The Kapustin-Li formula and the evaluation of (closed) $\mathfrak{sl}(N)$-foams

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Abstract

This notes intended to supplement a talk given by the author at the *Institut de Mathématiques de Jussieu - Université Paris 7* in January 2010. The purpose of the talk was to give an explanation of the use of Kapustin-Li formula in the evaluation of the closed foams used in the construction of the $\mathfrak{sl}(N)$-link homology. This notes contains material from the author's PhD thesis [17] as well as the explanation and the proof of some facts about the Kapustin-Li formula that were omitted from [11].

Contents

1 Introduction

This notes intended to supplement a talk given by the author at the *Institut de Mathématiques de Jussieu - Université Paris 7* in January 2010 when he was postdoc at the aforementioned institution[1]. The purpose of the talk was to give an explanation of the use of Kapustin-Li formula in the evaluation of the closed foams used in the construction of the $\mathfrak{sl}(N)$-link homology. This notes contains material from my thesis [17] as well as the explanation and the proof of some facts about the Kapustin-Li formula that were omitted from [11].

In the context of categorification foams first appeared in Khovanov's construction of a topological theory categorifying the $\mathfrak{sl}(3)$-link polynomial [6]. His construction uses cobordisms with singularities, called foams, modulo a finite set of relations. In [9] Khovanov and Rozansky (KR) categorified the $\mathfrak{sl}(N)$-link polynomial for arbitrary N, the 1-variable specializations of the 2-variable HOMFLY-PT polynomial. Their construction uses the theory of matrix factorizations, a mathematical tool introduced by Eisenbud in [3] (see also [2, 10, 18]) in the study of maximal Cohen-Macaulay modules over isolated hypersurface singularities and used by Kapustin and Li as boundary conditions for strings in Landau-Ginzburg models [5].

The goal of [11] was to construct a combinatorial topological definition of KR link homology, extending to all $N > 3$ the work of Khovanov [6] for $N = 3$ (see also [12]). Khovanov had to modify considerably his original setting for the construction of $\mathfrak{sl}(2)$ link homology in order to produce his $\mathfrak{sl}(3)$ link homology. It required the introduction of singular cobordisms with a particular type of singularity, which he called *foams*. The jump from $\mathfrak{sl}(3)$ to $\mathfrak{sl}(N)$, for $N > 3$, requires the introduction of a new type of singularity. The latter is needed for proving invariance under the third Reidemeister move. The introduction of the new singularities makes it much harder to evaluate closed foams and we do not know how to do it combinatorially. Instead we use the Kapustin-Li formula [5], which was introduced by A. Kapustin and Y. Li in [5] in the context of topological Landau-Ginzburg models with boundaries and adapted to foams by Khovanov and Rozansky [7]. The downside is that our construction does not yet allow us to deduce a (fast) algorithm for computing $\mathfrak{sl}(N)$ link homology. A positive side-effect is that it allows us to show that for any link the homology using foams is isomorphic to KR homology. Furthermore the combinatorics involved in establishing certain identities among foams gets much harder for arbitrary N. The theory of symmetric polynomials, in particular Schur polynomials, is used to handle that problem.

2 Matrix factorizations

This section contains a brief review of matrix factorizations and the properties that will be used throughout this notes. All the matrix factorizations in this notes are $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$-graded. Let R be a polynomial ring over $\mathbb{Q}$ in a finite number of variables. We take the $\mathbb{Z}$-degree of each polynomial to be twice its total degree. This way R is $\mathbb{Z}$-graded. Let W be a homogeneous element of R of degree $2m$. A matrix factorization of W over R is given by a $\mathbb{Z}/2\mathbb{Z}$-graded free

[1]The author thanks Christian Blanchet and the IMJ for the opportunity of giving this talk and their hospitality during the period from March 2009 to February 1010.

R-module $M = M_0 \oplus M_1$ with two R-homomorphisms of degree m

$$M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_0$$

such that $d_1 d_0 = W\,\mathrm{Id}_{M_0}$ and $d_0 d_1 = W\,\mathrm{Id}_{M_1}$. We call W the *potential*. The $\mathbb{Z}$-grading of R induces a $\mathbb{Z}$-grading on M. The shift functor $\{k\}$ acts on M as

$$M\{k\} = M_0\{k\} \xrightarrow{d_0} M_1\{k\} \xrightarrow{d_1} M_0\{k\},$$

where its action on the modules M_0, M_1 means an upward shift by k units on the $\mathbb{Z}$-grading.

A homomorphism $f\colon M \to M'$ of matrix factorizations of W is a pair of maps of the same degree $f_i\colon M_i \to M'_i$ $(i = 0,1)$ such that the diagram

$$\begin{tikzcd} M_0 \arrow[r, "d_0"] \arrow[d, "f_0"'] & M_1 \arrow[r, "d_1"] \arrow[d, "f_1"'] & M_0 \arrow[d, "f_0"'] \\ M'_0 \arrow[r, "d'_0"] & M'_1 \arrow[r, "d'_1"] & M'_0 \end{tikzcd}$$

commutes. It is an isomorphism of matrix factorizations if f_0 and f_1 are isomorphisms of the underlying modules. Denote the set of homomorphisms of matrix factorizations from M to M' by

$$\mathrm{Hom}_{\mathbf{MF}}(M, M').$$

It has an R-module structure with the action of R given by $r(f_0, f_1) = (rf_0, rf_1)$ for $r \in R$. Matrix factorizations over R with homogeneous potential W and homomorphisms of matrix factorizations form a graded additive category, which we denote by $\mathbf{MF}_R(W)$. If $W = 0$ we simply write $\mathbf{MF}_R$.

Another description of matrix factorizations is obtained by assembling the differentials d_0 and d_1 into an endomorphism D of the $\mathbb{Z}/2\mathbb{Z}$-graded free R-module $M = M_0 \oplus M_1$ such that

$$D = \begin{pmatrix} 0 & d_1 \\ d_0 & 0 \end{pmatrix} \qquad \deg_{\mathbb{Z}/2\mathbb{Z}} D = 1 \qquad D^2 = W\,\mathrm{Id}_M\,.$$

In this case we call D the *twisted differential*.

The free R-module $\mathrm{Hom}_R(M, M')$ of graded R-module homomorphisms from M to M' is a 2-complex

$$\mathrm{Hom}^0_R(M, M') \xrightarrow{d} \mathrm{Hom}^1_R(M, M') \xrightarrow{d} \mathrm{Hom}^0_R(M, M')$$

where

$$\mathrm{Hom}^0_R(M, M') = \mathrm{Hom}_R(M_0, M'_0) \oplus \mathrm{Hom}_R(M_1, M'_1)$$
$$\mathrm{Hom}^1_R(M, M') = \mathrm{Hom}_R(M_0, M'_1) \oplus \mathrm{Hom}_R(M_1, M'_0)$$

and for f in $\mathrm{Hom}^i_R(M, M')$ the differential acts as

$$df = d_{M'} f - (-1)^i f d_M.$$

We define

$$\mathrm{Ext}(M, M') = \mathrm{Ext}^0(M, M') \oplus \mathrm{Ext}^1(M, M') = \mathrm{Ker}\, d / \mathrm{Im}\, d,$$

and write $\mathrm{Ext}_{(m)}(M,M')$ for the elements of $\mathrm{Ext}(M,M')$ with $\mathbb{Z}$-degree m. Note that for $f \in \mathrm{Hom}_{\mathbf{MF}}(M,M')$ we have $df = 0$. We say that two homomorphisms $f,\ g \in \mathrm{Hom}_{\mathbf{MF}}(M,M')$ are homotopic if there is an element $h \in \mathrm{Hom}^1_R(M,M')$ such that $f - g = dh$.

Denote by $\mathrm{Hom}_{\mathbf{HMF}}(M,M')$ the R-module of homotopy classes of homomorphisms of matrix factorizations from M to M' and by $\mathbf{HMF}_R(W)$ the homotopy category of $\mathbf{MF}_R(W)$.

We denote by $M\langle 1\rangle$ and $M_\bullet$ the factorizations

$$M_1 \xrightarrow{-d_1} M_0 \xrightarrow{-d_0} M_1$$

and

$$(M_0)^* \xrightarrow{-(d_1)^*} (M_1)^* \xrightarrow{(d_0)^*} (M_0)^*$$

respectively. Factorization $M\langle 1\rangle$ has potential W while factorization $M_\bullet$ has potential $-W$. We call $M_\bullet$ the *dual factorization* of M.

We have

$$\mathrm{Ext}^0(M,M') \cong \mathrm{Hom}_{\mathbf{HMF}}(M,M')$$
$$\mathrm{Ext}^1(M,M') \cong \mathrm{Hom}_{\mathbf{HMF}}(M,M'\langle 1\rangle)$$

The tensor product $M \otimes_R M_\bullet$ has potential zero and is therefore a 2-complex. Denoting by $\mathrm{H}_{\mathbf{MF}}$ the homology of matrix factorizations with potential zero we have

$$\mathrm{Ext}(M,M') \cong \mathrm{H}_{\mathbf{MF}}(M' \otimes_R M_\bullet)$$

and, if M is a matrix factorization with $W = 0$,

$$\mathrm{Ext}(R,M) \cong \mathrm{H}_{\mathbf{MF}}(M).$$

Let $R = \mathbb{Q}[x_1,\dots,x_k]$ and $W \in R$. The Jacobi algebra of W is defined as

$$J_W = R/(\partial_1 W,\dots,\partial_k W), \tag{1}$$

where ∂_i means the partial derivative with respect to x_i. Writing the differential as a matrix and differentiating both sides of the equation $D^2 = W$ with respect to x_i we get $D(\partial_i D) + (\partial_i D)D = \partial_i W$. We thus see that multiplication by $\partial_i W$ is homotopic to the zero endomorphism and that the homomorphism

$$R \to \mathrm{End}_{\mathbf{HMF}}(M), \quad r \mapsto m(r)$$

factors through the Jacobi algebra of W.

