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NOT EVEN KHOVANOV HOMOLOGY

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We construct a supercategory that can be seen as a skew version of (thickened) KLR algebras for the type A quiver. We use our supercategory to construct homological invariants of tangles and show that for every link our invariant gives a link homology theory supercategorifying the Jones polynomial. Our homology is distinct from even Khovanov homology and we present evidence supporting the conjecture that it is isomorphic to odd Khovanov homology. We also show that cyclotomic quotients of our supercategory give supercategorifications of irreducible finite-dimensional representations of $\mathfrak{gl}_n$ of level 2.

1. Introduction

After the appearance of odd Khovanov homology in [Ozsváth et al. 2013] there has been a certain interest in odd categorified structures and supercategorification (see, for example, [Lauda and Egilmez 2018; Ellis et al. 2014; Ellis and Lauda 2016; Ellis and Qi 2016; Kang et al. 2013; 2014; Lauda and Russell 2014; Naisse and Vaz 2018]). In contrast to (even) Khovanov homology, odd Khovanov homology has an anticommutative feature. Both theories categorify the Jones polynomial and both agree modulo 2, but they are intrinsically distinct (see [Shumakovitch 2011] for a study of the properties of odd Khovanov homology and a comparison with even Khovanov homology).

A construction of odd Khovanov homology using higher representation theory is still missing. In the case of even Khovanov homology this question was solved in [Webster 2017] using categorification of tensor products and the WRT invariant and in [Lauda et al. 2015] using categorical Howe duality.

In this paper we construct a supercategorification of the Jones invariant for tangles using higher representation theory. In particular, we define a supercategory in the spirit of Khovanov and Lauda's diagrammatics that can be seen as a superalgebra version of KLR algebras [Khovanov and Lauda 2009; Rouquier 2008] of level 2 for the A_n quiver. We present our supercategory in the form of a graphical calculus reminiscent of the thick calculus for categorified $\mathfrak{sl}_2$ [Khovanov et al. 2012] and $\mathfrak{sl}_n$

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[Stošić 2019] (see also [Ellis et al. 2014] for a thick calculus for the odd nilHecke algebra). Our supercategory admits cyclotomic quotients that supercategorify irreducibles of $U_q(\mathfrak{gl}_k)$ of level 2.

We use cyclotomic quotients of our supercategories as input to Tubbenhauer's approach [2014] to Khovanov–Rozansky homologies. It is based on q-Howe duality and uses only the lower half of the quantum group $U_q(\mathfrak{gl}_k)$ to produce an invariant of tangles. In our case we obtain an invariant that shares several similarities with odd Khovanov homology when restricted to links. For example, it decomposes as a direct sum of two copies of a reduced homology and it produces chronological Frobenius algebras, analogous to the ones that can be extracted from [Ozsváth et al. 2013] (see [Putyra 2014a] for explanations). Both theories coincide over $\mathbb{Z}/2\mathbb{Z}$. We also give computational evidence that our invariant is distinct from even Khovanov homology and that support the conjecture that for every link L it coincides with the odd Khovanov homology of L.

2. The supercategory $\mathfrak{R}$

2A. ***The supercategory $\mathfrak{R}(\nu)$.*** We follow [Brundan and Ellis 2017] regarding supercategories. For objects X, Y in a supercategory $\mathcal{C}$ we write $\mathrm{Hom}^0_{\mathcal{C}}(X, Y)$ (resp. $\mathrm{Hom}^1_{\mathcal{C}}(X, Y)$) for its space of even (resp. odd) morphisms and we write $p(f)$ for the parity of $f \in \mathrm{Hom}^i_{\mathcal{C}}(X, Y)$. If $\mathcal{C}$ has additionally a $\mathbb{Z}$-grading we denote by $q^s X$ a grading shift up of X by s units and we consider only morphisms that preserve the $\mathbb{Z}$-grading. In this case we write $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \bigoplus_{s\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, q^s Y)$. We follow the grading conventions in [Lauda et al. 2015], which are aligned with the tradition in link homology. This means that a map of degree s from X to Y yields a degree zero map from X to $q^s Y$.

Fix a unital ring $\Bbbk$. Let $\alpha_1, \dots, \alpha_n$ denote the simple roots of $\mathfrak{sl}_n$ and $\langle -, - \rangle$ their inner product: $\langle \alpha_i, \alpha_i \rangle = 2$, $\langle \alpha_i, \alpha_{i\pm1} \rangle = -1$, and $\langle \alpha_i, \alpha_j \rangle = 0$ otherwise. Fix also a choice of scalars Q consisting of $r_i, t_{ij} \in \Bbbk^\times$ for all $i, j \in I := \{1, \dots, n\}$, such that $t_{ii} = 1$ and $t_{ij} = t_{ji}$ when $|i - j| \neq 1$. Let also p_{ij} be defined by $p_{ii} = p_{i+1,i} = 1$ and otherwise $p_{ij} = 0$.

For each $\nu = \sum_{i\in I} \nu_i.i \in \mathbb{N}_0[I]$, we consider the set of (colored) sequences of ν,

$$\mathrm{CSeq}(\nu) := \Big\{ i_1^{(\varepsilon_1)} \cdots i_r^{(\varepsilon_r)} \mid \varepsilon_s \in \{1, 2\},\ \sum_s \varepsilon_s i_s = \nu \Big\}.$$

By convention we write simply i_s for $i_s^{(1)}$. Two sequences $\boldsymbol{i} \in \mathrm{CSeq}(\nu)$ and $\boldsymbol{j} \in \mathrm{CSeq}(\nu')$ can be concatenated into a sequence $\boldsymbol{ij}$ in $\mathrm{CSeq}(\nu + \nu')$.

Definition 2.1. The supercategory $\mathfrak{R}(\nu)$ is defined by the following data:

(a) The objects of $\mathfrak{R}(\nu)$ are finite formal sums of grading shifts of elements of $\mathrm{CSeq}(\nu)$.

(b) The morphism space $\mathrm{Hom}_{\mathfrak{R}(\nu)}(\boldsymbol{i}, \boldsymbol{j})$ from $\boldsymbol{i}$ to $\boldsymbol{j}$ is the $\mathbb{Z}$-graded $\Bbbk$-supervector space generated by vertical juxtaposition and horizontal juxtaposition of the diagrams below. Composition consists of vertical concatenation of diagrams. By convention we read diagrams from bottom to top and so, ab consists of stacking the diagram for a atop the one for b. Diagrams are equipped with a Morse function that keeps trace of the relative height of the generators. We consider isotopy classes of such diagrams that do not change the relative height of generators.

Generators.

- Simple and double *identities*

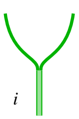
$\in \mathrm{Hom}^0_{\mathfrak{R}(\nu)}(i, i)$,

$\in \mathrm{Hom}^0_{\mathfrak{R}(\nu)}(i^{(2)}, i^{(2)})$,

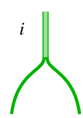
- *dots*

$\in \mathrm{Hom}^1_{\mathfrak{R}(\nu)}(i, q^2 i)$,

- *splitters*

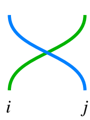


$\in \mathrm{Hom}^1_{\mathfrak{R}(\nu)}(i^{(2)}, q^{-1} ii)$,

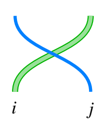


$\in \mathrm{Hom}^0_{\mathfrak{R}(\nu)}(ii, q^{-1} i^{(2)})$,

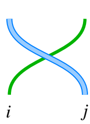
- and *crossings*



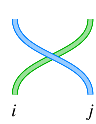
$\in \mathrm{Hom}^{p_{ij}}_{\mathfrak{R}(\nu)}(ij, q^{-\langle \alpha_i, \alpha_j \rangle} ji)$,



$\in \mathrm{Hom}^0_{\mathfrak{R}(\nu)}(i^{(2)} j, q^{-2\langle \alpha_i, \alpha_j \rangle} ji^{(2)})$,



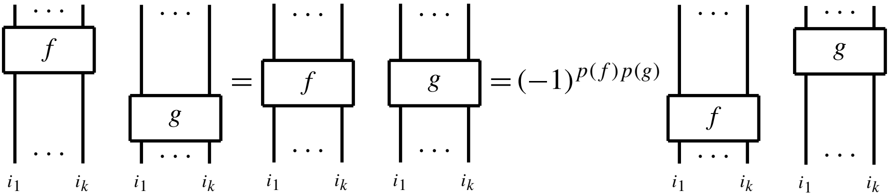
$\in \mathrm{Hom}^0_{\mathfrak{R}(\nu)}(ij^{(2)}, q^{-2\langle \alpha_i, \alpha_j \rangle} j^{(2)} i)$,



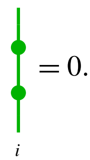
$\in \mathrm{Hom}^0_{\mathfrak{R}(\nu)}\big(i^{(2)} j^{(2)}, q^{-4\langle \alpha_i, \alpha_j \rangle} j^{(2)} i^{(2)}\big)$.

