_{v_k}),
\]

which proves the claim.

\[\square\]

### 5.4 Some computations

In this subsection we compute the KL evaluation of some closed foams.

**Spheres**

The values of dotted spheres are easy to compute. Note that for any sphere with dots \( f \) the KL formula gives

\[\varepsilon(f).\]

Therefore for a simple sphere we get 1 if \( f = x^{N-1} \), for a double sphere we get \(-1\) if \( f = \pi_{N-2,N-2} \) and for a triple sphere we get \(-1\) if \( f = \pi_{N-3,N-3,N-3} \).

Note that the evaluation of spheres corresponds to the trace on the cohomology of the Grassmannian \( H(\mathcal{G}_{i,N}) \) for \( i = 1, 2, 3 \) in Equation (7).

**Dot conversion and dot migration**

Since \( KL_u \) takes values in the tensor product of the Jacobi algebras of the potentials associated to the facets of \( u \), we see that for a simple facet we have \( x^N = 0 \), for a double facet \( \pi_{i,j} = 0 \) if \( i \geq N-1 \), and for a triple facet \( \pi_{p,q,r} = 0 \) if \( p \geq N-2 \). We call these the *dot conversion relations*:

\[
\begin{align*}
\begin{array}{c}
\text{i} \\\\\\\\\\\text{0} \quad \text{if} \quad i \geq N \\
\hline \\
\begin{array}{c}
\text{(k,m)} \\\\\\\\\\\text{0} \quad \text{if} \quad k \geq N-1 \\
\hline \\
\begin{array}{c}
\text{(p,q,r)} \\\\\\\\\\\text{0} \quad \text{if} \quad p \geq N-2 \\
\end{array}
\end{array}
\end{align*}
\]

The dot conversion relations are related to the relations defining the cohomology ring of the Grassmannian \( \mathcal{G}_{k,N} \) for \( k = 1, 2, 3 \) in Equation (5).
To each edge along which two simple facets with variables \( x \) and \( y \) and one double facet with the variables \( s \) and \( t \) are glued, we associated the matrix factorization \( MF_1 \) with entries \( x + y - s \) and \( xy - t \). Therefore \( \text{Ext}(MF_1, MF_1) \) is a module over \( R/(x + y - s, xy - t) \). Hence, we obtain the dot migration relations along this edge. Analogously, to the other type of singular edge along which are glued a simple facet with variable \( x \), a double facet with variable \( s \) and \( t \), and a triple facet with variables \( p, q \) and \( r \), we associated the matrix factorization \( MF_2 \). Note that \( \text{Ext}(MF_2, MF_2) \) is a module over \( R/(x + s - p, xs + t - q, xt - r) \), which gives us the dot migration relations along this edge:

\[
\begin{align*}
&\begin{array}{c}
\star \\
\circ
\end{array} =
\begin{array}{c}
\star \\
\circ
\end{array} + \begin{array}{c}
\star \\
\circ
\end{array} \\
&\begin{array}{c}
\circ
\end{array} =
\begin{array}{c}
\star \\
\circ
\end{array} + \begin{array}{c}
\star \\
\circ
\end{array} \\
&\begin{array}{c}
\star \\
\circ
\end{array} =
\begin{array}{c}
\star \\
\circ
\end{array} + \begin{array}{c}
\star \\
\circ
\end{array} \\
&\begin{array}{c}
\star \\
\circ
\end{array} =
\begin{array}{c}
\star \\
\circ
\end{array} + \begin{array}{c}
\star \\
\circ
\end{array}
\end{align*}
\]

The dot migration relations are related to the relations in the cohomology ring of the partial flag varieties \( Fl_{1,2,N} \) and \( Fl_{2,3,N} \) in Equation (8) under the projection maps in Equations (9) and (10).