Let $f,g \in \mathrm{End}(M)$. We define the *supercommutator* of f and g as

$$[f,g]_s = fg - (-1)^{\deg_{\mathbb{Z}/2\mathbb{Z}}(f)\deg_{\mathbb{Z}/2\mathbb{Z}}(g)} gf.$$

The *supertrace* of f is defined as

$$\mathrm{STr}(f) = \mathrm{Tr}\big((-1)^{\mathrm{gr}} f\big)$$

where the *grading operator* $(-1)^{\mathrm{gr}} \in \mathrm{End}(M_0 \oplus M_1)$ is given by

$$(m_0,m_1) \mapsto (m_0,-m_1), \quad m_0 \in M_0,\ m_1 \in M_1.$$

If f and g are homogeneous with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading we have that

$$\mathrm{STr}(fg) = (-1)^{\deg_{\mathbb{Z}/2\mathbb{Z}}(f)\deg_{\mathbb{Z}/2\mathbb{Z}}(g)}\,\mathrm{STr}(gf),$$

and

$$\mathrm{STr}\big([f,g]_s\big) = 0.$$

There is a canonical isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded R-modules

$$\mathrm{End}(M) \cong M \otimes_R M_\bullet.$$

Choose a basis $\{|\,i\,\rangle\}$ of M and define a dual basis $\{\langle\, j\,|\}$ of $M_\bullet$ by $\langle j|i\rangle = \delta_{i,j}$, where δ is the *Kronecker* symbol. There is a natural pairing map $M \otimes M_\bullet \to R$ called the *super-contraction* that is given on basis elements $|\,i\,\rangle\langle\, j\,|$ by

$$|\,i\,\rangle\langle\, j\,| \mapsto (-1)^{\deg_{\mathbb{Z}/2\mathbb{Z}}(|\,i\,\rangle)\deg_{\mathbb{Z}/2\mathbb{Z}}(\langle\, j\,|)}\langle j|i\rangle = \delta_{i,j}.$$

The super-contraction induces a map $\mathrm{End}(M) \to R$ which coincides with the supertrace. When M and $M_\bullet$ are factors in a tensor product $(M \otimes_R N) \otimes_R (M_\bullet \otimes_R N_\bullet)$ the super-contraction of M with $M_\bullet$ induces a map $\mathrm{STr}_M\colon\, \mathrm{End}(M \otimes_R N) \to \mathrm{End}(N)$ called the *partial super-trace* (w.r.t. M).

2.1 Koszul Factorizations

For a, b homogeneous elements of R, an *elementary Koszul factorization* $\{a,b\}$ over R with potential ab is a factorization of the form

$$R \xrightarrow{a} R\big\{\tfrac{1}{2}\big(\deg_{\mathbb{Z}} b - \deg_{\mathbb{Z}} a\big)\big\} \xrightarrow{b} R.$$

When we need to emphasize the ring R we write this factorization as $\{a,b\}_R$. The tensor product of matrix factorizations M_i with potentials W_i is a matrix factorization with potential $\sum_i W_i$. We restrict to the case where all the W_i are homogeneous of the same degree. Throughout this notes we use tensor products of elementary Koszul factorizations $\{a_j,b_j\}$ to build bigger matrix factorizations, which we write in the form of a *Koszul matrix* as

$$\begin{Bmatrix} a_1\ , & b_1 \\ \vdots & \vdots \\ a_k\ , & b_k \end{Bmatrix}$$

We denote by $\{\mathbf{a},\mathbf{b}\}$ the Koszul matrix which has columns $(a_1,\dots,a_k)$ and $(b_1,\dots,b_k)$. If $\sum_{i=1}^{k} a_i b_i = 0$ then $\{\mathbf{a},\mathbf{b}\}$ is a 2-complex whose homology is an $R/(a_1,\dots,a_k,b_1,\dots,b_k)$-module, since multiplication by a_i and b_i are null-homotopic endomorphisms of $\{\mathbf{a},\mathbf{b}\}$.

Note that the action of the shift $\langle 1\rangle$ on $\{\mathbf{a},\mathbf{b}\}$ is equivalent to switching terms in one line of $\{\mathbf{a},\mathbf{b}\}$:

$$\{\mathbf{a},\mathbf{b}\}\langle 1\rangle \cong \begin{Bmatrix} \vdots & \vdots \\ a_{i-1}\ , & b_{i-1} \\ -b_i\ , & -a_i \\ a_{i+1}\ , & b_{i+1} \\ \vdots & \vdots \end{Bmatrix}\big\{\tfrac{1}{2}\big(\deg_{\mathbb{Z}} b_i - \deg_{\mathbb{Z}} a_i\big)\big\}.$$

If we choose a different row to switch terms we get a factorization which is isomorphic to this one. We also have that

$$\{\mathbf{a},\mathbf{b}\}_\bullet \cong \{\mathbf{a},-\mathbf{b}\}\langle k\rangle\{s_k\},$$

where

$$s_k = \sum_{i=1}^{k} \deg_{\mathbb{Z}} a_i - \frac{k}{2}\deg_{\mathbb{Z}} W.$$

Let $R=\mathbb{Q}[x_1,\dots,x_k]$ and $R'=\mathbb{Q}[x_2,\dots,x_k]$. Suppose that $W=\sum_i a_i b_i \in R'$ and $x_1 - b_i \in R'$, for a certain $1 \le i \le k$. Let $c = x_1 - b_i$ and $\{\hat{\mathbf{a}}^i, \hat{\mathbf{b}}^i\}$ be the matrix factorization obtained from $\{\mathbf{a},\mathbf{b}\}$ by deleting the i-th row and substituting x_1 by c.

Lemma 2.1 (excluding variables)**.** *The matrix factorizations $\{\mathbf{a},\mathbf{b}\}$ and $\{\hat{\mathbf{a}}^i, \hat{\mathbf{b}}^i\}$ are homotopy equivalent.*

In [9] one can find the proof of this lemma and its generalization with several variables.

The following lemma contains three particular cases of Proposition 3 in [9]:

Lemma 2.2 (Row operations)**.** *We have the following isomorphisms of matrix factorizations*

$$\begin{Bmatrix} a_i\,, & b_i \\ a_j\,, & b_j \end{Bmatrix} \overset{[i,j]_\lambda}{\cong} \begin{Bmatrix} a_i - \lambda a_j\,, & b_i \\ a_j\,, & b_j + \lambda b_i \end{Bmatrix}, \qquad \begin{Bmatrix} a_i\,, & b_i \\ a_j\,, & b_j \end{Bmatrix} \overset{[i,j]'_\lambda}{\cong} \begin{Bmatrix} a_i + \lambda b_j\,, & b_i \\ a_j - \lambda b_i\,, & b_j \end{Bmatrix}$$

for $\lambda \in R$. If λ is invertible in R, we also have

$$\{a_i\,,\, b_j\} \overset{[i]_\lambda}{\cong} \{\lambda a_i\,,\, \lambda^{-1} b_i\}.$$

Proof. It is straightforward to check that the pairs of matrices

$$[i,j]_\lambda = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \right), \quad [i,j]'_\lambda = \left(\begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \quad \text{and} \quad [i]_\lambda = (1,\ \lambda)$$

define isomorphisms of matrix factorizations. □

Recall that a sequence $(a_1,a_2,\dots,a_k)$ is called *regular* in R if a_j is not a zero divisor in $R/(a_1,a_2,\dots,a_{j-1})$, for $j=1,\dots,k$. The proof of the following lemma can be found in [8].

Lemma 2.3. *Let $\mathbf{b}=(b_1,b_2,\dots,b_k)$, $\mathbf{a}=(a_1,a_2,\dots,a_k)$ and $\mathbf{a}'=(a'_1,a'_2,\dots,a'_k)$ be sequences in R. If $\mathbf{b}$ is regular and $\sum_i a_i b_i = \sum_i a'_i b_i$ then the factorizations*

$$\{\mathbf{a}\,,\mathbf{b}\} \ \text{and}\ \{\mathbf{a}'\,,\mathbf{b}\}$$

are isomorphic.

A factorization M with potential W is said to be *contractible* if it is isomorphic to a direct sum of factorizations of the form

$$R \xrightarrow{1} R\{\tfrac{1}{2}\deg_{\mathbb{Z}} W\} \xrightarrow{W} R \quad \text{and} \quad R \xrightarrow{W} R\{-\tfrac{1}{2}\deg_{\mathbb{Z}} W\} \xrightarrow{1} R.$$

3 Schur polynomials and the cohomology of partial flag varieties

In this section we recall some basic facts about Schur polynomials and the cohomology of partial flag varieties.

3.1 Schur polynomials

A nice basis for homogeneous symmetric polynomials is given by the Schur polynomials. If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition such that $\lambda_1 \geq \ldots \geq \lambda_k \geq 0$, then the Schur polynomial $\pi_\lambda(x_1, \ldots, x_k)$ is given by the following expression:

$$\pi_\lambda(x_1, \ldots, x_k) = \frac{\det\left(x_i^{\lambda_j+k-j}\right)}{\Delta}, \tag{2}$$

where $\Delta = \prod_{i<j}(x_i - x_j)$, and by $\det(x_i^{\lambda_j+k-j})$, we have denoted the determinant of the $k \times k$ matrix whose (i, j) entry is equal to $x_i^{\lambda_j+k-j}$. Note that the elementary symmetric polynomials are given by $\pi_{1,0,0,\ldots,0}, \pi_{1,1,0,\ldots,0}, \ldots, \pi_{1,1,1,\ldots,1}$. There are multiplication rules for the Schur polynomials which show that any $\pi_{\lambda_1,\lambda_2,\ldots,\lambda_k}$ can be expressed in terms of the elementary symmetric polynomials.

If we do not specify the variables of the Schur polynomial π_λ, we will assume that these are exactly $x_1, \ldots, x_k$, with k being the length of λ, i.e.

$$\pi_{\lambda_1,\ldots,\lambda_k} := \pi_{\lambda_1,\ldots,\lambda_k}(x_1, \ldots, x_k).$$

In this notes we only use Schur polynomials of two and three variables. In the case of two variables, the Schur polynomials are indexed by pairs of nonnegative integers (i, j), such that $i \geq j$, and (2) becomes

$$\pi_{i,j} = \sum_{\ell=j}^{i} x_1^\ell x_2^{i+j-\ell}.$$

Directly from *Pieri's formula* we obtain the following multiplication rule for the Schur polynomials in two variables:

$$\pi_{i,j}\pi_{a,b} = \sum \pi_{x,y}, \tag{3}$$

where the sum on the r.h.s. is over all indices x and y such that $x + y = i + j + a + b$ and $a + i \geq x \geq \max(a + j, b + i)$. Note that this implies $\min(a + j, b + i) \geq y \geq b + j$. Also, we shall write $\pi_{x,y} \in \pi_{i,j}\pi_{a,b}$ if $\pi_{x,y}$ belongs to the sum on the r.h.s. of (3). Hence, we have that $\pi_{x,x} \in \pi_{i,j}\pi_{a,b}$ iff $a + j = b + i = x$ and $\pi_{x+1,x} \in \pi_{i,j}\pi_{a,b}$ iff $a + j = x + 1$, $b + i = x$ or $a + j = x$, $b + i = x + 1$.

We shall need the following combinatorial result which expresses the Schur polynomial in three variables as a combination of Schur polynomials of two variables. For $i \geq j \geq k \geq 0$, and the triple (a, b, c) of nonnegative integers, we define

$$(a, b, c) \sqsubset (i, j, k),$$

if $a + b + c = i + j + k$, $i \geq a \geq j$ and $j \geq b \geq k$. We note that this implies that $i \geq c \geq k$, and hence $\max\{a, b, c\} \leq i$.