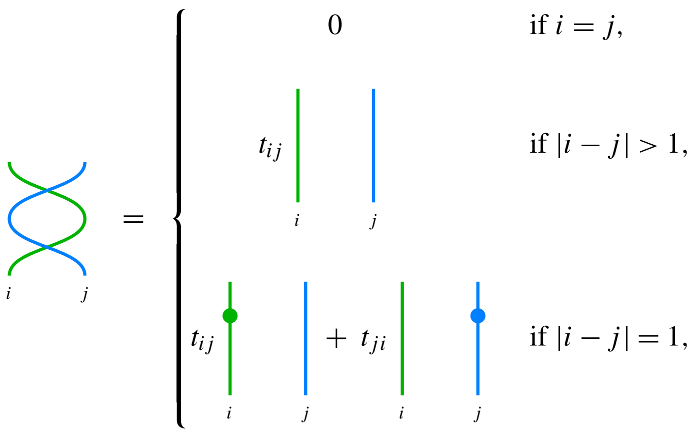
Relations. Morphisms are subject to the local relations (1) to (14) below.

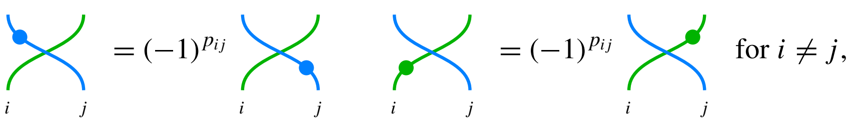
- For all f, g:

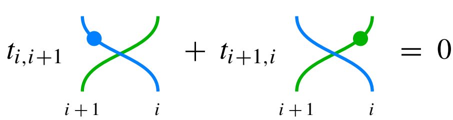
(1) 

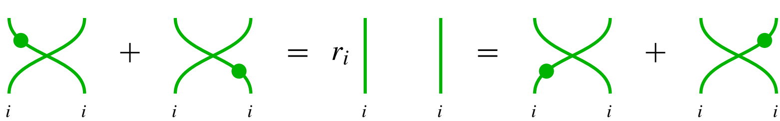
- For all $i, j, k \in I$:

(2) 

(3) 

(4) 

(5) 

(6) 

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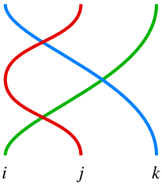
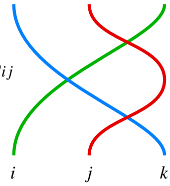
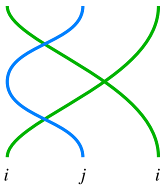


$= (-1)^{p_{jk}p_{ik}+p_{jk}p_{ij}+p_{ik}p_{ij}}$


unless $i = k$ and $|i-j| = 1$,

(8)


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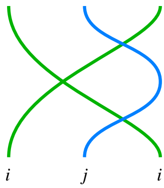
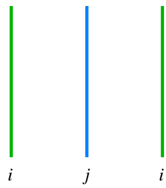
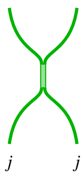


$= r_i t_{ij}$


if $|i-j| = 1$,

(9)


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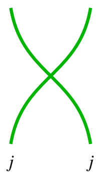

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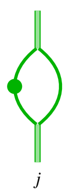

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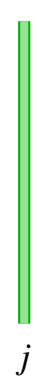

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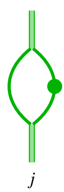
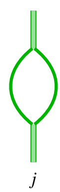
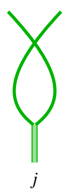
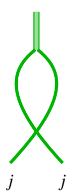




$= 0$

(11)


$= 0 =$


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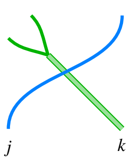

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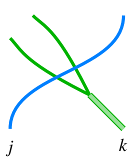

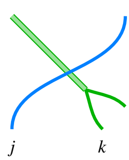

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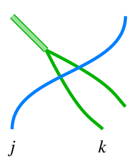

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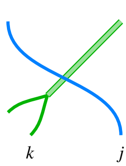

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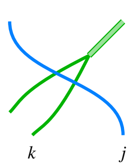

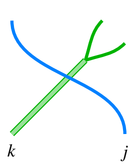

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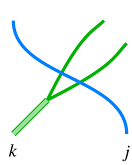
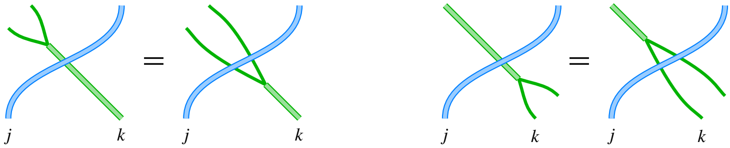


(14)



This ends the definition of $\mathfrak{R}(\nu)$.

In Section 2E below we show that $\mathfrak{R}(\nu)$ acts on a supercommutative ring.

Definition 2.2. We define the monoidal supercategory

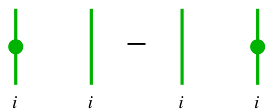
$$\mathfrak{R} = \bigoplus_{\nu \in \mathbb{N}_0[I]} \mathfrak{R}(\nu),$$

the monoidal structure given by horizontal composition of diagrams.

2B. *Further relations in $\mathfrak{R}(\nu)$.* We have several consequences of the defining relations.

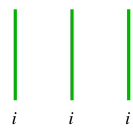
Lemma 2.3. *For all $i \in I$,*

(15)



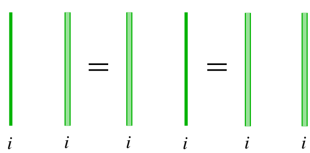
$= 0,$

(16)



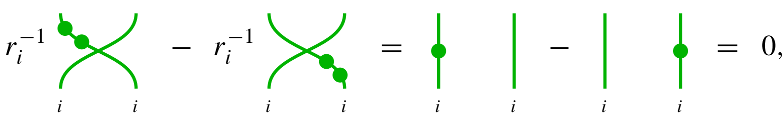
$= 0,$

(17)



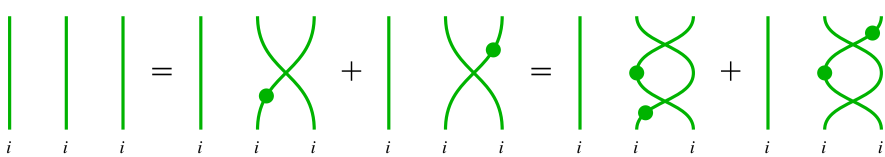
$= 0.$

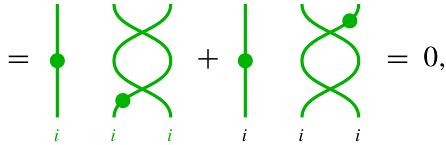
Proof. By (2) and (6),



which proves (15).