**The \((1,1,2)\)-theta foam**

Recall that \( W(s,t) \) is the polynomial such that \( W(x+y,xy) = x^{N+1} + y^{N+1} \). More precisely, we have

\[
W(s,t) = \sum_{i+2j=N+1} a_{ij}s^it^j,
\]

with \( a_{N+1,0} = 1, a_{N+1-2j,j} = (-1)^j(N+1)\binom{N-j}{j-1}, \) for \( 2 \leq 2j \leq N + 1 \), and \( a_{ij} = 0 \) otherwise. In particular \( a_{N-1,1} = -(N+1) \). We have

\[
\begin{align*}
W_1'(s,t) &= \sum_{i+2j=N+1} ia_{ij}s^{i-1}t^j, \\
W_2'(s,t) &= \sum_{i+2j=N+1} ja_{ij}s^it^{i-1}.
\end{align*}
\]

By \( W_1'(s,t) \) and \( W_2'(s,t) \), we denote the partial derivatives of \( W(s,t) \) with respect to the first and the second variable, respectively.
Consider the \((1, 1, 2)\)-theta foam of Figure 9. According to the conventions of Subsection 5.2 we have variables \(x\) and \(y\) on the lower and upper simple facets respectively, and the variables \(s\) and \(t\) on the double facet. To the singular circle we assign the matrix factorization

\[
MF_1 = \left\{ A', \begin{array}{c} x+y-s \\ xy-t \end{array} \right\},
\]

Recall that

\[
A' = \frac{W(x+y,xy) - W(s,xy)}{x+y-s}, \tag{23}
\]

\[
B' = \frac{W(s,xy) - W(s,t)}{xy-t}. \tag{24}
\]

Hence, the differential of this matrix factorization is given by the following 4 by 4 matrix:

\[
D = \begin{pmatrix} 0 & D_1 \\ D_0 & 0 \end{pmatrix},
\]

where

\[
D_0 = \begin{pmatrix} A' & xy-t \\ B' & s-x-y \end{pmatrix}, \quad D_1 = \begin{pmatrix} x+y-s, & xy-t \\ B', & -A' \end{pmatrix}.
\]

Note that we are using a convention for tensor products of matrix factorizations that is different from the one in [11]. The KL formula assigns the polynomial, \(KL_{\Theta_1}(x,y,s,t)\), which is given by the supertrace of the twisted differential of \(D\)

\[
KL_{\Theta_1} = STr \left( \partial_3 D_3 D_2 \left( \frac{1}{2} (\partial_2 D_2 D - \partial_2 D_3 D) \right) \right).
\]

Straightforward computation gives

\[
KL_{\Theta_1} = -B_s'(A' \cdot -A'_y) - (A'_x + A'_y)(B'_y + xB'_x) + (A'_y + A'_x)(B'_x + yB'_y), \tag{25}
\]

where by \(A'_i\) and \(B'_i\) we have denoted the partial derivatives with respect to the variable \(i\). From
the definitions (23) and (24) we have

\[
\begin{align*}
A'_x - A'_y &= (y - x) \frac{W'_2(x + y, xy) - W'_2(s, xy)}{x + y - s}, \\
A'_x + A'_y &= \frac{W'_1(x + y, xy) - W'_1(s, xy) + y(W'_2(x + y, xy) - W'_2(s, xy))}{x + y - s}, \\
A'_y + A'_s &= \frac{W'_1(x + y, xy) - W'_1(s, xy) + x(W'_2(x + y, xy) - W'_2(s, xy))}{x + y - s}, \\
B'_x &= \frac{W'_2(s, xy) - W'_1(s, t)}{xy - t}, \\
B'_x + yB'_t &= y \frac{W'_2(s, xy) - W'_2(s, t)}{xy - t}, \\
B'_y + xB'_t &= x \frac{W'_2(s, xy) - W'_2(s, t)}{xy - t}.
\end{align*}
\]