Lemma 3.1.
$$\pi_{i,j,k}(x_1,x_2,x_3) = \sum_{(a,b,c)\sqsubset(i,j,k)} \pi_{a,b}(x_1,x_2)x_3^c.$$

Proof. From the definition of the Schur polynomial, we have

$$\pi_{i,j,k}(x_1,x_2,x_3) = \frac{(x_1x_2x_3)^k}{(x_1-x_2)(x_1-x_3)(x_2-x_3)} \det \begin{pmatrix} x_1^{i-k+2} & x_1^{j-k+1} & 1 \\ x_2^{i-k+2} & x_2^{j-k+1} & 1 \\ x_3^{i-k+2} & x_3^{j-k+1} & 1 \end{pmatrix}.$$

After subtracting the last row from the first and the second one of the last determinant, we obtain

$$\pi_{i,j,k} = \frac{(x_1x_2x_3)^k}{(x_1-x_2)(x_1-x_3)(x_2-x_3)} \det \begin{pmatrix} x_1^{i-k+2} - x_3^{i-k+2} & x_1^{j-k+1} - x_3^{j-k+1} \\ x_2^{i-k+2} - x_3^{i-k+2} & x_2^{j-k+1} - x_3^{j-k+1} \end{pmatrix},$$

and so
$$\pi_{i,j,k} = \frac{(x_1x_2x_3)^k}{x_1-x_2} \det \begin{pmatrix} \sum_{m=0}^{i-k+1} x_1^m x_3^{i-k+1-m} & \sum_{n=0}^{j-k} x_1^n x_3^{j-k-n} \\ \sum_{m=0}^{i-k+1} x_2^m x_3^{i-k+1-m} & \sum_{n=0}^{j-k} x_2^n x_3^{j-k+n} \end{pmatrix}.$$

Finally, after expanding the last determinant we obtain

$$\pi_{i,j,k} = \frac{(x_1x_2x_3)^k}{x_1-x_2} \sum_{m=0}^{i-k+1} \sum_{n=0}^{j-k} (x_1^m x_2^n - x_1^n x_2^m) x_3^{i+j-2k+1-m-n}. \tag{4}$$

We split the last double sum into two: the first one when m goes from 0 to $j-k$, denoted by S_1, and the other one when m goes from $j-k+1$ to $i-k+1$, denoted by S_2. To show that $S_1 = 0$, we split the double sum further into three parts: when $m < n$, $m = n$ and $m > n$. Obviously, each summand with $m = n$ is equal to 0, while the summands of the sum for $m < n$ are exactly the opposite of the summands of the sum for $m > n$. Thus, by replacing only S_2 instead of the double sum in (4) and after rescaling the indices $a = m+k-1$, $b = n+k$, we get

$$\begin{aligned} \pi_{i,j,k} &= \frac{(x_1x_2x_3)^k}{x_1-x_2} \sum_{m=j-k+1}^{i-k+1} \sum_{n=0}^{j-k} (x_1^m x_2^n - x_1^n x_2^m) x_3^{i+j-2k+1-m-n} \\ &= \sum_{a=j}^{i} \sum_{b=k}^{j} \pi_{a,b} x_3^{i+j+k-a-b} = \sum_{(a,b,c)\sqsubset(i,j,k)} \pi_{a,b} x_3^c, \end{aligned}$$

as wanted. □

Of course there is a multiplication rule for three-variable Schur polynomials which is compatible with (3) and the lemma above, but we do not want to discuss it here. For details see [4].

3.2 The cohomology of partial flag varieties

In this notes the rational cohomology rings of partial flag varieties play an essential role. The partial flag variety $Fl_{d_1,d_2,\ldots,d_l}$, for $1 \le d_1 < d_2 < \ldots < d_l = N$, is defined by

$$Fl_{d_1,d_2,\ldots,d_l} = \{V_{d_1} \subset V_{d_2} \subset \ldots \subset V_{d_l} = \mathbb{C}^N \mid \dim(V_i) = i\}.$$

A special case is $Fl_{k,N}$, the Grassmannian variety of all k-planes in $\mathbb{C}^N$, also denoted $\mathscr{G}_{k,N}$. The dimension of the partial flag variety is given by

$$\dim Fl_{d_1,d_2,\dots,d_l} = N^2 - \sum_{i=1}^{l-1} (d_{i+1}-d_i)^2 - d_1^2.$$

The rational cohomology rings of the partial flag varieties are well known and we only recall those facts that we need in this notes.

Lemma 3.2. *$\mathrm{H}(\mathscr{G}_{k,N})$ is isomorphic to the vector space generated by all $\pi_{i_1,i_2,\dots,i_k}$ modulo the relations*

$$\pi_{N-k+1,0,\dots,0} = 0, \quad \pi_{N-k+2,0,\dots,0} = 0, \quad \dots \quad , \pi_{N,0,\dots,0} = 0, \tag{5}$$

where there are exactly $k-1$ zeros in the multi-indices of the Schur polynomials.

A consequence of the multiplication rules for Schur polynomials is that

Corollary 3.3. *The Schur polynomials $\pi_{i_1,i_2,\dots,i_k}$, for $N-k \geq i_1 \geq i_2 \geq \ldots \geq i_k \geq 0$, form a basis of $\mathrm{H}(\mathscr{G}_{k,N})$*

Thus, the dimension of $\mathrm{H}(\mathscr{G}_{k,N})$ is $\binom{N}{k}$, and up to a degree shift, its graded dimension is $\begin{bmatrix} N \\ k \end{bmatrix}$.

Another consequence of the multiplication rules is that

Corollary 3.4. *The Schur polynomials $\pi_{1,0,0,\dots,0}, \pi_{1,1,0,\dots,0}, \dots, \pi_{1,1,1,\dots,1}$ (the elementary symmetric polynomials) generate $\mathrm{H}(\mathscr{G}_{k,N})$ as a ring.*

Furthermore, we can introduce a non-degenerate trace form on $\mathrm{H}(\mathscr{G}_{k,N})$ by giving its values on the basis elements

$$\varepsilon(\pi_\lambda) = \begin{cases} (-1)^{\lfloor \frac{k}{2} \rfloor}, & \lambda = (N-k,\dots,N-k) \\ 0, & \text{else} \end{cases}. \tag{6}$$

This makes $\mathrm{H}(\mathscr{G}_{k,N})$ into a commutative Frobenius algebra. One can compute the basis dual to $\{\pi_\lambda\}$ in $\mathrm{H}(\mathscr{G}_{k,N})$, with respect to ε. It is given by

$$\widehat{\pi}_{\lambda_1,\dots,\lambda_k} = (-1)^{\lfloor \frac{k}{2} \rfloor} \pi_{N-k-\lambda_k,\dots,N-k-\lambda_1}. \tag{7}$$

We can also express the cohomology rings of the partial flag varieties $Fl_{1,2,N}$ and $Fl_{2,3,N}$ in terms of Schur polynomials. Indeed, we have

$$\begin{aligned} \mathrm{H}(Fl_{1,2,N}) &= \mathbb{Q}[x_1,x_2]/(\pi_{N-1,0}, \pi_{N,0}), \\ \mathrm{H}(Fl_{2,3,N}) &= \mathbb{Q}[x_1+x_2, x_1x_2, x_3]/(\pi_{N-2,0,0}, \pi_{N-1,0,0}, \pi_{N,0,0}). \end{aligned} \tag{8}$$

The natural projection map $p_1 : Fl_{1,2,N} \to \mathscr{G}_{2,N}$ induces

$$p_1^* : \mathrm{H}(\mathscr{G}_{2,N}) \to \mathrm{H}(Fl_{1,2,N}), \tag{9}$$

which is just the inclusion of the polynomial rings. Analogously, the natural projection map $p_2 : Fl_{2,3,N} \to \mathscr{G}_{3,N}$, induces

$$p_2^* : \mathrm{H}(\mathscr{G}_{3,N}) \to \mathrm{H}(Fl_{2,3,N}), \tag{10}$$

which is also given by the inclusion of the polynomial rings.

4 Foams

In this section we begin to define the foams we will work with (foams were called pre-foams in [11] and in [17]. This distinction is irrelevant for the purposes of this notes). The philosophy behind these foams will be explained in Section 5. The basic examples of foams are given in Figure 1. These foams are composed of three types of facets: simple, double and triple facets. The double facets are coloured and the triple facets are marked to show the difference. Intersecting such a foam with a plane results in a web, as long as the plane avoids the singularities

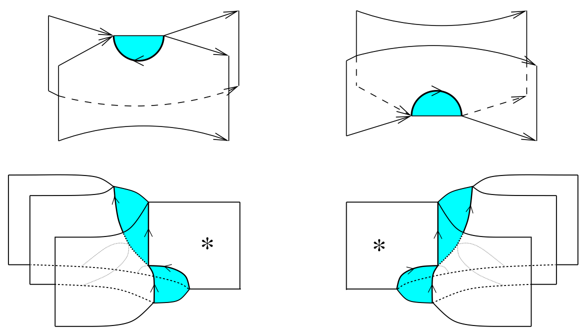


Figure 1: Some elementary foams

where six facets meet, such as on the right in Figure 1. Recall that a *web* is a planar trivalent graph with three types of edges: simple, double and triple which contain closed loops (simple, double, triple) and that only the simple edges are equipped with an orientation.

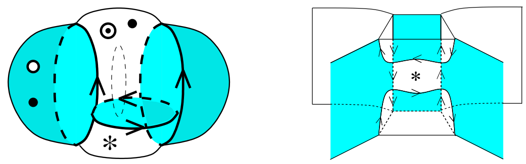
We adapt the definition of a world-sheet foam given in [15] to our setting.

Definition 4.1. Let s_γ be a finite closed oriented 4-valent graph, which may contain disjoint circles. We assume that all edges of s_γ are oriented. A cycle in s_γ is defined to be a circle or a closed sequence of edges which form a piece-wise linear circle. Let Σ be a compact orientable possibly disconnected surface, whose connected components are white, coloured or marked, also denoted by simple, double or triple. Each component can have a boundary consisting of several disjoint circles and can have additional decorations which we discuss below. A closed *foam* u is the identification space Σ/s_γ obtained by glueing boundary circles of Σ to cycles in s_γ such that every edge and circle in s_γ is glued to exactly three boundary circles of Σ and such that for any point $p \in s_\gamma$:

1. if p is an interior point of an edge, then p has a neighborhood homeomorphic to the letter Y times an interval with exactly one of the facets being double, and at most one of them being triple. For an example see Figure 1;
2. if p is a vertex of s_γ, then it has a neighborhood as shown on the r.h.s. in Figure 1.

We call s_γ the *singular graph*, its edges and vertices *singular arcs* and *singular vertices*, and the connected components of $u - s_\gamma$ the *facets*.