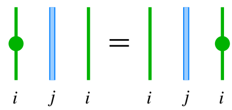
Also,



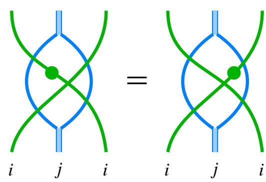


and this proves (16). Relations (17) are an easy consequence of (10) together with (16). □

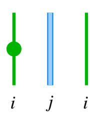
Lemma 2.4. *For all $i, j \in I$ with $|i - j| = 1$,*



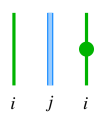
Proof. Start from the equality



Sliding up the dot on the left-hand side using (4) and (1), followed by (8) to pass the ii-crossing to the left, and simplifying using (3) and (10) gives

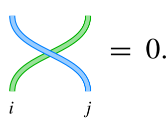
$-r_i t_{ij} t_{ji}$ 

Proceeding similarly on the right-hand side, but sliding the ii-crossing to the right gives

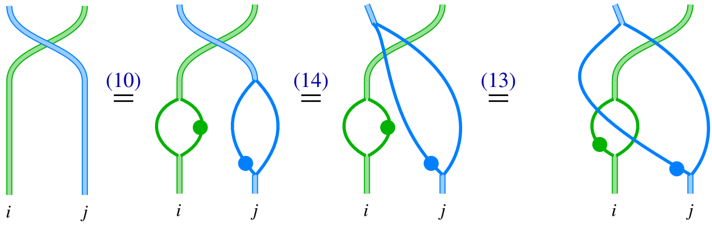
$-r_i t_{ij} t_{ji}$ 

and the claim follows. □

Lemma 2.5. *For all $i, j \in I$ with $|i - j| = 1$,*



Proof. We compute:

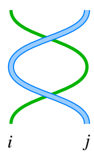


which is zero if $i = j \pm 1$ by (4), (5) and (2). □

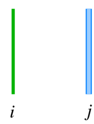
The following are easy consequences of the defining relations of $\mathfrak{R}(\nu)$.

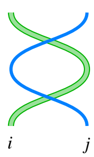
Lemma 2.6. *For all $i, j \in I$,*

Lemma 2.7. *For all $i, j \in I$,*

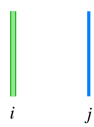


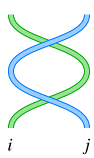
$=$ $\begin{cases} t_{ij}^2 & \text{if } |i-j|>1, \\ 0 & \text{otherwise,} \end{cases}$



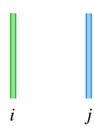


$=$ $\begin{cases} t_{ij}^2 & \text{if } |i-j|>1, \\ 0 & \text{otherwise,} \end{cases}$

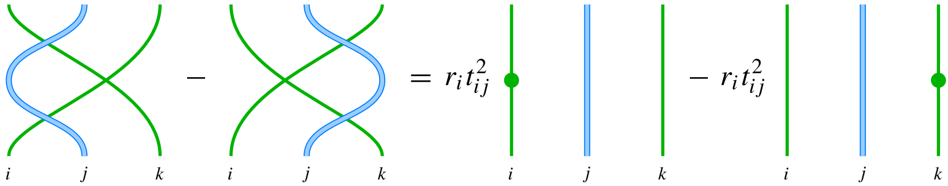




$=$ $\begin{cases} t_{ij}^4 & \text{if } |i-j|>1, \\ 0 & \text{otherwise.} \end{cases}$



Lemma 2.8. *If* $|i-j|=1$ *and* $i=k$*,*



If $i \neq j \neq k$ *and at least one of the strands is double, then the right hand side is zero.*

Let

$$\mathrm{Seq}(\nu) := \left\{ i_1^{(\varepsilon_1)} \cdots i_r^{(\varepsilon_r)} \in \mathrm{CSeq}(\nu) \mid \varepsilon_s = 1 \right\} \subset \mathrm{CSeq}(\nu).$$

The superalgebra

$$\overline{\mathfrak{R}}(\nu) = \bigoplus_{\boldsymbol{i},\boldsymbol{j} \in \mathrm{Seq}(\nu)} \mathrm{Hom}_{\mathfrak{R}(\nu)}(\boldsymbol{i}, \boldsymbol{j}),$$

is the subsuperalgebra of the Hom-superalgebra of $\mathfrak{R}(\nu)$ consisting of all diagrams having only simple strands. If we interpret $\overline{\mathfrak{R}}(\nu)$ as a superalgebra version of a level 2 cyclotomic KLR algebra for $\mathfrak{sl}_n$ then $\mathfrak{R}(\nu)$ can be seen as version of the thick calculus [Khovanov et al. 2012; Stošić 2019] for this superalgebra. It is not hard to see that both the center and the supercenter of $\overline{\mathfrak{R}}(\nu)$ are zero.

2C. *Cyclotomic quotients.* Fix a $\mathfrak{sl}_n$-weight Λ and denote by $R^\Lambda(\nu)$, $\overline{\mathfrak{R}}^\Lambda(\nu)$ and $\mathfrak{R}^\Lambda(\nu)$ the cyclotomic quotients of $R(\nu)$, $\overline{\mathfrak{R}}(\nu)$ and $\mathfrak{R}(\nu)$. The following is immediate.

Lemma 2.9. *If* Λ *is of level* 2 *then the algebras*

$$\overline{\mathfrak{R}}^\Lambda(\nu) \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \quad \textit{and} \quad R^\Lambda(\nu) \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$$

are isomorphic (*after collapsing the* $\mathbb{Z}/2\mathbb{Z}$ *grading of* $\overline{\mathfrak{R}}^\Lambda(\nu)$).

We depict a morphism of $\mathfrak{R}^\lambda(\nu)$ by decorating the rightmost region of each diagram D with the weight Λ. This defines weights for all regions of D.

The supercategory $\mathfrak{R}^\Lambda := \bigoplus_{\nu \in \mathbb{N}_0[I]} \mathfrak{R}^\Lambda(\nu)$ is not monoidal anymore, but it is a (left) module category over $\mathfrak{R}$, where $\mathfrak{R}$ acts by adding diagrams of $\mathfrak{R}$ to the left of diagrams from $\mathfrak{R}^\Lambda$. This is expressed by a bifunctor

$$\Phi\colon \mathfrak{R} \times \mathfrak{R}^\lambda \to \mathfrak{R}^\lambda. \tag{18}$$

2D. ***A super 2-category.*** There is a super 2-category around $\mathfrak{R}(\nu)$, paralleling the case of Khovanov–Lauda and Rouquier. An element $\boldsymbol{i} = i_1^{(\varepsilon_1)} \cdots i_r^{(\varepsilon_r)}$ in CSeq(ν) corresponds to a root $\alpha_{\boldsymbol{i}} := \sum_s \varepsilon_s \alpha_s$. Let

$$\Lambda(n,d) := \big\{\mu \in \{0,1,2\}^n \mid \mu_1 + \cdots + \mu_n = d\big\}.$$

Define $\mathcal{R}(n,d)$ as the super 2-category with objects the elements of $\Lambda(n,d)$ and with morphism supercategories $\mathrm{HOM}_{\mathcal{R}(n,d)}(\mu,\mu')$ the various $\mathfrak{R}(\nu)$. In other words, a 1-morphisms $\mu \to \mu'$ is a sequence $\boldsymbol{i}$ such that $\mu' - \mu = \alpha_{\boldsymbol{i}}$ and the 2-morphism space $\boldsymbol{i} \to \boldsymbol{j}$ is $\mathrm{Hom}_{\mathfrak{R}(\nu)}(\boldsymbol{i},\boldsymbol{j})$.

Similarly we define the super 2-category $\mathcal{R}^\Lambda(n,d)$ by using the cyclotomic quotient with respect with the integral dominant weight Λ. Both super 2-categories $\mathcal{R}^\Lambda(n,d)$ have diagrammatic presentations with regions labeled by objects Λ. The 2-morphisms in $\mathcal{R}^\lambda(n,d)$ are presented as a collection of 2-morphisms in $\mathcal{R}(n,d)$ with rightmost region decorated with Λ, subjected to the same relations together with the cyclotomic condition. This defines a label for every region of a diagram of $\mathcal{R}^\Lambda(n,d)$.

For later use, we denote

$$F_{\boldsymbol{i}}\lambda := F_{i_1^{(\varepsilon_1)}\cdots i_r^{(\varepsilon_r)}}\lambda := F_{i_1}^{(\varepsilon_1)} \cdots F_{i_r}^{(\varepsilon_r)}\lambda$$

the 1-morphisms of $\mathcal{R}^\Lambda(n,d)$ and, by abuse of notation, the objects of $\mathfrak{R}^\Lambda$.

2E. ***Action on a supercommutative ring.*** We now construct an action of $\mathfrak{R}(\nu)$ on exterior spaces.

2E1. *Demazure operators on an exterior algebra.* Let $V = \bigwedge(y_1,\dots,y_d)$ be the exterior algebra in d variables. This algebra is naturally graded by word length. Denote by $|z|$ the degree of the homogeneous element z.

The symmetric group $\mathfrak{S}_d$ acts on V by the permutation action,

$$wy_i = y_{w(i)}$$

for all $w \in \mathfrak{S}_d$.