After substituting this back into (25), we obtain

\[
(26) \quad KL_{\Theta_1} = (x - y) \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},
\]

where

\[
\begin{align*}
\alpha &= \frac{W'_1(x + y, xy) - W'_1(s, xy)}{x + y - s}, \\
\beta &= \frac{W'_2(x + y, xy) - W'_2(s, xy)}{x + y - s}, \\
\gamma &= \frac{W'_1(s, xy) - W'_1(s, t)}{xy - t}, \\
\delta &= \frac{W'_2(s, xy) - W'_2(s, t)}{xy - t}.
\end{align*}
\]

From this formula we see that \(KL_{\Theta_1}\) is homogeneous of degree \(4N - 6\) (remember that \(\text{deg}\, x = \text{deg}\, y = \text{deg}\, s = 2\) and \(\text{deg}\, t = 4\)).

Since the evaluation is in the product of the Grassmannians corresponding to the three disks, i.e. in the ring \(\mathbb{Q}_s[x]/(x^N) \times \mathbb{Q}_y/(y^N) \times \mathbb{Q}_t/(W'_1(s, t), W'_2(s, t))\), we have \(x^N = y^N = 0 = W'_1(s, t) = W'_2(s, t)\). Also, we can express the monomials in \(s\) and \(t\) as linear combinations of the Schur polynomials \(\pi_{k,l}\) (writing \(s = \pi_{1,0}\) and \(t = \pi_{1,1}\)), and we have \(W'_1(s, t) = (N + 1)\pi_{N,0}\) and \(W'_2(s, t) = -(N + 1)\pi_{N-1,0}\). Hence, we can write \(KL_{\Theta_1}\) as

\[
KL_{\Theta_1} = (x - y) \sum_{N-2 \geq k \geq l \geq 0} \pi_{k,l} p_{kl}(x, y),
\]

with \(p_{kl}\) being a polynomial in \(x\) and \(y\). We want to determine which combinations of dots on the simple facets give rise to non-zero evaluations, so our aim is to compute the coefficient of \(\pi_{N-2,N-2}\) in the sum on the r.h.s. of the above equation (i.e. in the determinant in (26)). For degree reasons, this coefficient is of degree zero, and so we shall only compute the parts of \(\alpha\),
\[ \bar{\alpha} = (N + 1)s^{N-1}, \]
\[ \bar{\beta} = -(N + 1)s^{N-2}, \]
\[ \bar{\gamma} = \sum_{i+2j=N+1, j \geq 1} ia_j s^{i-1} t^{j-1}, \]
\[ \bar{\delta} = \sum_{i+2j=N+1, j \geq 1} ja_j s^i t^{j-2}. \]

Note that we have
\[ t \bar{\gamma} + (N + 1)s^N = W'_1(s, t), \]
and
\[ t \bar{\delta} - (N + 1)s^{N-1} = W'_2(s, t), \]
and so in the cohomology ring of the Grassmannian \( \mathcal{G}_{2,N} \), we have \( t \bar{\gamma} = -(N + 1)s^N \) and \( t \bar{\delta} = (N + 1)s^{N-1} \). On the other hand, by using \( s = \pi_{1,0} \) and \( t = \pi_{1,1} \), we obtain that in \( H(\mathcal{G}_{2,N}) \cong \mathbb{Q}[s, t]/(\pi_{N-1,0}, \pi_{N,0}) \), the following holds:
\[ s^{N-2} = \pi_{N-2,0} + tq(s, t), \]
for some polynomial \( q \), and so
\[ s^{N-1} = s^{N-2} s = \pi_{N-1,0} + \pi_{N-2,1} + stq(s, t) = t(\pi_{N-3,0} + sq(s, t)). \]

Thus, we have
\[
\det \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix} = (N + 1)(\pi_{N-3,0} + sq(s, t))t \bar{\delta} + (N + 1)\pi_{N-2,0} \bar{\gamma} + (N + 1)q(s, t) t \bar{\gamma} \\
= (N + 1)^2(\pi_{N-3,0} + sq(s, t))s^{N-1} + (N + 1)\pi_{N-2,0} \bar{\gamma} - (N + 1)^2 q(s, t)s^N \\
= (N + 1)^2 \pi_{N-3,0} s^{N-1} + (N + 1)\pi_{N-2,0} \bar{\gamma}.
\]