Furthermore the facets can be decorated with dots. A simple facet can only have black dots (●), a double facet can also have white dots (○), and a triple facet besides black and white dots can have double dots (⊙). Dots can move freely on a facet but are not allowed to cross singular arcs. See Figure 2 for examples of foams.



a) *b*)

Figure 2: a) A foam. b) An open foam

Note that the cycles to which the boundaries of the simple and the triple facets are glued are always oriented, whereas the ones to which the boundaries of the double facets are glued are not. Note also that there are two types of singular vertices. Given a singular vertex v, there are precisely two singular edges which meet at v and bound a triple facet: one oriented toward v, denoted e_1, and one oriented away from v, denoted e_2. If we use the "left hand rule", then the cyclic ordering of the facets incident to e_1 and e_2 is either $(3,2,1)$ and $(3,1,2)$ respectively, or the other way around. We say that v is of type I in the first case and of type II in the second case (see Figure 3). When we go around a triple facet we see that there have to be as many singular

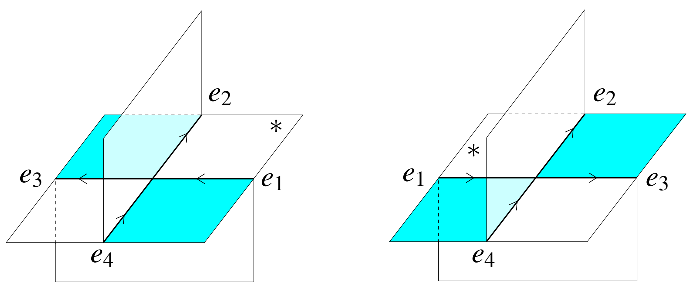


Figure 3: Singular vertices of type I and type II

vertices of type I as there are of type II for the cyclic orderings of the facets to match up. This shows that for a closed foam the number of singular vertices of type I is equal to the number of singular vertices of type II.

We can intersect a foam u generically by a plane W in order to get a web, as long as the plane avoids the vertices of s_γ. The orientation of s_γ determines the orientation of the simple edges of the web according to the convention in Figure 4.

Suppose that for all but a finite number of values $i \in]0,1[$, the plane $W \times i$ intersects u generically. Suppose also that $W \times 0$ and $W \times 1$ intersect u generically and outside the vertices of s_γ. We call $W \times I \cap u$ an *open* foam. Interpreted as morphisms we read open foams from bottom to top, and their composition consists of placing one foam on top of the other, as long as their boundaries are isotopic and the orientations of the simple edges coincide.

We now define the q-degree of a foam. Let u be a foam, u_1, u_2 and u_3 the disjoint union of its simple and double and marked facets respectively and $s_\gamma(u)$ its singular graph. Define the

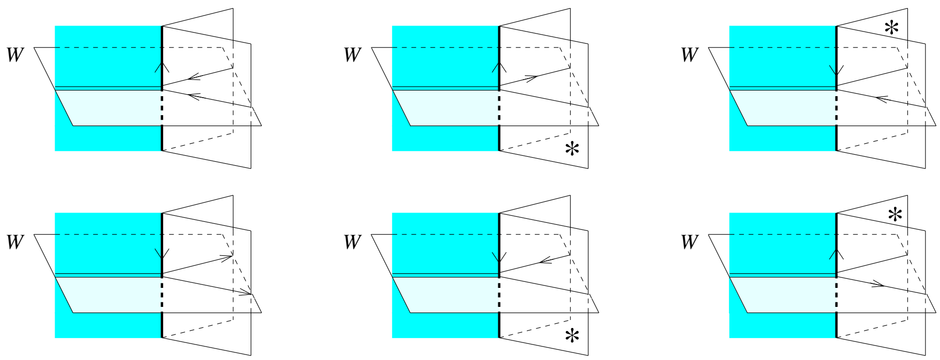


Figure 4: Orientations near a singular arc

partial q-gradings of u as

$$q_i(u) = \chi(u_i) - \frac{1}{2}\chi(\partial u_i \cap \partial u), \qquad i = 1,2,3$$
$$q_{s_\gamma}(u) = \chi(s_\gamma(u)) - \frac{1}{2}\chi(\partial\, s_\gamma(u)).$$

where χ is the Euler characteristic and ∂ denotes the boundary.

Definition 4.2. Let u be a foam with $d_\bullet$ dots of type $\bullet$, $d_\circ$ dots of type $\circ$ and $d_\odot$ dots of type $\odot$. The q-grading of u is given by

$$q(u) = -\sum_{i=1}^{3} i(N-i)q_i(u) - 2(N-2)q_{s_\gamma}(u) + 2d_\bullet + 4d_\circ + 6d_\odot.$$

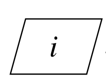
The following result is a direct consequence of the definitions.

Lemma 4.3. *$q(u)$ is additive under the glueing of foams.*

5 The KL formula and the evaluation of closed foams

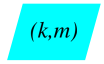
Let us briefly recall the philosophy behind the foams. Losely speaking, to each closed foam should correspond an element in the cohomology ring of a configuration space of planes in some big $\mathbb{C}^M$. The singular graph imposes certain conditions on those planes. The evaluation of a foam should correspond to the evaluation of the corresponding element in the cohomology ring. Of course one would need to find a consistent way of choosing the volume forms on all of those configuration spaces for this to work. However, one encounters a difficult technical problem when working out the details of this philosophy. Without explaining all the details, we can say that the problem can only be solved by figuring out what to associate to the singular vertices. Ideally we would like to find a combinatorial solution to this problem, but so far it has eluded us. That is the reason why we are forced to use the KL formula.

We denote a simple facet with i dots by

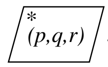


.

Recall that $\pi_{k,m}$ can be expressed in terms of $\pi_{1,0}$ and $\pi_{1,1}$. In the philosophy explained above, the latter should correspond to ● and ○ on a double facet respectively. We can then define



as being the linear combination of dotted double facets corresponding to the expression of $\pi_{k,m}$ in terms of $\pi_{1,0}$ and $\pi_{1,1}$. Analogously we expressed $\pi_{p,q,r}$ in terms of $\pi_{1,0,0}$, $\pi_{1,1,0}$ and $\pi_{1,1,1}$ (see Section 3). The latter correspond to ●, ○ and ⊙ on a triple facet respectively, so we can make sense of



.

In the sequel, we shall give a definition of the KL formula for the evaluation of foams and state some of its basic properties. The KL formula was introduced by A. Kapustin and Y. Li [5] to generalize Vafa's work [16] in the context of the evaluation of 2-dimensional TQFTs to the case of smooth surfaces with boundary. It was later extended to the case of foams by M. Khovanov and L. Rozansky in [7], who interpreted singular arcs as boundary conditions as in [5]. Khovanov and Rozansky adapted the KL formula to a general sort of foam. In this notes we have to specify the input data which allows us to use it for the evaluation of our foams. The normalization is ours and is used to obtain integral relations.

5.1 The general framework

Let $u = \Sigma/s_\gamma$ be a closed foam with singular graph s_γ and without any dots on it. Let F denote an arbitrary i-facet, $i \in \{1,2,3\}$, with a 1-facet being a simple facet, a 2-facet being a double facet and a 3-facet being a triple facet.

Each i-facet can be decorated with dots, which correspond to generators of the rational cohomology ring of the Grassmannian $\mathscr{G}_{i,N}$, i.e. $\mathrm{H}(\mathscr{G}_{i,N},\mathbb{Q})$. Alternatively, we can associate to every i-facet F, i variables $x_1^F \dots, x_i^F$, with $\deg x_i^F = 2i$, and the potential $W(x_1^F,\dots,x_i^F)$, which is the polynomial defined such that

$$W(\sigma_1,\dots,\sigma_i) = y_1^{N+1} + \dots + y_i^{N+1},$$

where σ_j is the j-th elementary symmetric polynomial in the variables $y_1,\dots,y_i$. The Jacobi algebra J_W

$$J_W = \mathbb{Q}[x_1^F,\dots,x_i^F]/(\partial_i W),$$

where $\partial_i W$ denote the ideal generated by the partial derivatives of W, is isomorphic to the rational cohomology ring of the Grassmannian $\mathscr{G}_{i,N}$. Note that up to a multiple the top degree nonvanishing element in this Jacobi algebra is $\pi_{N-i,\dots,N-i}$ (multiindex of length i), i.e. the polynomial in variables $x_1^F,\dots,x_i^F$ which gives $\pi_{N-i,\dots,N-i}$ after replacing the variable x_j^F by $\pi_{1,\dots,1,0,\dots,0}$ with exactly j 1's, $1 \le j \le i$ (see also Subsection 3.1). We define the trace (volume) form ε on $\mathrm{H}(\mathscr{G}_{i,N},\mathbb{Q})$ by giving it on the basis of the Schur polynomials:

$$\varepsilon(\pi_{j_1,\dots,j_i}) = \begin{cases} (-1)^{\lfloor \frac{i}{2} \rfloor} & \text{if} \quad (j_1,\dots,j_i) = (N-i,\dots,N-i) \\ 0 & \text{else} \end{cases}.$$

The KL formula associates to u an element in the product of the cohomology rings of the Jacobi algebras J, over all the facets in the foam. Alternatively, we can see this element as a polynomial, $KL_u \in J$, in all the variables associated to the facets. Now, let us put some dots on u. Recall that a dot corresponds to an elementary symmetric polynomial. So a linear combination of dots on u is equivalent to a polynomial, f, in the variables of the dotted facets. Let ε denote the product of the trace forms ε_{J_i} over all facets of u. The value of this dotted foam we define to be

$$\langle u\rangle_{KL} := \varepsilon\left(\prod_F \frac{\det(\partial_i\partial_j W_F)^{g(F)}}{(N+1)^{g'(F)}} KL_u f\right). \tag{11}$$

The product is over all facets F and W_F is the potential associated to F. For any i-facet F, $i = 1,2,3$, the symbol $g(F)$ denotes the genus of F and $g'(F) = ig(F)$. If u is a closed surface without singularities we define $KL_u = 1$ and $\langle\ \rangle_{KL}$ reduces to an extension to colored closed surfaces of the formula introduced by Vafa in [16]. The *Vafa factor*

$$\prod_F \frac{\det(\partial_i\partial_j W_F)^{g(F)}}{(N+1)^{g'(F)}}$$

computes the contribution of the handles in the facets of u.

Having explained the general idea, we are left with defining the element KL_u for a dotless foam. For that we have to explain Khovanov and Rozansky's extension of the KL formula to foams [7], which uses the theory of matrix factorizations.

5.2 Decoration of foams

As we said, to each facet we associate certain variables (depending on the type of facet), a potential and the corresponding Jacobi algebra. If the variables associated to a facet F are $x_1,\dots,x_i$, then we define $R_F = \mathbb{Q}[x_1,\dots,x_i]$. It is immediate that the KL formula gives zero if the argument of ε in Equation 11 contains an element of $\partial_i W_F$: for any $Q \in \bigotimes_F R_F$ we have that

$$\varepsilon(Q\partial_i W_F) = 0. \tag{12}$$

Now we consider the edges. To each edge we associate a matrix factorization whose potential is equal to the signed sum of the potentials of the facets that are glued along this edge. We define it to be a certain tensor product of Koszul factorizations. In the cases we are interested in there are always three facets glued along an edge, with two possibilities: either two simple facets and one double facet, or one simple, one double and one triple facet. In the first case, we denote the variables of the two simple facets by x and y and take the potentials to be x^{N+1} and y^{N+1} respectively, according to the convention in Figure 5. To the double facet we associate the variables s and t and the potential $W(s,t)$. To the edge we associate the matrix factorization which is the tensor product of Koszul factorizations given by

$$MF_1 = \begin{Bmatrix} A', & x+y-s \\ B', & xy-t \end{Bmatrix}, \tag{13}$$

where A' and B' are given by

$$A' = \frac{W(x+y,xy) - W(s,xy)}{x+y-s},$$
$$B' = \frac{W(s,xy) - W(s,t)}{xy-t}.$$

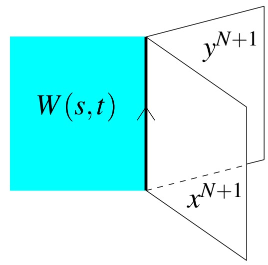


Figure 5: Singular edge of type $(1,1,2)$

Note that $(x+y-s)A'+(xy-t)B' = x^{N+1}+y^{N+1}-W(s,t)$.