Define operators ∂_i for $i = 1,\dots,d-1$ on V by the following rules:

$$\partial_i(y_k) = \begin{cases} 1 & i = k, k+1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{and} \quad \partial_i(fg) = \partial_i(f)g + (-1)^{|f|} f \partial_i(g),$$

for all $f, g \in V$ such that $fg \neq 0$.

The following can be checked through a simple computation.

Lemma 2.10. *The operators ∂_i satisfy the relations $\partial_i^2 = 0$, $\partial_i\partial_j + \partial_j\partial_i = 0$ if $|i-j| > 1$, and $\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$.*

2E2. *An action of $\mathfrak{R}(\nu)$ on supercommutative rings.* For $\boldsymbol{i} \in \mathrm{CSeq}(\nu)$ let

$$P\boldsymbol{i} = \bigwedge\bigl(x_{1,1}, x_{1,\varepsilon_1}, \ldots, x_{d,1}, x_{d,\varepsilon_d}\bigr)\boldsymbol{i},$$

be an exterior algebra in $\sum_i \nu_i$ generators, and set

$$P(\nu) = \bigoplus_{\boldsymbol{i} \in \mathrm{CSeq}(\nu)} P\boldsymbol{i}.$$

We extend the action of $\mathfrak{S}_d$ from V to $P(\nu)$ by declaring that

$$w x_{r,1} = x_{w(r),1}, \qquad w x_{r,\varepsilon_r} = x_{w(r),\varepsilon_{r+1}},$$

or $w \in \mathfrak{S}_d$.

Below we denote by $\partial_{u,z}$ the Demazure operator with respect to the variables u and z.

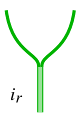
To the object $\boldsymbol{i} \in \mathfrak{R}(\nu)$ we associate the idempotent $\boldsymbol{i} \in P\boldsymbol{i}$. The defining generators of $\mathfrak{R}(\nu)$ act on P as follows. A diagram D acts as zero on $P\boldsymbol{i}$ unless the sequence of labels in the bottom of D is $\boldsymbol{i}$.

- Dots

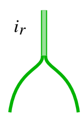
$$: p\boldsymbol{i} \mapsto x_{r,1} p\boldsymbol{i},$$

- Splitters

(19)

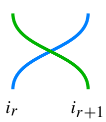


$$: p\boldsymbol{i} \mapsto \partial_{x_{r,1},x_{r,2}}(p)\boldsymbol{i},$$



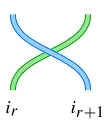
$$: p\boldsymbol{i} \mapsto x_{r,1}\partial_{x_{r,1},x_{r,2}}(p)\boldsymbol{i},$$

- Crossings



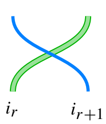
$$: p\boldsymbol{i} \mapsto \begin{cases} r_{i_r}\partial_{x_{r,1},x_{r+1,1}}(p)\boldsymbol{i} & \text{if} \quad i_r = i_{r+1}, \\ (t_{i_{r+1}i_r}x_{r,1} + t_{i_r i_{r+1}}x_{r+1,1})s_r(p\boldsymbol{i}) & \text{if} \quad i_r = i_{r+1}+1, \\ s_r(p\boldsymbol{i}) & \text{else,} \end{cases}$$

(20)

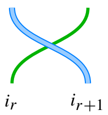


$$: p\boldsymbol{i} \mapsto \begin{cases} 0 & \text{if} \quad i_r = i_{r+1}, \quad \text{or} \quad i_s = i_{s+1}+1, \\ s_r(p\boldsymbol{i}) & \text{else,} \end{cases}$$

(21)



$$: p\boldsymbol{i} \mapsto \begin{cases} 0 & \text{if} \quad i_r = i_{r+1}, \\ f_{2,1}(x_{r,1}, x_{r,2}, x_{r+1,1})s_r(p\boldsymbol{i}) & \text{if} \quad i_s = i_{s+1}+1, \\ s_r(p\boldsymbol{i}) & \text{else,} \end{cases}$$

(22)  $: p\boldsymbol{i} \mapsto \begin{cases} 0 & \text{if} \quad i_r = i_{r+1}, \\ f_{1,2}(x_{r,1}, x_{r+1,1}, x_{r+1,2}) s_r(p\boldsymbol{i}) & \text{if} \quad i_s = i_{s+1} + 1, \\ s_r(p\boldsymbol{i}) & \text{else,} \end{cases}$

where

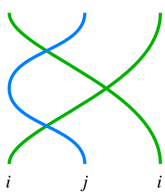
$$f_{2,1}(x_{r,1}, x_{r,2}, x_{r+1,1}) \\ = t_{i_r i_{r+1}} t_{i_{r+1} i_r} x_{r,1} x_{r+1,1} + t_{i_r i_{r+1}} t_{i_{r+1} i_r} x_{r,1} x_{r,2} + t^2_{i_{r+1} i_r} x_{r,2} x_{r+1,1}.$$

$$f_{1,2}(x_{r,1}, x_{r+1,1}, x_{r+1,2}) \\ = -t^2_{i_r i_{r+1}} x_{r,1} x_{r,2} + t_{i_r i_{r+1}} t_{i_{r+1} i_r} x_{r,2} x_{r+1,1} - t_{i_{r+1} i_r} t_{i_{r+1} i_r} x_{r,1} x_{r+1,1}.$$

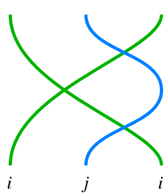
Proposition 2.11. *The assignment above defines an action of* $\mathfrak{R}(\nu)$ *on* $P(\nu)$.

Proof. By a long and rather tedious computation one can check that the operators above satisfy the defining relations of $\mathfrak{R}(\nu)$.

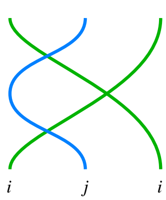
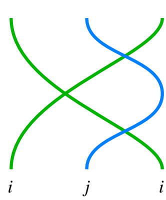
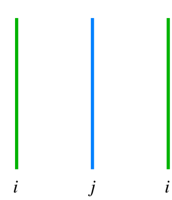
The relations involving the action of the generators of $\overline{\mathfrak{R}}(\nu)$ are easy to check by direct computation. For example, for $\nu = 2i + j$, with $j = i + 1$ we have

 $(f) = (t_{ij} x_1 + t_{ji} x_2) s_1 r_i \partial_2 s_1 (f),$

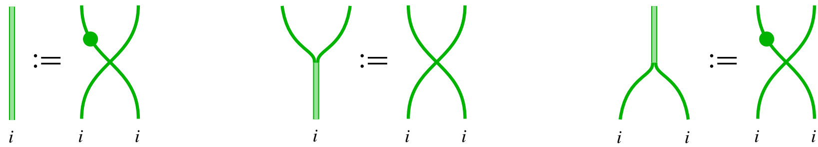
and

 $(f) = s_2 r_i \partial_1 (t_{ij} x_2 + t_{ji} x_3) s_2 (f) = r_i t_{ij} f - (t_{ij} x_1 + t_{ji} x_2) s_1 r_i \partial_2 s_1 (f),$

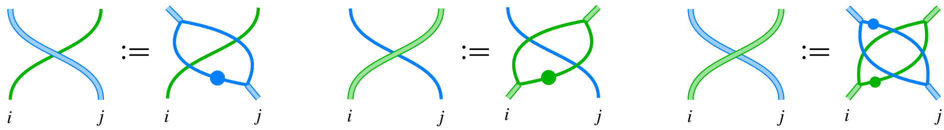
and so, for any $f(x_1, x_2, x_3) \in Piji$,

 (f) $+$  (f) $=$ $r_i t_{ij}$  $(f).$

Setting as in [Khovanov et al. 2012],



and



then it follows that the action of the generators of $\mathfrak{R}(\nu)$ on $P(\nu)$ is given by the operators (19), (20), (21) and (22) and satisfy the defining relations of $\mathfrak{R}(\nu)$. □

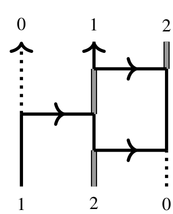
3. A topological invariant

In [Tubbenhauer 2014] q-skew Howe duality is used to show how to write as a web in a form that uses only the lower part of $U_q(\mathfrak{gl}_k)$. In this language, the formula for the $\mathfrak{sl}_2$-commutator becomes one of Lusztig's higher quantum Serre relations [1993, §7]. It is also proved in [Tubbenhauer 2014] that this results in a well defined evaluation of closed webs allowing to write any link diagram as a linear combination of words in the various F_i in $U^- := U_q^-(\mathfrak{gl}_k)$.