(27)

Since
\[ \bar{\gamma} = (N - 1)a_{N-1,1}s^{N-2} + tr(s, t) \]
holds in the cohomology ring of the Grassmannian \( \mathcal{G}_{2,N} \) for some polynomial \( r(s, t) \), we have
\[ \pi_{N-2,0} \bar{\gamma} = \pi_{N-2,0}(N - 1)a_{N-1,1}s^{N-2} = -\pi_{N-2,0}(N - 1)(N + 1)s^{N-2}. \]

Also, we have that for every \( k \geq 2 \),
\[ s^k = \pi_{k,0} + (k - 1)\pi_{k-1,1} + t^2w(s, t), \]
for some polynomial \( w \). Replacing this in (27) and bearing in mind that \( \pi_{i,j} = 0 \), for \( i \geq N - 1 \), we get
\[
\det \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix} = (N + 1)^2 s^{N-2}(\pi_{N-2,0} + \pi_{N-3,1} - (N - 1)\pi_{N-2,0}) \\
= (N + 1)^2(\pi_{N-2,0} + (N - 3)\pi_{N-3,1} + \pi_{2,2}w(s, t))(\pi_{N-3,1} - (N - 2)\pi_{N-2,0}) \\
= - (N + 1)^2 \pi_{N-2,0}. \]
Hence, we have
\[ KL_{θ_1} = (N + 1)^2 (y - x) π_{N-2,N-2} + \sum_{N-2\geq i \geq 0} c_{i,j,k,l} π_{i,j} x^i y^j. \]

Recall that in the product of the Grassmannians corresponding to the three disks, i.e. in the ring \( \mathbb{Q}[x]/(x^N) \times \mathbb{Q}[y]/(y^N) \times \mathbb{Q}[s,t]/(π_{N-1,0}, π_{N,0}) \), we have
\[ \varepsilon(x^{N-1} y^{N-1} π_{N-2,N-2}) = -1. \]
Therefore the only monomials \( f \) in \( x \) and \( y \) such that \( (KL_{θ_1}f)_{KL} \neq 0 \) are \( f_1 = x^{N-1} y^{N-2} \) and \( f_2 = x^{N-2} y^{N-1} \), and \( (KL_{θ_1}f_1)_{KL} = -(N + 1)^2 \) and \( (KL_{θ_1}f_2)_{KL} = (N + 1)^2 \). Thus, we have that the value of the theta foam with unlabelled 2-facet is nonzero only when the upper 1-facet has \( N - 2 \) dots and the lower one has \( N - 1 \) dots (and has the value \((N + 1)^2\)) and when the upper 1-facet has \( N - 1 \) dots and the lower one has \( N - 2 \) dots (and has the value \(-(N + 1)^2\)). The evaluation of this theta foam with other labellings can be obtained from the result above by dot migration.

Up to normalization the KL evaluation of the \((1,1,2)\)-theta foam corresponds to the trace on the cohomology ring of the partial flag variety \( Fl_{1,2,N} \) in Equation (8) given by \( ε(x_1^{N-2} x_2^{N-1}) = 1 \), and where \( x_1 \) and \( x_2 \) correspond to the dots in the upper and lower facet respectively.