In the second case, the variable of the simple facet is x and the potential is x^{N+1}, the variables of the double facet are s and t and the potential is $W(s,t)$, and the variables of the triple face are p, q and r and the potential is $W(p,q,r)$.

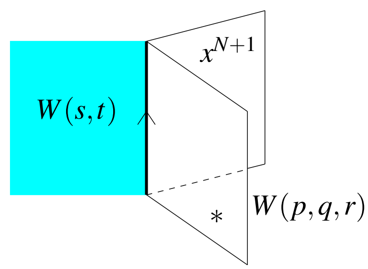


Figure 6: Singular edge of type $(1,2,3)$

Define the polynomials

$$A = \frac{W(x+s,xs+t,xt)-W(p,xs+t,xt)}{x+s-p}, \tag{14}$$

$$B = \frac{W(p,xs+t,xt)-W(p,q,xt)}{xs+t-q}, \tag{15}$$

$$C = \frac{W(p,q,xt)-W(p,q,r)}{xt-r}, \tag{16}$$

so that

$$(x+s-p)A+(xs+t-q)B+(xt-r)C = x^{N+1}+W(s,t)-W(p,q,r).$$

To such an edge we associate the matrix factorization given by the following tensor product of Koszul factorizations:

$$MF_2 = \begin{Bmatrix} A, & x+s-p \\ B, & xs+t-q \\ C, & xt-r \end{Bmatrix}. \tag{17}$$

In both cases, if the edges have the opposite orientation we associate the matrix factorizations $(MF_1)_\bullet$ and $(MF_2)_\bullet$ respectively.

Next we explain what we associate to a singular vertex. First of all, for each vertex v, we define its local graph γ_v to be the intersection of a small sphere centered at v with the foam. Then the vertices of γ_v correspond to the edges of u that are incident to v, to which we had associated matrix factorizations. In this notes all local graphs γ_v are in fact tetrahedrons. However, recall that there are two types of vertices (see the remarks below Definition 4.1). Label the six facets that are incident to a vertex v by the numbers $1,2,3,4,5$ and 6. Furthermore, denote the edge along which are glued the facets i, j and k by (ijk). Denote the matrix factorization associated to the edge (ijk) by M_{ijk}, if the edge points toward v, and by $(M_{ijk})_\bullet$, if the edge points away from v. Note that M_{ijk} and $(M_{ijk})_\bullet$ are both defined over $R_i \otimes R_j \otimes R_k$.

Now we can take the tensor product of these four matrix factorizations, over the polynomial rings of the facets of the foam, that correspond to the vertices of γ_v. This way we obtain the matrix factorization M_v, whose potential is equal to 0, and so it is a 2-complex and we can take its homology.

To each vertex v we associate an element $O_v \in \mathrm{H}_{\mathbf{MF}}(M_v)$. More precisely, if v is of type I, then

$$\begin{aligned}\mathrm{H}_{\mathbf{MF}}(M_v) \cong \mathrm{Ext}\,(&MF_1(x,y,s_1,t_1)\otimes_{s_1,t_1} MF_2(z,s_1,t_1,p,q,r)\,,\\ &MF_1(y,z,s_2,t_2)\otimes_{s_2,t_2} MF_2(x,s_2,t_2,p,q,r))\,.\end{aligned}\tag{18}$$

If v is of type II, then

$$\begin{aligned}\mathrm{H}_{\mathbf{MF}}(M_v) \cong \mathrm{Ext}\,(&MF_1(y,z,s_2,t_2)\otimes_{s_2,t_2} MF_2(x,s_2,t_2,p,q,r)\,,\\ &MF_1(x,y,s_1,t_1)\otimes_{s_1,t_1} MF_2(z,s_1,t_1,p,q,r))\,.\end{aligned}\tag{19}$$

Both isomorphisms hold up to a global shift in q. Note that

$$MF_1(x,y,s_1,t_1)\otimes_{s_1,t_1} MF_2(z,s_1,t_1,p,q,r) \simeq MF_1(y,z,s_2,t_2)\otimes_{s_2,t_2} MF_2(x,s_2,t_2,p,q,r),$$

because both tensor products are homotopy equivalent to the factorization

$$\left\{\begin{matrix} *, & x+y+z-p \\ *, & xy+xz+yz-q \\ *, & xyz-r \end{matrix}\right\}.$$

We have not specified the l.h.s. of the latter Koszul matrix, because of Lemma 2.3. If v is of type I, we take O_v to be the cohomology class of a fixed degree 0 homotopy equivalence

$$w_v\colon MF_1(x,y,s_1,t_1)\otimes_{s_1,t_1} MF_2(z,s_1,t_1,p,q,r) \to MF_1(y,z,s_2,t_2)\otimes_{s_2,t_2} MF_2(x,s_2,t_2,p,q,r).$$

The choice of O_v is unique up to a scalar, because the graded dimension of the Ext-group in (18) is equal to

$$q^{3N-6}\,\mathrm{qdim}(\mathrm{H}(M_v)) = q^{3N-6}[N][N-1][N-2] = 1+q(\dots),$$

where $(\dots)$ is a polynomial in q. Note that M_v is homotopy equivalent to the matrix factorization which corresponds to the closure of Υ in [9], which allows one to compute the graded dimension above using the results in the latter paper. If v is of type II, we take O_v to be the cohomology class of the homotopy inverse of w_v. Note that a particular choice of w_v fixes O_v for both types of vertices and that the value of the KL formula for a closed foam does not depend on that choice because there are as many singular vertices of type I as there are of type II (see the remarks below Definition 4.1). We do not know an explicit formula for O_v. Although such a formula would be very interesting to have, we do not need it for the purposes of this notes.

5.3 The KL derivative and the evaluation of closed foams

From the definition, every boundary component of each facet F is either a circle or a cyclicly ordered finite sequence of edges, such that the beginning of the next edge corresponds to the end of the previous edge. For every boundary component choose an edge e and denote the differential of the matrix factorization associated to this edge by D_e. Let $R_F = \mathbb{Q}[x_1,\dots,x_k]$. The KL derivative of D_e in the variables $x_1,\dots,x_k$ associated to the facet F, is an element from $\mathrm{End}(M) \cong M \otimes M_\bullet$, given by:

$$O_{F,e} = \partial \hat{D_e} = \frac{1}{k!}\sum_{\sigma\in S_k} (\operatorname{sgn}\sigma)\partial_{\sigma(1)}D_e\partial_{\sigma(2)}D_e \dots \partial_{\sigma(k)}D_e, \tag{20}$$

where S_k is the symmetric group on k letters, and $\partial_i D$ is the partial derivative of D with respect to the variable x_i. For all the other edges e' in the boundary of F we take $O_{F,e'}$ to be the identity. Denote the set of facets whose boundary contains e by $F(e)$. For every edge define $O_e \in \mathrm{End}(M)$ as the composite

$$O_e = \prod_{F\in F(e)} O_{F,e}.$$

The order of the factors in O_e is irrelevant as we will prove it in Lemma 5.2.

Let $\mathscr{V}$ and $\mathscr{E}$ be the sets of all vertices and all edges of the singular graph s_γ of a foam u. Denote the matrix factorization associated to an edge e by M_e ($M_e = MF_1$ if e is of type $(1,1,2)$ and $M_e = MF_2$ if e is of type $(1,2,3)$). Recall that the factorization M_v associated to a singular vertex is the tensor product of the matrix factorizations associated to the edges that are incident to v. Consider the factorization M_{s_γ} given by the tensor product

$$M_{\mathrm{s}_\gamma} = \left(\bigotimes_{v\in\mathscr{V}} M_v\right) \otimes \left(\bigotimes_{e\in\mathscr{E}} M_e \otimes (M_e)_\bullet\right). \tag{21}$$

From the definition of M_v we see that we can group all the factorizations involved in pairs of mutually dual factorizations: for every edge e we can pair M_e coming from $M_e \otimes (M_e)_\bullet$ with $(M_e)_\bullet$ coming from M_v and $(M_e)_\bullet$ from $M_e \otimes (M_e)_\bullet$ can be paired with M_e coming from M_v. Using super-contraction on each pair we get a map

$$\phi_\gamma \colon M_{\mathrm{s}_\gamma} \to \mathbb{Q}[\mathbf{x}_u],$$

where $\mathbf{x}_u$ is the set of variables associated to all the facets of u.

Definition 5.1. $KL_u = \phi_\gamma\left(\big(\bigotimes_{v\in\mathscr{V}} O_v\big) \otimes \big(\bigotimes_{e\in\mathscr{E}} O_e\big)\right).$

Note that the O_e and O_v can be seen as tensors with indices associated to the facets that meet at e and v respectively. So we can super-contract all the tensor factors O_e and O_v, with respect to a particular facet F, along a cycle that bounds F. From Definition 5.1 we see that if we do this for all boundary components of all facets we also get KL_u.

Lemma 5.2. *KL_u does not depend on the order of the factors in O_e.*

Proof. Let e be an edge in the boundary of facets F and F'. Since the potential W_e is a sum of the individual potentials associated to the facets that are glued along e, each depending on its own set of variables, we have $\partial_i \partial'_j W_e = 0$. Therefore, applying $\partial_i \partial'_j$ to both sides of the relation $D_e^2 = W_e$ gives

$$[\partial_i D_e, \partial'_j D_e]_s = -[D_e, \partial_i \partial'_j D_e]_s,$$

and the term on the r.h.s. is annihilated after the super-contraction because it is a coboundary. This means that the KL derivatives of D w.r.t. different facets super-commute. □

Lemma 5.3. *KL_u does not depend on the choice of the preferred edges.*

Proof. It suffices to prove the claim for only one facet F with one boundary component. Label the edges that bound F by $e_1, \ldots, e_k$ and take e_1 as the preferred edge of F. Suppose first that F is a simple or a triple facet, so that its boundary consists of an oriented cycle of s_γ. Suppose also that e_i is oriented from v_i to v_{i+1}. Since $[O_{F,e}, O_{F',e}]_s = 0$ for every $F' \neq F$ we can assume that $O_{e_1} = O_{F,e_1}$ without loss of generality. The contribution to KL_u of the facet F is given by

$$\mathrm{STr}_{W_F}\left(\partial D_{\hat{e_1}} O_{v_1} O_{v_2} \ldots O_{v_k}\right),$$

where STr_{W_F} is the partial supertrace w.r.t. the indices associated to F.