This allows a categorification of webs using only (cyclotomic) KLR algebras [Khovanov and Lauda 2009; Rouquier 2008] instead of the whole 2-quantum group $\mathcal{U}(\mathfrak{gl}_k)$ [Khovanov and Lauda 2010; Rouquier 2008]. In this context, the unit and counit maps of the several adjunctions in $\mathcal{U}(\mathfrak{gl}_k)$ that are used as differentials in the Khovanov–Rozansky chain complex can be written as composition with elements of the KLR algebra. Taking cyclotomic KLR algebras of level 2 gives Khovanov homology. The approach in [Tubbenhauer 2014] is easily adapted to tangles, which we do in this section for level 2 in the context of the supercategories introduced in Section 2.

3A. *Supercategorification of $\mathfrak{gl}_2$-webs and flat tangles.*

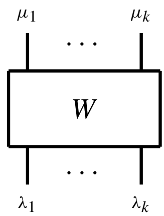
Our webs have strands labeled from {0, 1, 2} which we depict as "invisible," "simple," and "double," as in the example below. All the strands point either up or to the right and sometimes we omit the orientations in the pictures.



For $\lambda = (\lambda_1, \dots, \lambda_k) \in \{0, 1, 2\}^k$ and $\epsilon \in \{0, 1\}$ with $|\lambda| = 2\ell + \epsilon$, we put $\Lambda = (2)^\ell \epsilon = (2, \dots, 2, \epsilon, 0, \dots, 0)$ and we define

$$\mathfrak{W}(\lambda) = \mathrm{HOM}_{\mathcal{R}^\Lambda(k,|\lambda|)}(\Lambda, \lambda).$$

Let W be a $\mathfrak{gl}_2$-web with all ladders pointing to the right. Suppose that W has the bottom boundary labeled λ and the top boundary labeled μ, with $\lambda, \mu \in \{0, 1, 2\}^k$ and $|\lambda| = |\mu|$. We write W as a word in the F_i in $U_q^-(\mathfrak{gl}_k)$ applied to a vector v_λ of $\mathfrak{gl}_k$-weight λ.



$$= F_{i_1} \cdots F_{i_r}(v_\lambda).$$

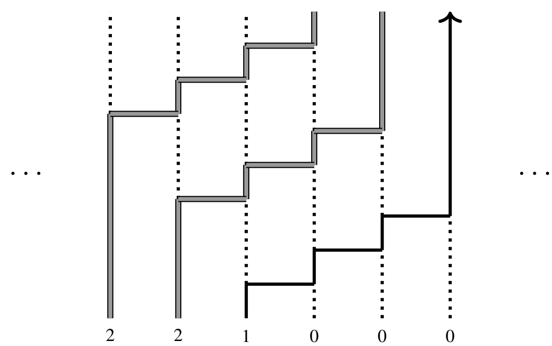
This gives a 1-morphism $F(W)$ in $\mathcal{R}(k, |\lambda|)$. Composition of 1-morphisms in $\mathcal{R}(k, |\lambda|)$ defines a superfunctor

$$\mathfrak{F}(W)\colon \mathfrak{W}(\lambda) \to \mathfrak{W}(\mu).$$

If λ is dominant and μ is antidominant then $\mathfrak{F}(W)$ is a superfunctor from $\Bbbk$-smod to $\Bbbk$-smod that is, a direct sum of grading shifts of the identity superfunctor. In this case, there is a canonical 1-morphism $F_{\mathrm{can}}(W)$ in $\mathrm{Hom}_{\mathcal{R}^\Lambda(k,|\lambda|)}(\lambda, \mu)$

$$F_{\mathrm{can}} = F_{(k-\ell-1)^{(2)}\dots(1)^{(2)}} \cdots F_{(k-3)^{(2)}\dots(\ell-1)^{(2)}} F_{(k-2)^{(2)}\dots\ell^{(2)}} F_{(k-1)^{(\epsilon)}\dots(\ell+1)^{(\epsilon)}} (2)^\ell \epsilon, \tag{23}$$

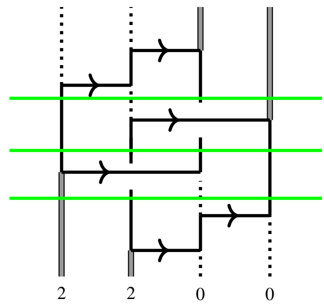
which in terms of webs takes the form of the following example:



We have that $\mathfrak{F}(W) = \mathrm{Hom}_{\mathcal{R}^\lambda(k,|\lambda|)}(\lambda, \mu)$ is isomorphic to the graded $\Bbbk$-supervector space $\mathrm{Hom}_{\mathfrak{R}^\Lambda}(F_{\mathrm{can}}(W), F(W))$.

3B. ***The chain complex.*** As explained in [Tubbenhauer 2014] any oriented tangle diagram T can be written in the form of a web W_T with all horizontal strands pointing to the right. In this case we say that T *is in F-form.*

Example 3.1. For the Hopf link we have the following web diagram.



Suppose the bottom boundary of W_T is $(\lambda_1, \dots, \lambda_k)$ and the top boundary is $(\mu_1, \dots, \mu_k)$. Let $\mathrm{Kom}(\lambda, \mu)$ be the category of complexes of

$$\mathrm{HOM}_{\mathcal{R}(k,|\lambda|)}(\mathfrak{W}(\lambda), \mathfrak{W}(\mu))$$

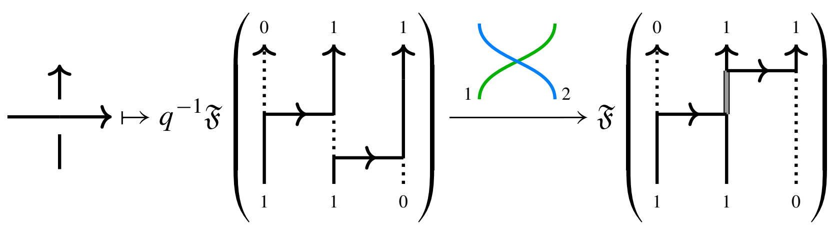
generated monoidally by tensor products of complexes of length 2, and $\mathrm{Kom}_{/h}(\lambda, \mu)$ its homotopy category (these are not supercategories). The usual constructions with chain complexes (homomorphisms, homotopies, cones, etc.) work in the same way as with nonsupercategories. Since we are in a supercategory, some signs have to be introduced (further details will appear in a follow-up paper). To each tangle in F-form as above we associate an object in $\mathrm{Kom}_{/h}(\lambda, \mu)$ as follows.

We first chop the diagram vertically in such way that each slice contains either a web without crossings, or a single crossing together with vertical pieces (as in Example 3.1). Each slice then gives either a superfunctor or a complex of superfunctors, as explained below. By composition we get a complex $\mathfrak{F}(W_T)$ of superfunctors from $\mathfrak{W}(\lambda)$ to $\mathfrak{W}(\mu)$.

3B1. *Basic tangles.*

- If T is a flat tangle, then we're done by Section 3A.
- To the positive crossing we associate the chain complex

(24)

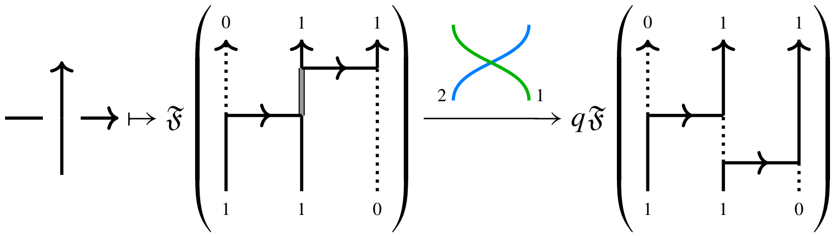


with the leftmost term in homological degree zero. Algebraically this can be written

$$\beta_+ \mapsto q^{-1} F_1 F_2(1, 1, 0) \xrightarrow{\tau_1} F_2 F_1(1, 1, 0),$$

where τ is the diagram above.