**The \((1,2,3)\)-theta foam**

For the theta foam in Figure 10 the method is the same as in the previous case, just the computations are more complicated. In this case, we have one 1-facet, to which we associate the variable \( x \), one 2-facet, with variables \( s \) and \( t \) and the 3-facet with variables \( p, q \) and \( r \). Recall that the polynomial \( W(p,q,r) \) is such that \( W(a+b+c,ab+bc+ac,abc) = a^{N+1} + b^{N+1} + c^{N+1} \). We denote by \( W'_i(p,q,r) \), \( i = 1,2,3 \), the partial derivative of \( W \) with respect to \( i \)-th variable. Also, let \( A, B \) and \( C \) be the polynomials given by
\[
\begin{align*}
A &= \frac{W(x + s, xs + t, xt) - W(p, xs + t, xt)}{x + s - p}, \\
B &= \frac{W(p, xs + t, xt) - W(p, q, xt)}{xs + t - q}, \\
C &= \frac{W(p, q, xt) - W(p, q, r)}{xt - r}.
\end{align*}
\]

To the singular circle of this theta foam, we associated the matrix factorization (see Equations (14)-(17)):
\[
MF_2 = \left\{ A, \begin{array}{l} \frac{x + s - p}{x + s - p} \\ B, \frac{xs + t - q}{xs + t - q} \\ C, \frac{xt - r}{xt - r} \end{array} \right\}.
\]


The differential of this matrix factorization is the 8 by 8 matrix

\[ D = \begin{pmatrix} 0 & D_1 \\ D_0 & 0 \end{pmatrix}, \]

where

\[ D_0 = \begin{pmatrix} d_0 \\ (xt - r) \text{Id}_2 \end{pmatrix}, \quad D_1 = \begin{pmatrix} d_1 \\ -d_0 \end{pmatrix}. \]

Here \( d_0 \) and \( d_1 \) are the differentials of the matrix factorization

\[ \begin{cases} A, & x + s - p \\ B, & xs + t - q \end{cases}, \]

i.e.

\[ d_0 = \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{and} \quad d_1 = \begin{pmatrix} B \\ A \end{pmatrix}. \]

The KL formula assigns to this theta foam the polynomial \( KL \Theta_2(x,s,t,p,q,r) \) given as the supertrace of the twisted differential of \( D \), i.e.

\[ KL \Theta_2 = \text{STr} \left( \partial_3 D \frac{1}{2} (\partial_i D \partial_j D - \partial_i D \partial_j D) \partial_3 D^t \right), \]

where

\[ \partial_3 D^t = \frac{1}{3!} \left( \partial_p D \partial_q D \partial_i D - \partial_i D \partial_p D \partial_q D + \partial_q D \partial_p D \partial_i D - \partial_i D \partial_q D \partial_p D \right). \]

After straightforward computations and some grouping, we obtain

\[ KL \Theta_2 = (A_p + A_s) \left[ (B_t + B_q)(C_t + tC_r) - (B_s + sB_q)(C_t + xC_r) - (B_x - sB_t)C_q \right] \]
\[ + (A_p + A_s) \left[ (B_s + xB_q)(C_t + xC_r) + (B_s - xB_t)C_q \right] \]
\[ + (A_s - A_t) \left[ B_p(C_t + xC_r) - (B_t + B_q)C_p + B_p C_q \right] \]
\[ - A_1 \left[ ((B_s + xB_q) + B_p)(C_t + tC_r) + ((B_s + xB_q) + B_p)C_q \right] \]
\[ - (B_s + sB_q))C_p + ((sB_s - xB_s) + (s - x)B_p)C_q \].

In order to simplify this expression, we introduce the following polynomials

\[ a_{1i} = W_i'(x + s, xs + t, xt) - W_i'(p, xs + t, xt), \quad i = 1, 2, 3, \]
\[ a_{2i} = W_i'(p, xs + t, xt) - W_i'(p, q, xt), \quad i = 1, 2, 3, \]
\[ a_{3i} = W_i'(p, q, xt) - W_i'(p, q, r), \quad i = 1, 2, 3. \]

Then from (28)-(30), we have

\[ A_s + A_p = a_{11} + sa_{12} + ta_{13}, \quad A_p + A_s = a_{11} + xa_{12}, \]
\[ A_s - A_t = (s - x)a_{12} + ta_{13}, \quad A_t = a_{12} + xa_{13}, \]
\[ B_p = a_{21}, \quad B_s - xB_t = -x^2 a_{23}, \]