The relevant part of a small neighborhood of the vertex v_1 is depicted in Figure 7, where only the facet F is shown. From Equation (18) it follows that O_v can be seen as a homomorphism from

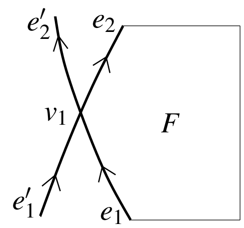


Figure 7: Singular vertex

$M_e(e_1) \otimes M_e(e'_1)$ to $M_e(e_2) \otimes M_e(e'_2)$, where (e_i) denotes the variables associated to the facets that are glued along e. Therefore we have that $[D, O_v]_s = 0$, where $D = D_{e_1} + D_{e'_1} + D_{e_2} + D_{e'_2}$ and we are using the convention that the composite of two non-composable homomorphisms is zero. Note that $\partial_i D = \partial_i D_{e_1} + \partial_i D_{e_2}$ since e'_1 and e'_2 are not variables associated to F. Therefore $[D, O_v]_s = 0$ implies

$$[\partial_i D, O_v]_s = -[D, \partial_i O_v]_s \tag{22}$$

by partial differentiation w.r.t. a variable of F. This implies

$$\mathrm{STr}_{W_F}\left(\partial D_{\hat{e_1}} O_{v_1} O_{v_2} \ldots O_{v_k}\right) = \mathrm{STr}_{W_F}\left(O_{v_1} \partial D_{\hat{e_2}} O_{v_2} \ldots O_{v_k}\right),$$

since terms involving the r.h.s. of Equation (22) get killed by STr.

Now suppose that F is a double facet. The boundary of F is not an oriented cycle in s_γ. Suppose a small neighborhood of v has a part as depicted in Figure 8. In this case O_v can be seen as a homomorphism from $M_e(e_1) \otimes M_e(e'_1)_\bullet$ to $M_e(e_2) \otimes M_e(e'_2)_\bullet$, so that D and O_v

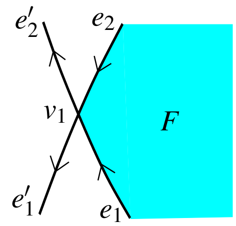


Figure 8: Double facet near a singular vertex

super-commute, where $D = D_{e_1} + (D_{e'_1})_\bullet + (D_{e_2})_\bullet + D_{e'_2}$. Taking a partial derivative of both sides of the relation $[D, O_v]_s$ relative to a variable associated to F we obtain that

$$\mathrm{STr}_{W_F}\left(\partial D_{\hat{e_1}} O_{v_1} O_{v_2} \dots O_{v_k}\right) = \mathrm{STr}_{W_F}\left(O_{v_1} \partial (D_{\hat{e_2}})_\bullet O_{v_2} \dots O_{v_k}\right),$$

which proves the claim. $\square$

5.4 Some computations

In this subsection we compute the KL evaluation of some closed foams.

Spheres

The values of dotted spheres are easy to compute. Note that for any sphere with dots f the KL formula gives

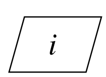
$$\varepsilon(f).$$

Therefore for a simple sphere we get 1 if $f = x^{N-1}$, for a double sphere we get -1 if $f = \pi_{N-2,N-2}$ and for a triple sphere we get -1 if $f = \pi_{N-3,N-3,N-3}$.

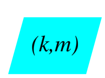
Note that the evaluation of spheres corresponds to the trace on the cohomology of the Grassmannian $\mathrm{H}(\mathscr{G}_{i,N})$ for $i = 1, 2, 3$ in Equation (7).

Dot conversion and dot migration

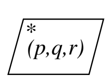
Since KL_u takes values in the tensor product of the Jacobi algebras of the potentials associated to the facets of u, we see that for a simple facet we have $x^N = 0$, for a double facet $\pi_{i,j} = 0$ if $i \geq N-1$, and for a triple facet $\pi_{p,q,r} = 0$ if $p \geq N-2$. We call these the *dot conversion relations*:



$= 0 \quad \text{if} \quad i \geq N$



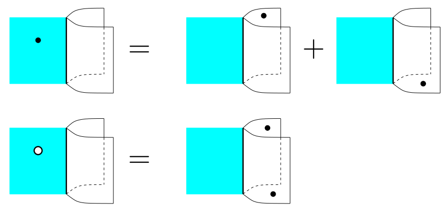
$= 0 \quad \text{if} \quad k \geq N-1$

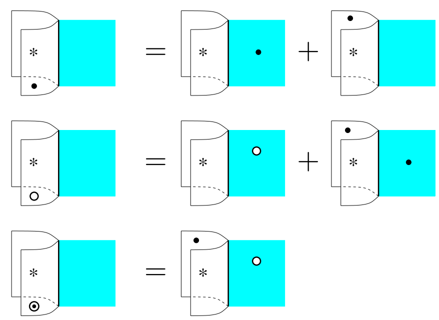


$= 0 \quad \text{if} \quad p \geq N-2$

The dot conversion relations are related to the relations defining the cohomology ring of the Grassmannian $\mathscr{G}_{k,N}$ for $k = 1, 2, 3$ in Equation (5).

To each edge along which two simple facets with variables x and y and one double facet with the variables s and t are glued, we associated the matrix factorization MF_1 with entries $x+y-s$ and $xy-t$. Therefore $\mathrm{Ext}(MF_1, MF_1)$ is a module over $R/(x+y-s, xy-t)$. Hence, we obtain the *dot migration relations* along this edge. Analogously, to the other type of singular edge along which are glued a simple facet with variable x, a double facet with variable s and t, and a triple facet with variables p, q and r, we associated the matrix factorization MF_2. Note that $\mathrm{Ext}(MF_2, MF_2)$ is a module over $R/(x+s-p, xs+t-q, xt-r)$, which gives us the *dot migration relations* along this edge:





The dot migration relations are related to the relations in the cohomology ring of the partial flag varieties $Fl_{1,2,N}$ and $Fl_{2,3,N}$ in Equation (8) under the projection maps in Equations (9) and (10).

The $(1,1,2)$-theta foam

Recall that $W(s,t)$ is the polynomial such that $W(x+y, xy) = x^{N+1} + y^{N+1}$. More precisely, we have

$$W(s,t) = \sum_{i+2j=N+1} a_{ij} s^i t^j,$$

with $a_{N+1,0} = 1$, $a_{N+1-2j,j} = \frac{(-1)^j}{j}(N+1)\binom{N-j}{j-1}$, for $2 \le 2j \le N+1$, and $a_{ij} = 0$ otherwise. In particular $a_{N-1,1} = -(N+1)$. We have

$$W_1'(s,t) = \sum_{i+2j=N+1} i a_{ij} s^{i-1} t^j,$$

$$W_2'(s,t) = \sum_{i+2j=N+1} j a_{ij} s^i t^{j-1}.$$

By $W_1'(s,t)$ and $W_2'(s,t)$, we denote the partial derivatives of $W(s,t)$ with respect to the first and the second variable, respectively.

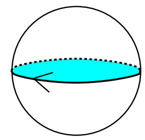


Figure 9: A dotless (1,1,2)-theta foam

Consider the $(1,1,2)$-theta foam of Figure 9. According to the conventions of Subsection 5.2 we have variables x and y on the lower and upper simple facets respectively, and the variables s and t on the double facet. To the singular circle we assign the matrix factorization

$$MF_1 = \begin{Bmatrix} A', & x+y-s \\ B', & xy-t \end{Bmatrix}.$$

Recall that

$$A' = \frac{W(x+y,xy) - W(s,xy)}{x+y-s}, \tag{23}$$

$$B' = \frac{W(s,xy) - W(s,t)}{xy-t}. \tag{24}$$

Hence, the differential of this matrix factorization is given by the following 4 by 4 matrix:

$$D = \begin{pmatrix} 0 & D_1 \\ D_0 & 0 \end{pmatrix},$$

where

$$D_0 = \begin{pmatrix} A', & xy-t \\ B', & s-x-y \end{pmatrix}, \quad D_1 = \begin{pmatrix} x+y-s, & xy-t \\ B', & -A' \end{pmatrix}.$$

Note that we are using a convention for tensor products of matrix factorizations that is different from the one in [11]. The KL formula assigns the polynomial, $KL_{\Theta_1}(x,y,s,t)$, which is given by the supertrace of the twisted differential of D

$$KL_{\Theta_1} = \mathrm{STr}\left(\partial_x D \partial_y D \frac{1}{2}(\partial_s D \partial_t D - \partial_t D \partial_s D)\right).$$

Straightforward computation gives

$$KL_{\Theta_1} = -B'_s(A'_x - A'_y) - (A'_x + A'_s)(B'_y + xB'_t) + (A'_y + A'_s)(B'_x + yB'_t), \tag{25}$$

where by A'_i and B'_i we have denoted the partial derivatives with respect to the variable i. From the definitions (23) and (24) we have

$$
\begin{aligned}
A'_x - A'_y &= (y-x)\frac{W'_2(x+y,xy)-W'_2(s,xy)}{x+y-s},\\
A'_x + A'_s &= \frac{W'_1(x+y,xy)-W'_1(s,xy)+y(W'_2(x+y,xy)-W'_2(s,xy))}{x+y-s},\\
A'_y + A'_s &= \frac{W'_1(x+y,xy)-W'_1(s,xy)+x(W'_2(x+y,xy)-W'_2(s,xy))}{x+y-s},\\
B'_s &= \frac{W'_1(s,xy)-W'_1(s,t)}{xy-t},\\
B'_x + yB'_t &= y\frac{W'_2(s,xy)-W'_2(s,t)}{xy-t},\\
B'_y + xB'_t &= x\frac{W'_2(s,xy)-W'_2(s,t)}{xy-t}.
\end{aligned}
$$

After substituting this back into (25), we obtain

$$
KL_{\Theta_1} = (x-y)\det\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \tag{26}
$$

where

$$
\begin{aligned}
\alpha &= \frac{W'_1(x+y,xy)-W'_1(s,xy)}{x+y-s},\\
\beta &= \frac{W'_2(x+y,xy)-W'_2(s,xy)}{x+y-s},\\
\gamma &= \frac{W'_1(s,xy)-W'_1(s,t)}{xy-t},\\
\delta &= \frac{W'_2(s,xy)-W'_2(s,t)}{xy-t}.
\end{aligned}
$$

From this formula we see that KL_{Θ_1} is homogeneous of degree $4N-6$ (remember that $\deg x = \deg y = \deg s = 2$ and $\deg t = 4$).