- To the negative crossing we associate the chain complex

(25) 

with the rightmost term in homological degree zero. Algebraically

$$\beta_- \mapsto F_2F_1(1,1,0) \xrightarrow{\tau_1} qF_1F_2(1,1,0).$$

Remark 3.2. Caution should be taken when applying (24) and (25): when passing from a tangle diagram to it's F-form some crossings may change from positive to negative and vice versa. To have an invariant of all tangles some grading shifts have to be introduced locally whenever this occurs. We shift (25) by -1 in the q-grading and 1 in the homological grading when it comes from a positive crossing and the opposite whenever (25) comes from a positive crossing.

3B2. *The normalized complex.* Let $n_\pm$ be the number of positive/negative crossings in W_T and let $w = n_+ - n_-$ be the writhe of W_T. We define the normalized complex

$$\mathfrak{F}(W_T) := q^{2w}\overline{\mathfrak{F}}(W_T). \tag{26}$$

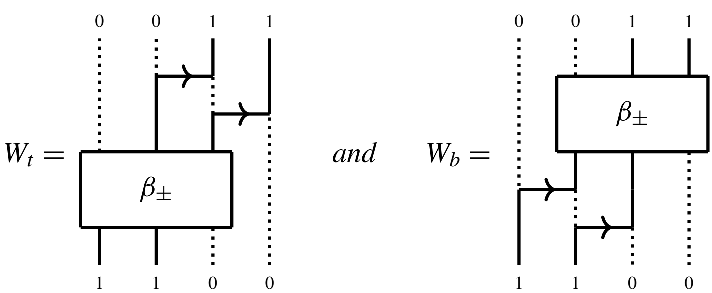
3C. *Topological invariance.*

Theorem 3.3. *For every tangle diagram T the homotopy type of $\mathfrak{F}(W_T)$ is invariant under the Reidemeister moves.*

Theorem 3.4. *For every link L the homology of $\mathfrak{F}(L)$ is a $\mathbb{Z}$-graded supermodule over $\mathbb{Z}$ whose graded Euler characteristic equals the Jones polynomial.*

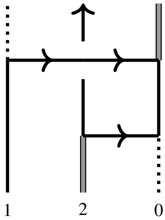
Proof of Theorem 3.3. The following is immediate.

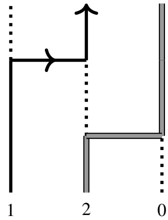
Lemma 3.5. *For $\beta_\pm$ a positive/negative crossing let W_t and W_b be the following tangles in F-form:*



Then the complexes $\mathfrak{F}(W_t)$ *and* $\mathfrak{F}(W_b)$ *are isomorphic.*

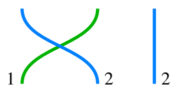
Lemma 3.6 (Reidemeister I). *Consider diagrams* D_1^+ *and* D_0 *that differ as below.*

$D_1^+ =$ 

$D_0 =$ 

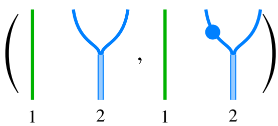
Then $\mathfrak{F}(D_1^+)$ *and* $\mathfrak{F}(D_0)$ *are isomorphic in* $\mathrm{Kom}_{/h}\big((1,2,0),(0,1,2)\big)$.

Proof. We have



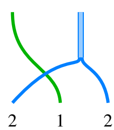
$$\overline{\mathfrak{F}}(D_1^+) = q^{-1}F_1F_2F_2(1,2,0) \longrightarrow F_1F_1F_2(1,2,0).$$

The first term is isomorphic to $F_1F_2^{(2)}(1,2,0) \oplus q^{-2}F_1F_2^{(2)}(1,2,0)$ via the map



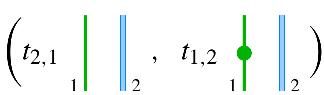
$$F_1F_2^{(2)}(1,2,0)\oplus q^{-2}F_1F_2^{(2)}(1,2,0) \xrightarrow[\simeq]{} q^{-1}F_1F_2^2(1,2,0),$$

while for the second term there is an isomorphism



$$F_2F_1F_2(1,2,0) \xrightarrow[\simeq]{} F_1F_2^{(2)}(1,2,0),$$

so that $\overline{\mathfrak{F}}(D_1^+)$ is isomorphic to the complex

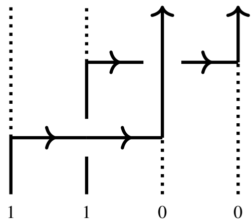


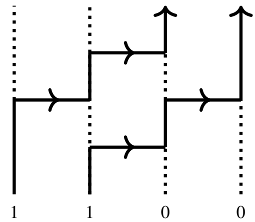
$$\begin{pmatrix} F_1F_2^{(2)}(1,2,0) \\ q^{-2}F_1F_2^{(2)}(1,2,0) \end{pmatrix} \longrightarrow F_1F_2^{(2)}(1,2,0).$$

By Gaussian elimination one gets that the complex $\overline{\mathfrak{F}}(D_1^+)$ is homotopy equivalent to the one term complex $q^{-2}F_1F_2^{(2)}(1,2,0)$ concentrated in homological degree zero, which after normalization is $\mathfrak{F}(D_0)$. □

The other types of Reidemeister I move can be verified similarly. For example, replacing the positive crossing by a negative crossing in Lemma 3.6 and using the inverses of the various isomorphisms above results in a complex isomorphic to $\overline{\mathfrak{F}}(D_1^-)$ that is homotopy equivalent to the 1-term complex $q^2F_1F_2^{(2)}(1,2,0)$ concentrated in homological degree zero.

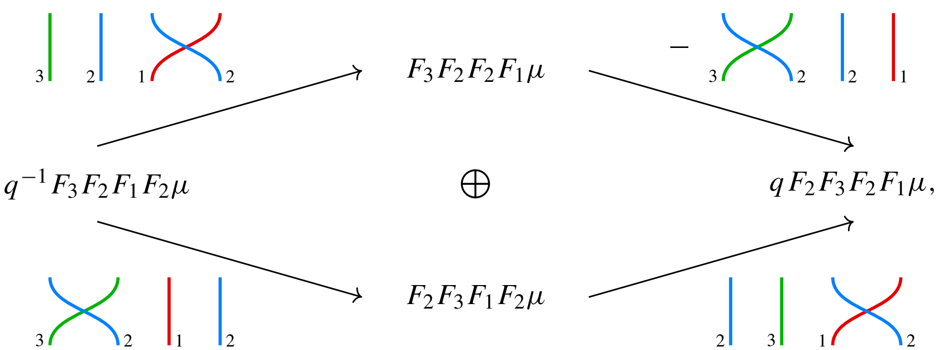
Lemma 3.7 (Reidemeister IIa). *Consider diagrams D_1 and D_0 that differ as below.*

$D_1 =$ 

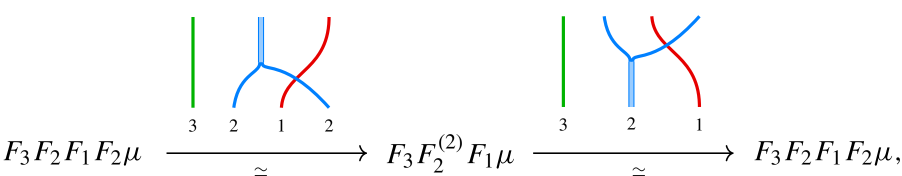
$D_0 =$ 

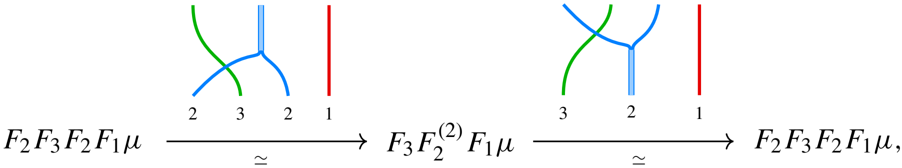
Then $\mathfrak{F}(D_1)$ and $\mathfrak{F}(D_0)$ are isomorphic in $\mathrm{Kom}_{/h}\big((1,1,0,0),(0,0,1,1)\big)$.