25
\[
\begin{align*}
    sB_s - xB_x &= xt a_{23}, \\
    B_t + B_q &= a_{22} + xa_{23}, \\
    C_p &= a_{31}, \\
    C_x + tC_r &= ta_{33},
\end{align*}
\]

Using this \( KL_{\Theta_2} \) becomes

\[
(31) \quad KL_{\Theta_2} = (t - sx + x^2) \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.
\]

Now the last part follows analogously as in the case of the \((1, 1, 2)\)-theta foam. For degree reasons the coefficient of \( \pi_{N-3,N-3,N-3} \) in the latter determinant is of degree zero, and one can obtain that it is equal to \((N + 1)^3\). Thus, the coefficient of \( \pi_{N-3,N-3,N-3} \) in \( KL_{\Theta_2} \) is \((N + 1)^3(t - sx + x^2)\) from which we obtain the value of the theta foam when the 3-facet is undotted. For example, we see that

\[
\epsilon(KL_{\Theta_2}, \pi_{1,1}(s,t)^{N-3}x^{N-1}) = (N + 1)^3.
\]

It is then easy to obtain the values when the 3-facet is labelled by \( \pi_{N-3,N-3,N-3}(p,q,r) \) using dot migration. The example above implies that

\[
\epsilon(KL_{\Theta_2}, \pi_{N-3,N-3,N-3}(p,q,r)x^2) = (N + 1)^3.
\]

Up to normalization the KL evaluation of the \((1,2,3)\)-theta foam corresponds to the trace on the cohomology ring of the partial flag variety \( F_{2,3,N} \) in Equation (8) given by \( \epsilon(x^2 \pi_{N-3,N-3,N-3}) = 1 \), where \( \pi_{N-3,N-3,N-3} \) correspond to a linear combination of dots in the triple facet and \( x \) corresponds to a dot in the upper simple facet (see Section 3).

For \( N = 3 \), using the explicit formula for \( W(p,q,r) \) we see that the determinant (31) is zero, which means that the \((1,2,3)\)-theta foams would evaluate to zero, independently of the dots they may have. That is why we restrict the construction in this notes to the case of \( N \geq 4 \).

### 5.5 Normalization

It will be convenient to normalize the KL evaluation. Let \( u \) be a closed foam with graph \( \Gamma \). Note that \( \Gamma \) has two types of edges: the ones incident to two simple facets and one double facet and the ones incident to one simple, one double and one triple facet. Edges of the same type form cycles in \( \Gamma \). Let \( e_{112}(u) \) be the total number of cycles in \( \Gamma \) with edges of the first type and \( e_{123}(u) \) the total number of cycles with edges of the second type. We normalize the KL formula by dividing \( KL_u \) by

\[
(N + 1)^{2e_{112} + 3e_{123}}.
\]

In the sequel we only use this normalized KL evaluation keeping the same notation \( \langle u \rangle_{KL} \). Note that with this normalization the KL-evaluation in the examples above always gives 0, -1 or 1.
5.6 The glueing property