Since the evaluation is in the product of the Grassmannians corresponding to the three disks, i.e. in the ring $\mathbb{Q}[x]/(x^N)\times\mathbb{Q}[y]/(y^N)\times\mathbb{Q}[s,t]/(W'_1(s,t),W'_2(s,t))$, we have $x^N = y^N = 0 = W'_1(s,t) = W'_2(s,t)$. Also, we can express the monomials in s and t as linear combinations of the Schur polynomials $\pi_{k,l}$ (writing $s=\pi_{1,0}$ and $t=\pi_{1,1}$)), and we have $W'_1(s,t) = (N+1)\pi_{N,0}$ and $W'_2(s,t) = -(N+1)\pi_{N-1,0}$. Hence, we can write KL_{Θ_1} as

$$
KL_{\Theta_1} = (x-y)\sum_{N-2\geq k\geq l\geq 0}\pi_{k,l}\,p_{kl}(x,y),
$$

with p_{kl} being a polynomial in x and y. We want to determine which combinations of dots on the simple facets give rise to non-zero evaluations, so our aim is to compute the coefficient of $\pi_{N-2,N-2}$ in the sum on the r.h.s. of the above equation (i.e. in the determinant in (26)). For degree reasons, this coefficient is of degree zero, and so we shall only compute the parts of α,

β, γ and δ which do not contain x and y. We shall denote these parts by putting a bar over the Greek letters. Thus we have

$$\begin{aligned}\bar{\alpha} &= (N+1)s^{N-1},\\ \bar{\beta} &= -(N+1)s^{N-2},\\ \bar{\gamma} &= \sum_{i+2j=N+1,\, j\geq 1} ia_{ij}s^{i-1}t^{j-1},\\ \bar{\delta} &= \sum_{i+2j=N+1,\, j\geq 2} ja_{ij}s^{i}t^{j-2}.\end{aligned}$$

Note that we have

$$t\bar{\gamma}+(N+1)s^N = W_1'(s,t),$$

and

$$t\bar{\delta}-(N+1)s^{N-1} = W_2'(s,t),$$

and so in the cohomology ring of the Grassmannian $\mathscr{G}_{2,N}$, we have $t\bar{\gamma} = -(N+1)s^N$ and $t\bar{\delta} = (N+1)s^{N-1}$. On the other hand, by using $s = \pi_{1,0}$ and $t = \pi_{1,1}$, we obtain that in $\mathrm{H}(\mathscr{G}_{2,N}) \cong \mathbb{Q}[s,t]/(\pi_{N-1,0}, \pi_{N,0})$, the following holds:

$$s^{N-2} = \pi_{N-2,0} + tq(s,t),$$

for some polynomial q, and so

$$s^{N-1} = s^{N-2}s = \pi_{N-1,0}+\pi_{N-2,1}+stq(s,t) = t(\pi_{N-3,0}+sq(s,t)).$$

Thus, we have

$$\begin{aligned}\det\begin{pmatrix}\bar{\alpha} & \bar{\beta}\\ \bar{\gamma} & \bar{\delta}\end{pmatrix} &= (N+1)(\pi_{N-3,0}+sq(s,t))t\bar{\delta}+(N+1)\pi_{N-2,0}\bar{\gamma}+(N+1)q(s,t)t\bar{\gamma}\\ &= (N+1)^2(\pi_{N-3,0}+sq(s,t))s^{N-1}+(N+1)\pi_{N-2,0}\bar{\gamma}-(N+1)^2q(s,t)s^N\\ &= (N+1)^2\pi_{N-3,0}s^{N-1}+(N+1)\pi_{N-2,0}\bar{\gamma}. \qquad (27)\end{aligned}$$

Since

$$\bar{\gamma} = (N-1)a_{N-1,1}s^{N-2}+tr(s,t)$$

holds in the cohomology ring of the Grassmannian $\mathscr{G}_{2,N}$ for some polynomial $r(s,t)$, we have

$$\pi_{N-2,0}\bar{\gamma} = \pi_{N-2,0}(N-1)a_{N-1,1}s^{N-2} = -\pi_{N-2,0}(N-1)(N+1)s^{N-2}.$$

Also, we have that for every $k \geq 2$,

$$s^k = \pi_{k,0}+(k-1)\pi_{k-1,1}+t^2w(s,t),$$

for some polynomial w. Replacing this in (27) and bearing in mind that $\pi_{i,j} = 0$, for $i \geq N-1$, we get

$$\begin{aligned}\det\begin{pmatrix}\bar{\alpha} & \bar{\beta}\\ \bar{\gamma} & \bar{\delta}\end{pmatrix} &= (N+1)^2s^{N-2}(\pi_{N-2,0}+\pi_{N-3,1}-(N-1)\pi_{N-2,0})\\ &= (N+1)^2(\pi_{N-2,0}+(N-3)\pi_{N-3,1}+\pi_{2,2}w(s,t))(\pi_{N-3,1}-(N-2)\pi_{N-2,0})\\ &= -(N+1)^2\pi_{N-2,N-2}.\end{aligned}$$

Hence, we have

$$KL_{\Theta_1} = (N+1)^2(y-x)\pi_{N-2,N-2} + \sum_{\substack{N-2\geq k\geq l\geq 0\\ N-2>l}} c_{i,j,k,l}\pi_{k,l}x^i y^j.$$

Recall that in the product of the Grassmannians corresponding to the three disks, i.e. in the ring $\mathbb{Q}[x]/(x^N)\times\mathbb{Q}[y]/(y^N)\times\mathbb{Q}[s,t]/(\pi_{N-1,0},\pi_{N,0})$, we have

$$\varepsilon(x^{N-1}y^{N-1}\pi_{N-2,N-2}) = -1.$$

Therefore the only monomials f in x and y such that $\langle KL_{\Theta_1} f\rangle_{KL}\neq 0$ are $f_1 = x^{N-1}y^{N-2}$ and $f_2 = x^{N-2}y^{N-1}$, and $\langle KL_{\Theta_1} f_1\rangle_{KL} = -(N+1)^2$ and $\langle KL_{\Theta_1} f_2\rangle_{KL} = (N+1)^2$. Thus, we have that the value of the theta foam with unlabelled 2-facet is nonzero only when the upper 1-facet has $N-2$ dots and the lower one has $N-1$ dots (and has the value $(N+1)^2$) and when the upper 1-facet has $N-1$ dots and the lower one has $N-2$ dots (and has the value $-(N+1)^2$). The evaluation of this theta foam with other labellings can be obtained from the result above by dot migration.

Up to normalization the KL evaluation of the $(1,1,2)$-theta foam corresponds to the trace on the cohomology ring of the partial flag variety $Fl_{1,2,N}$ in Equation (8) given by $\varepsilon(x_1^{N-2}x_2^{N-1}) = 1$, and where x_1 and x_2 correspond to the dots in the upper and lower facet respectively.

The $(1,2,3)$-theta foam

For the theta foam in Figure 10 the method is the same as in the previous case, just the computations are more complicated. In this case, we have one 1-facet, to which we associate the

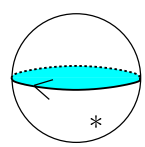


Figure 10: A dotless (1,2,3)-theta foam

variable x, one 2-facet, with variables s and t and the 3-facet with variables p, q and r. Recall that the polynomial $W(p,q,r)$ is such that $W(a+b+c, ab+bc+ac, abc) = a^{N+1}+b^{N+1}+c^{N+1}$. We denote by $W_i'(p,q,r)$, $i=1,2,3$, the partial derivative of W with respect to i-th variable. Also, let A, B and C be the polynomials given by

$$A = \frac{W(x+s,xs+t,xt)-W(p,xs+t,xt)}{x+s-p}, \tag{28}$$

$$B = \frac{W(p,xs+t,xt)-W(p,q,xt)}{xs+t-q}, \tag{29}$$

$$C = \frac{W(p,q,xt)-W(p,q,r)}{xt-r}. \tag{30}$$

To the singular circle of this theta foam, we associated the matrix factorization (see Equations (14)-(17)):

$$MF_2 = \begin{Bmatrix} A, & x+s-p \\ B, & xs+t-q \\ C, & xt-r \end{Bmatrix}.$$

The differential of this matrix factorization is the 8 by 8 matrix

$$D = \begin{pmatrix} 0 & D_1 \\ D_0 & 0 \end{pmatrix},$$

where

$$D_0 = \left(\begin{array}{c|c} d_0 & (xt-r)\,\mathrm{Id}_2 \\ \hline C\,\mathrm{Id}_2 & -d_1 \end{array} \right), \qquad D_1 = \left(\begin{array}{c|c} d_1 & (xt-r)\,\mathrm{Id}_2 \\ \hline C\,\mathrm{Id}_2 & -d_0 \end{array} \right).$$

Here d_0 and d_1 are the differentials of the matrix factorization

$$\begin{Bmatrix} A, & x+s-p \\ B, & xs+t-q \end{Bmatrix},$$

i.e.

$$d_0 = \begin{pmatrix} A & xs+t-q \\ B & p-x-s \end{pmatrix}, \qquad d_1 = \begin{pmatrix} x+s-p & xs+t-q \\ B & -A \end{pmatrix}.$$

The KL formula assigns to this theta foam the polynomial $KL_{\Theta_2}(x,s,t,p,q,r)$ given as the supertrace of the twisted differential of D, i.e.

$$KL_{\Theta_2} = \mathrm{STr}\left(\partial_x D \frac{1}{2}(\partial_s D \partial_t D - \partial_t D \partial_s D)\partial_3 D\hat{} \right),$$

where

$$\partial_3 D\hat{} = \frac{1}{3!}\,(\partial_p D \partial_q D \partial_r D - \partial_p D \partial_r D \partial_q D + \partial_q D \partial_r D \partial_p D$$
$$-\partial_q D \partial_p D \partial_r D + \partial_r D \partial_p D \partial_q D - \partial_r D \partial_q D \partial_p D).$$

After straightforward computations and some grouping, we obtain

$$\begin{aligned} KL_{\Theta_2} = {} & (A_p + A_s)\big[(B_t+B_q)(C_x+tC_r) - (B_x+sB_q)(C_t+xC_r) - (B_x - sB_t)C_q\big] \\ & + (A_p+A_x)\big[(B_s+xB_q)(C_t+xC_r) + (B_s - xB_t)C_q\big] \\ & + (A_x - A_s)\big[B_p(C_t+xC_r) - (B_t+B_q)C_p + B_pC_q\big] \\ & - A_t\big[((B_s+xB_q)+B_p)(C_x+tC_r) + ((B_s+xB_q) \\ & - (B_x+sB_q))C_p + ((sB_s - xB_x) + (s-x)B_p)C_q\big]. \end{aligned}$$

In order to simplify this expression, we introduce the following polynomials

$$\begin{aligned} a_{1i} &= \frac{W_i'(x+s,xs+t,xt) - W_i'(p,xs+t,xt)}{x+s-p}, \quad && i=1,2,3, \\ a_{2i} &= \frac{W_i'(p,xs+t,xt) - W_i'(p,q,xt)}{xs+t-q}, && i=1,2,3, \\ a_{3i} &= \frac{W_i'(p,q,xt) - W_i'(p,q,r)}{xt-r}, && i=1,2,3. \end{aligned}$$