Proof. In the following we write μ instead of $(1,1,0,0)$. The complex $\overline{\mathfrak{F}}(D_1)$ is

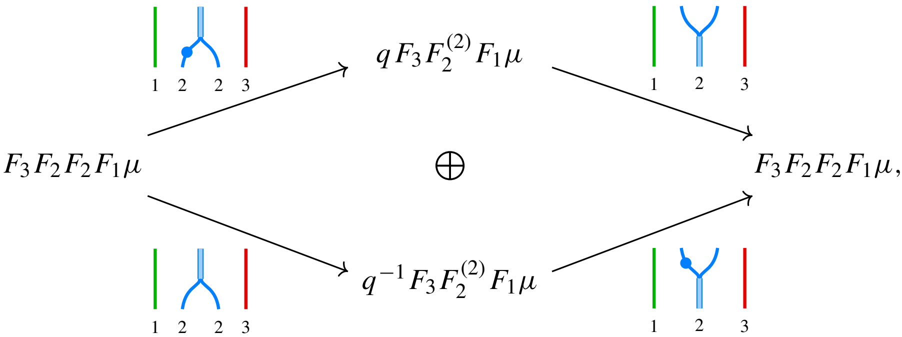


From the isomorphisms

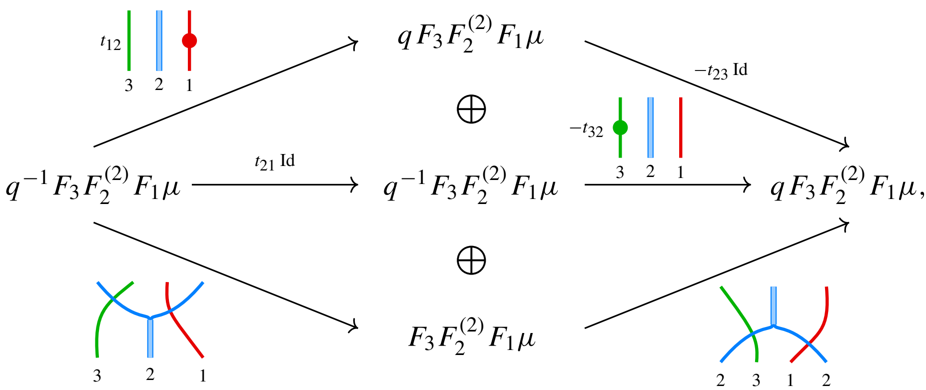




and



and simplifying the maps using the relations in $\mathfrak{R}(\nu)$ one gets that $\mathfrak{F}(D_1)$ is isomorphic to the complex



By Gaussian elimination of the acyclic two-term complexes

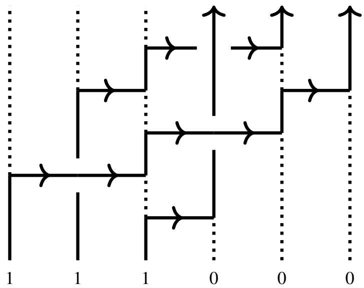
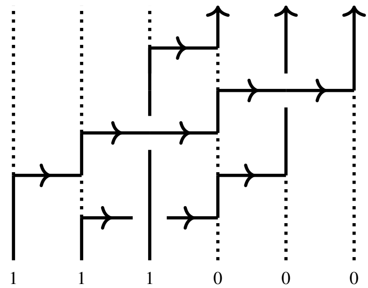
$$q^{-1}F_3F_2^{(2)}F_1\mu \xrightarrow{t_{21}\,\mathrm{Id}} q^{-1}F_3F_2^{(2)}F_1\mu \quad \text{and} \quad qF_3F_2^{(2)}F_1\mu \xrightarrow{-t_{23}\,\mathrm{Id}} qF_3F_2^{(2)}F_1\mu$$

one obtains that $\overline{\mathfrak{F}}(D_1)$ is homotopy equivalent to the complex

$$0 \longrightarrow F_3F_2^{(2)}F_1\mu \longrightarrow 0,$$

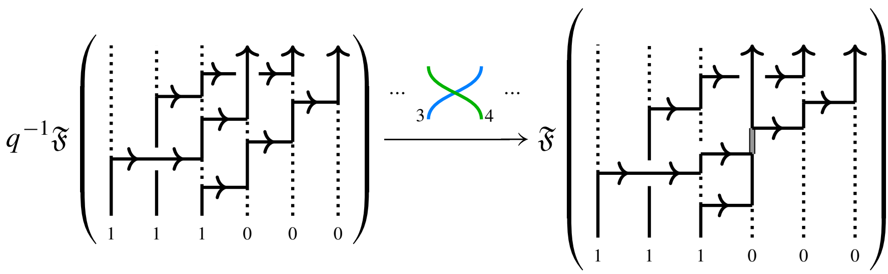
with the middle-term in homological degree zero. □

Lemma 3.8 (Reidemeister III). *Consider diagrams D_L and D_R that differ as below.*

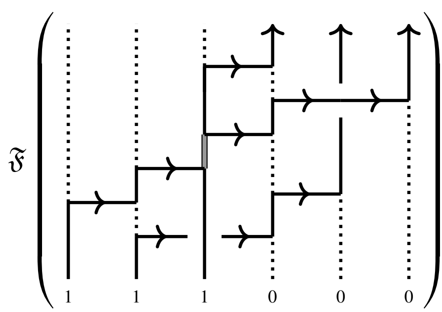
$D_L =$  $D_R =$ 

Then $\mathfrak{F}(D_L)$ *and* $\mathfrak{F}(D_R)$ *are isomorphic in* $\mathrm{Kom}_{/h}\big((1,1,1,0,0,0),(0,0,0,1,1,1)\big)$.

Proof. The proof is inspired by [Putyra 2014a, Lemma 7.9] (see also [Putyra 2014b, §4.3.3] for further details). The complex associated to D_L is the mapping cone of the map



An easy exercise shows that the second complex is isomorphic to the complex



In [Putyra 2014b, §4.3.3] it is explained in detail how to use an isomorphism like this together with the maps associated to two Reidemeister 2 moves on the first complex to prove that $\mathfrak{F}(D_L)$ is homotopy equivalent to $\mathfrak{F}(D_R)$. □

This finishes the proof of Theorem 3.3. □

3D. ***Not even Khovanov homology.*** We now show that for links the invariant $\mathcal{H}(L)$ is distinct from even Khovanov homology and shares common properties with odd Khovanov homology.

3D1. *Reduced homology.*

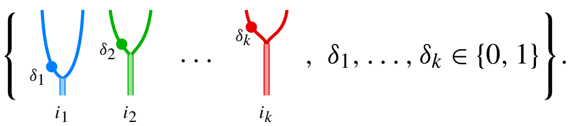
Theorem 3.9. *For every link L there is an invariant $H_{\text{reduced}}(L)$ with the property*

$$H(L) \simeq q\,H_{\text{reduced}}(L) \oplus q^{-1} H_{\text{reduced}}(L).$$

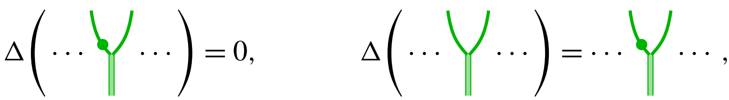
The proof of Theorem 3.9 follows a reasoning analogous to the proof of Theorem 3.2.A. in [Shumakovitch 2014], for the analogous decomposition for Khovanov homology over $\mathbb{Z}/2\mathbb{Z}$ in terms of reduced Khovanov homology.

Before proving the theorem we do some preparation. Recall that for D a diagram of L the chain groups of $\mathfrak{F}(D)$ are the various $\Bbbk$-supervector spaces $\operatorname{Hom}_{\mathcal{R}^\wedge}(F_{\text{can}}, F(W))$, where W runs over all the resolutions of D.

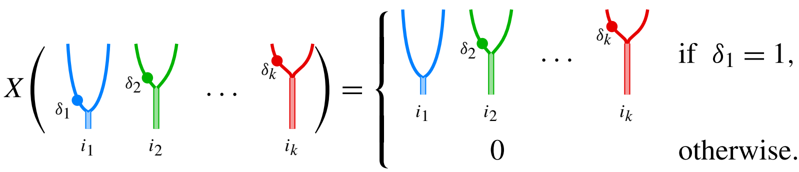
If we write $F_{\text{can}} = F_{i_1^{(2)} i_2^{(2)} \cdots i_k^{(2)}}$ then $\operatorname{Hom}_{\mathcal{R}^\wedge}(F_{\text{can}}, F_{i_1 i_1 i_2 i_2 \cdots i_k i_k})$ is spanned by



Introduce linear maps X and Δ on $\operatorname{Hom}_{\mathcal{R}^\wedge}(F_{\text{can}}, F_{i_1 i_1 i_2 i_2 \cdots i_k i_k})$ as follows. Map Δ is defined on the factors as



and extended to $\operatorname{Hom}_{\mathcal{R}^\wedge}(F_{\text{can}}, F_{i_1 i_1 i_2 i_2 \cdots i_k i_k})$ using the Leibniz rule. The map X is defined by



Since

$$\operatorname{Hom}_{\mathcal{R}^\wedge}(F_{\text{can}}, F(W)) \simeq \operatorname{Hom}_{\mathcal{R}^\wedge}(F_{\text{can}}, F_{i_1 i_1 i_2 i_2 \cdots i_k i_k}) \times \operatorname{Hom}_{\mathcal{R}^\wedge}(F_{i_1 i_1 i_2 i_2 \cdots i_k i_k}, F(W))$$

the maps Δ and X induce maps on $\operatorname{Hom}_{\mathcal{R}^\wedge}(F_{\text{can}}, F(W))$, denoted by the same symbols.