We now consider the glueing property of the KL formula, which is an important property of TQFT’s. Suppose that $u$ is a foam with boundary $\Gamma$. We decorate the facets, singular arcs and singular vertices of $u$ as in Subsection 5.2. Recall that the orientations of the singular arcs of $u$ induce an orientation of $\Gamma$ (see Figure 4). To each vertex $\nu$ of $\Gamma$ we associate the matrix factorization which is the matrix factorization associated to the singular arc of $u$ that is bounded by $\nu$. To each circle in $\Gamma$ we associate the Jacobi algebra of the corresponding facet in $\mathbb{Z}/2\mathbb{Z}$-degree $i \pmod{2}$, where $i = 1, 2, 3$. Then define the matrix factorization $M_\Gamma$ as the tensor product of all the matrix factorizations of its vertices as given above and Jacobi algebras $J_i$ in $\mathbb{Z}/2\mathbb{Z}$-degree $i \pmod{2}$ for all (if any) circles in $\Gamma$. The tensor product is taken over suitable rings so that $M_\Gamma$ is a free module over $R$ of finite rank, where $R$ is the polynomial ring with rational coefficients in the variables of the facets of $u$ that are bounded by $\Gamma$. The factorization $M_\Gamma$ has potential zero, since for every edge $e$ of $\Gamma$ the individual potential $W_e$ appears twice in $W_\Gamma$ (one for each vertex bounding $e$) with opposite signs. The homology

\begin{equation}
\text{H}_{\text{MF}}(M_\Gamma) \cong \text{Ext}(R, M_\Gamma)
\end{equation}

is finite-dimensional and coincides with the one in [9] after using Lemma 2.1 to exclude the variables associated to all double and triple edges of $\Gamma$.

Let $u$ be an open foam whose boundary consists of two parts $\Gamma_1$ and $\Gamma_2$, and denote by $M_1$ and $M_2$ the matrix factorizations associated to $\Gamma_1$ and $\Gamma_2$ respectively. We say that $F$ is an interior facet of $u$ if $\partial F \cap \partial u = \emptyset$. Restricting $KL_u$ to the interior facets of $u$ and doing the same to $\varepsilon$ in Equation (11) we see that the KL formula associates to $u$ an element of $\text{Ext}(M_1, M_2)$.

If $u'$ is another foam whose boundary consists of $\Gamma_2$ and $\Gamma_3$, then it corresponds to an element of $\text{Ext}(M_2, M_3)$, while the element associated to the foam $uu'$, which is obtained by glueing the foams $u$ and $u'$ along $\Gamma_2$, is equal to the composite of the elements associated to $u$ and $u'$.

On the other hand, we can also see $u$ as a morphism from the empty web to its boundary $\Gamma = \Gamma_2 \sqcup \Gamma_1^\ast$, where $\Gamma_1^\ast$ is equal to $\Gamma_1$ but with the opposite orientation. In that case, the KL formula associates to it an element from

\begin{equation}
\text{Ext}(R, M_{\Gamma_2} \otimes (M_{\Gamma_1}^\ast)) \cong \text{H}_{\text{MF}}(\Gamma).
\end{equation}

Both ways of applying the KL formula are equivalent up to a global $q$-shift by corollary 6 in [9].

In the case of a foam $u$ with corners, i.e. a foam with two horizontal boundary components $\Gamma_1$ and $\Gamma_2$ which are connected by vertical edges, one has to “pinch” the vertical edges. This way one can consider $u$ to be a morphism from the empty set to $\Gamma_2 \sqcup \nu \Gamma_1^\ast$, where $\sqcup \nu$ means that the webs are glued at their vertices. The same observations as above hold, except that $M_{\Gamma_2} \otimes (M_{\Gamma_1}^\ast)$ is now the tensor product over the polynomial ring in the variables associated to the horizontal edges with corners.

The KL formula also has a general property that will be useful later. The KL formula defines a duality pairing between $\text{Hom}_{\text{Foam}_\nu}(\emptyset, \Gamma)$ and $\text{Hom}_{\text{Foam}_\nu}(\Gamma, \emptyset)$ as