Then from (28)-(30), we have

$$\begin{aligned} A_x + A_p &= a_{11} + sa_{12} + ta_{13}, \qquad & A_p + A_s &= a_{11} + xa_{12}, \\ A_x - A_s &= (s-x)a_{12} + ta_{13}, & A_t &= a_{12} + xa_{13}, \\ B_p &= a_{21}, & B_s - xB_t &= -x^2 a_{23}, \end{aligned}$$

$$\begin{aligned} sB_s - xB_x &= xta_{23}, & B_x - sB_t &= (t-sx)a_{23}, \\ B_t + B_q &= a_{22} + xa_{23}, & B_x + sB_q &= sa_{22} + ta_{23}, B_s + xB_q = xa_{22}, \\ C_p &= a_{31}, & C_q &= a_{32}, \\ C_x + tC_r &= ta_{33}, & C_t + xC_r &= xa_{33}. \end{aligned}$$

Using this KL_{Θ_2} becomes

$$KL_{\Theta_2} = (t - sx + x^2) \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{31}$$

Now the last part follows analogously as in the case of the $(1,1,2)$-theta foam. For degree reasons the coefficient of $\pi_{N-3,N-3,N-3}$ in the latter determinant is of degree zero, and one can obtain that it is equal to $(N+1)^3$. Thus, the coefficient of $\pi_{N-3,N-3,N-3}$ in KL_{Θ_2} is $(N+1)^3(t-sx+x^2)$ from which we obtain the value of the theta foam when the 3-facet is undotted. For example, we see that

$$\varepsilon\big(KL_{\Theta_2}\pi_{1,1}(s,t)^{N-3}x^{N-1}\big) = (N+1)^3.$$

It is then easy to obtain the values when the 3-facet is labelled by $\pi_{N-3,N-3,N-3}(p,q,r)$ using dot migration. The example above implies that

$$\varepsilon\big(KL_{\Theta_2}\pi_{N-3,N-3,N-3}(p,q,r)x^2\big) = (N+1)^3.$$

Up to normalization the KL evaluation of the $(1,2,3)$-theta foam corresponds to the trace on the cohomology ring of the partial flag variety $Fl_{2,3,N}$ in Equation (8) given by $\varepsilon(x^2\pi_{N-3,N-3,N-3}) = 1$, where $\pi_{N-3,N-3,N-3}$ correspond to a linear combination of dots in the triple facet and x corresponds to a dot in the upper simple facet (see Section 3).

For $N=3$, using the explicit formula for $W(p,q,r)$ we see that the determinant (31) is zero, which means that the $(1,2,3)$-theta foams would evaluate to zero, independently of the dots they may have. That is why we restrict the construction in this notes to the case of $N \geq 4$.

5.5 Normalization

It will be convenient to normalize the KL evaluation. Let u be a closed foam with graph Γ. Note that Γ has two types of edges: the ones incident to two simple facets and one double facet and the ones incident to one simple, one double and one triple facet. Edges of the same type form cycles in Γ. Let $e_{112}(u)$ be the total number of cycles in Γ with edges of the first type and $e_{123}(u)$ the total number of cycles with edges of the second type. We normalize the KL formula by dividing KL_u by

$$(N+1)^{2e_{112}+3e_{123}}.$$

In the sequel we only use this normalized KL evaluation keeping the same notation $\langle u\rangle_{KL}$. Note that with this normalization the KL-evaluation in the examples above always gives $0, -1$ or 1.

5.6 The glueing property

We now consider the glueing property of the KL formula, which is an important property of TQFT's.

Suppose that u is a foam with boundary Γ. We decorate the facets, singular arcs and singular vertices of u as in Subsection 5.2. Recall that the orientations of the singular arcs of u induce an orientation of Γ (see Figure 4). To each vertex v of Γ we associate the matrix factorization which is the matrix factorization associated to the singular arc of u that is bounded by v. To each circle in Γ we associate the Jacobi algebra of the corresponding facet in $\mathbb{Z}/2\mathbb{Z}$-degree i (mod 2), where $i = 1,2,3$. Then define the matrix factorization M_Γ as the tensor product of all the matrix factorizations of its vertices as given above and Jacobi algebras J_i in $\mathbb{Z}/2\mathbb{Z}$-degree i (mod 2) for all (if any) circles in Γ. The tensor product is taken over suitable rings so that M_Γ is a free module over R of finite rank, where R is the polynomial ring with rational coefficients in the variables of the facets of u that are bounded by Γ. The factorization M_Γ has potential zero, since for every edge e of Γ the individual potential W_e appears twice in W_Γ (one for each vertex bounding e) with opposite signs. The homology

$$\mathrm{H}_{\mathbf{MF}}(M_\Gamma) \cong \mathrm{Ext}(R, M_\Gamma) \tag{32}$$

is finite-dimensional and coincides with the one in [9] after using Lemma 2.1 to exclude the variables associated to all double and triple edges of Γ.

Let u be an open foam whose boundary consists of two parts Γ_1 and Γ_2, and denote by M_1 and M_2 the matrix factorizations associated to Γ_1 and Γ_2 respectively. We say that F is an *interior facet* of u if $\partial F \cap \partial u = \emptyset$. Restricting KL_u to the interior facets of u and doing the same to ε in Equation (11) we see that the KL formula associates to u an element of $\mathrm{Ext}(M_1, M_2)$.

If u' is another foam whose boundary consists of Γ_2 and Γ_3, then it corresponds to an element of $\mathrm{Ext}(M_2, M_3)$, while the element associated to the foam uu', which is obtained by glueing the foams u and u' along Γ_2, is equal to the composite of the elements associated to u and u'.

On the other hand, we can also see u as a morphism from the empty web to its boundary $\Gamma = \Gamma_2 \sqcup \Gamma_1^*$, where Γ_1^* is equal to Γ_1 but with the opposite orientation. In that case, the KL formula associates to it an element from

$$\mathrm{Ext}\big(R, M_{\Gamma_2} \otimes \big(M_{\Gamma_1}\big)_\bullet\big) \cong \mathrm{H}_{\mathbf{MF}}(\Gamma).$$

Both ways of applying the KL formula are equivalent up to a global q-shift by corollary 6 in [9].

In the case of a foam u with corners, i.e. a foam with two horizontal boundary components Γ_1 and Γ_2 which are connected by vertical edges, one has to "pinch" the vertical edges. This way one can consider u to be a morphism from the empty set to $\Gamma_2 \cup_v \Gamma_1^*$, where $\cup_v$ means that the webs are glued at their vertices. The same observations as above hold, except that $M_{\Gamma_2} \otimes \big(M_{\Gamma_1}\big)_\bullet$ is now the tensor product over the polynomial ring in the variables associated to the horizontal edges with corners.

The KL formula also has a general property that will be useful later. The KL formula defines a duality pairing between $\mathrm{Hom}_{\mathbf{Foam}_N}(\emptyset, \Gamma)$ and $\mathrm{Hom}_{\mathbf{Foam}_N}(\Gamma, \emptyset)$ as

$$(a, a') = \langle a'a \rangle_{KL}, \tag{33}$$

for $a \in \mathrm{Hom}_{\mathbf{Foam}_N}(\emptyset,\Gamma)$ and $a' \in \mathrm{Hom}_{\mathbf{Foam}_N}(\Gamma,\emptyset)$. From the duality pairing it follows that

$$\mathrm{Hom}_{\mathbf{Foam}_N}(\emptyset,\Gamma^*) = \mathrm{Hom}_{\mathbf{Foam}_N}(\Gamma,\emptyset).$$

The duality pairing also defines a canonical element

$$\psi_{\Gamma,\Gamma^*} \in \mathrm{Hom}_{\mathbf{Foam}_N}(\emptyset,\Gamma^*) \otimes \mathrm{Hom}_{\mathbf{Foam}_N}(\emptyset,\Gamma)$$

by

$$(\psi_{\Gamma,\Gamma^*}, a \otimes a') = (a,a')$$

Introducing a basis $\{a_i\}$ of $\mathrm{Hom}_{\mathbf{Foam}_N}(\emptyset,\Gamma)$ and its dual basis $\{a_j^*\}$ of $\mathrm{Hom}_{\mathbf{Foam}_N}(\Gamma,\emptyset)$ we have

$$\psi_{\Gamma,\Gamma^*} = \sum_j a_j \otimes a_j^*.$$

Suppose that a closed foam u contains two points p_1 and p_2 such that intersecting u with disjoint spheres centered in p_1 and p_2 result in two webs Γ_1 and Γ_2 and that $\Gamma_2 = \Gamma_1^*$. If we remove the parts inside those spheres from u and glue the boundary components Γ_1 and Γ_2 onto each other we obtain a new closed foam u' and the KL evaluations of u and u' are related by (see [7])

$$\langle u' \rangle_{KL} = \langle \psi_{\Gamma_1,\Gamma_1^*} u \rangle_{KL} = \sum_j \langle a_j^* u a_j \rangle_{KL}. \tag{34}$$

6 Odds and ends

6.1 Size of the kernel of the KL evaluation

The set of relations between foams given in [17] and [11] is clearly smaller that the kernel of the KL evaluation. Characterizing the kernel of the KL evaluation is interesting for several reasons. Having a finite set of relations generating the kernel would establish that the link homology of [11, 17] is purely combinatorial. This problem is related to the problem of finding a presentation by generators and relations of the category of tensor products of exterior powers of the fundamental representation of quantum $\mathfrak{sl}(N)$ and intertwiner maps. (a complete set of relations is conjectured in [14]). A solution of one of these problems could help solving the other.

6.2 Integrality of the KL evaluation

In all the examples above the Kapustin-Li evaluation always returns an integer. This motivates the following conjecture.

Conjecture 1. *For a closed foam u we have $\varepsilon\big(\langle u \rangle_{KL}\big) \in \mathbb{Z}$.*

Being true, this conjecture would imply that the link homology using foams is integral that is, all the $\mathfrak{sl}(N)$-homology groups would be modules over $\mathbb{Z}$ instead of $\mathbb{Q}$-vector spaces. Integrality of the $\mathfrak{sl}(N)$-link homology was conjectured in [17] where is was shown that being integral it would have torsion of order of at least N.

6.3 Representation theoretic interpretation

In [13] V. Mazorchuk and C. Stroppel gave a construction of the $\mathfrak{sl}(N)$-link homology using representation theory. In their interpretation a web corresponds to a composite of certain projective functors between parabolic singular blocks of category $\mathscr{O}$ and a foam to a natural transformation between projective functors. A closed foam u yields an endomorphism of the functor associated to the empty web and therefore an element of the ground field. Up to a normalization we expect these evaluations to be equal.

More recently C. Blanchet [1] informed me of another method to evaluate closed foams. His method consists of realizing the singular vertex as a certain 4-step partial flag manifold and the evaluation of a closed foam is obtained by taking a Frobenius trace in some configuration space, in the same spirit as the ε we used in the Kapustin-Li evaluation. This evaluation is defined over the integers. We also expect this evaluation to equal $\varepsilon\big(\langle u\rangle_{KL}\big)$.

Conjecture 2. *The Mazorchuk-Stroppel and the Blanchet evaluations of a closed foam u are both equal to* $\varepsilon\big(\langle u\rangle_{KL}\big)$.

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