\begin{equation}
(a, a') = \langle d'a' \rangle_{KL},
\end{equation}

27
for $a \in \text{Hom}_{\text{Foam}}(\emptyset, \Gamma)$ and $a' \in \text{Hom}_{\text{Foam}}(\Gamma, \emptyset)$. From the duality pairing it follows that
\[
\text{Hom}_{\text{Foam}}(\emptyset, \Gamma^*) = \text{Hom}_{\text{Foam}}(\Gamma, \emptyset).
\]
The duality pairing also defines a canonical element
\[
\psi_{\Gamma, \Gamma^*} \in \text{Hom}_{\text{Foam}}(\emptyset, \Gamma^*) \otimes \text{Hom}_{\text{Foam}}(\emptyset, \Gamma)
\]
by
\[
(\psi_{\Gamma, \Gamma^*}, a \otimes a') = (a, a').
\]
Introducing a basis $\{a_i\}$ of $\text{Hom}_{\text{Foam}}(\emptyset, \Gamma)$ and its dual basis $\{a^*_j\}$ of $\text{Hom}_{\text{Foam}}(\Gamma, \emptyset)$ we have
\[
\psi_{\Gamma, \Gamma^*} = \sum_j a_j \otimes a^*_j.
\]
Suppose that a closed foam $u$ contains two points $p_1$ and $p_2$ such that intersecting $u$ with disjoint spheres centered in $p_1$ and $p_2$ result in two webs $\Gamma_1$ and $\Gamma_2$ and that $\Gamma_2 = \Gamma_1^*$. If we remove the parts inside those spheres from $u$ and glue the boundary components $\Gamma_1$ and $\Gamma_2$ onto each other we obtain a new closed foam $u'$ and the KL evaluations of $u$ and $u'$ are related by (see [7])
\[
\langle u' \rangle_{KL} = \langle \psi_{\Gamma_1, \Gamma_1^*} u \rangle_{KL} = \sum_j \langle a^*_j u a_j \rangle_{KL}.
\]

6 Odds and ends

6.1 Size of the kernel of the KL evaluation

The set of relations between foams given in [17] and [11] is clearly smaller than the kernel of the KL evaluation. Characterizing the kernel of the KL evaluation is interesting for several reasons. Having a finite set of relations generating the kernel would establish that the link homology of [11, 17] is purely combinatorial. This problem is related to the problem of finding a presentation by generators and relations of the category of tensor products of exterior powers of the fundamental representation of quantum $\mathfrak{sl}(N)$ and intertwiner maps. (a complete set of relations is conjectured in [14]). A solution of one of these problems could help solving the other.

6.2 Integrality of the KL evaluation

In all the examples above the Kapustin-Li evaluation always returns an integer. This motivates the following conjecture.

Conjecture 1. For a closed foam $u$ we have $\varepsilon(\langle u \rangle_{KL}) \in \mathbb{Z}$. Being true, this conjecture would imply that the link homology using foams is integral that is, all the $\mathfrak{sl}(N)$-homology groups would be modules over $\mathbb{Z}$ instead of $\mathbb{Q}$-vector spaces. Integrality of the $\mathfrak{sl}(N)$-link homology was conjectured in [17] where is was shown that being integral it would have torsion of order of at least $N$.

28
6.3 Representation theoretic interpretation

In [13] V. Mazorchuk and C. Stroppel gave a construction of the $\mathfrak{sl}(N)$-link homology using representation theory. In their interpretation a web corresponds to a composite of certain projective functors between parabolic singular blocks of category $\mathcal{O}$ and a foam to a natural transformation between projective functors. A closed foam $u$ yields an endomorphism of the functor associated to the empty web and therefore an element of the ground field. Up to a normalization we expect these evaluations to be equal.

More recently C. Blanchet [1] informed me of another method to evaluate closed foams. His method consists of realizing the singular vertex as a certain 4-step partial flag manifold and the evaluation of a closed foam is obtained by taking a Frobenius trace in some configuration space, in the same spirit as the $\varepsilon$ we used in the Kapustin-Li evaluation. This evaluation is defined over the integers. We also expect this evaluation to equal $\varepsilon(\langle u \rangle_{KL})$.

**Conjecture 2.** The Mazorchuk-Stroppel and the Blanchet evaluations of a closed foam $u$ are both equal to $\varepsilon(\langle u \rangle_{KL})$.

**References**


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