Lemma 3.10. *Both maps X and Δ commute with the differential of $\mathfrak{F}(D)$, $\Delta^2 = 0$, and moreover $X\Delta + \Delta X = \operatorname{Id}_{\mathfrak{F}(D)}$.*

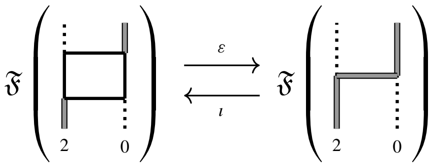
Proof. Straightforward. □

Proof of Theorem 3.9. We have that Δ is acyclic and therefore

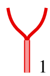
$$\mathfrak{F}(D) \simeq \ker(\Delta) \oplus q^2 \ker(\Delta),$$

and so the claim follows by setting $\mathfrak{F}_{\text{reduced}}(D) = q\ker(\Delta)$. □

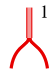
3D2. *A chronological Frobenius algebra.* We now examine the behavior of the functor $\mathfrak{F}$ under merge and splitting of circles. First define maps ι and ε,



as



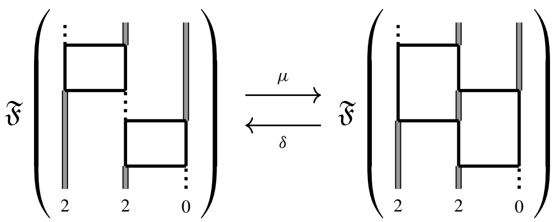
$$\iota\colon F_1^{(2)}(2,0) \longrightarrow F_1^2(2,0) \qquad \varepsilon\colon F_1^2(2,0) \longrightarrow F_1^{(2)}(2,0).$$



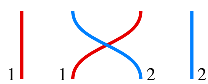
Note that, contrary to [Ozsváth et al. 2013], $p(\iota) = 1$ and $p(\varepsilon) = 0$.

We now consider the following two cases (*a*) and (*b*) below.

(*a*)

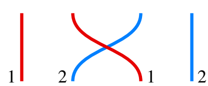


The maps μ and δ are given by



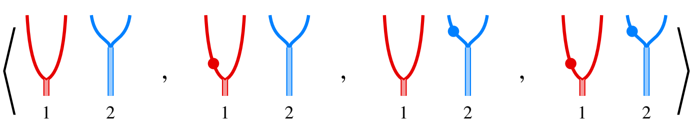
$$\mu\colon F_1^2F_2^2(2,2,0) \longrightarrow F_1F_2F_1F_2(2,2,0),$$

and



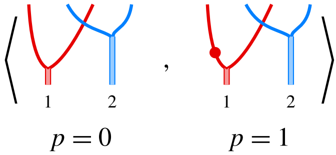
$$\delta\colon F_1F_2F_1F_2(2,2,0) \longrightarrow F_1^2F_2^2(2,2,0).$$

We have $p(\mu) = 0$ and $p(\delta) = 1$. Decomposing $F_1^2F_2^2(2,2,0)$ and $F_1F_2F_1F_2(2,2,0)$ into a direct sum of several copies of $F_1^{(2)}F_2^{(2)}(2,2,0)$ with the appropriate grading shifts we fix bases

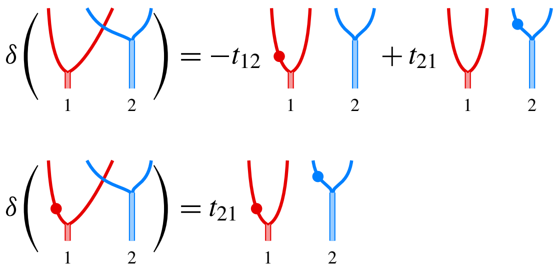


$p = 0$ $\quad$ $p = 1$ $\quad$ $p = 1$ $\quad$ $p = 0$

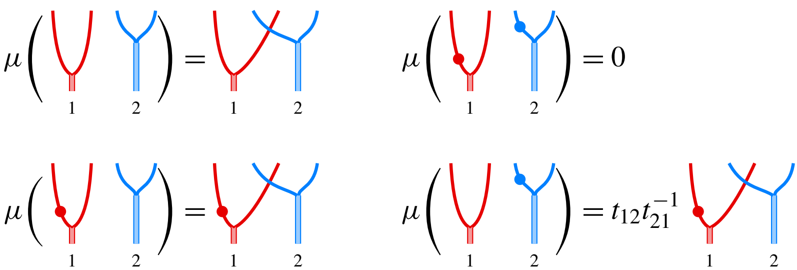
of $F_1^2 F_2^2(2,2,0)$, and



of $F_1 F_2 F_1 F_2(2,2,0)$. Then we compute



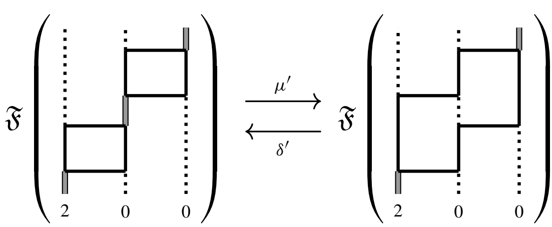
and



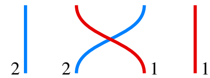
Using this one sees that easily that $\mu\delta = 0$, as in the case of the odd Khovanov homology of [Ozsváth et al. 2013].

Setting to 1 all t_{ij} and renaming $\langle 1, a_1, a_2, a_1 \wedge a_2\rangle$ the basis vectors of $F_1^2 F_2^2(2,0,0)$ and $\langle 1, a_1 = a_2\rangle$ the basis vectors of $F_1 F_2 F_1 F_2(2,0,0)$ one can give the maps δ, μ, ι and ε a form that coincides with the corresponding maps in [Ozsváth et al. 2013, §1.1]. Note though, that while the parities of δ and μ coincide with the corresponding maps in [Ozsváth et al. 2013], the parities of ι and ε are reversed with respect to [Ozsváth et al. 2013].

(*b*)

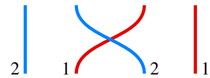


The maps μ' and δ' are given by



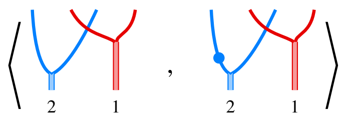
$$\mu'\colon\ F_2^2F_1^2(2,0,0) \longrightarrow F_2F_1F_2F_1(2,0,0),$$

and



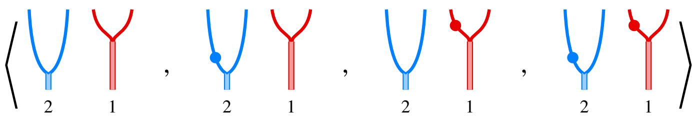
$$\delta'\colon\ F_2F_1F_2F_1(2,0,0) \longrightarrow F_2^2F_1^2(2,0,0).$$

Proceeding as above we fix a basis



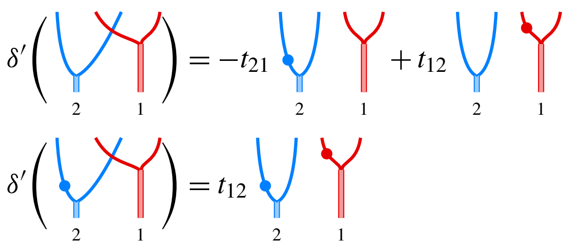
$p=1 \qquad p=0$

of $F_2F_1F_2F_1(2,0,0)$ and

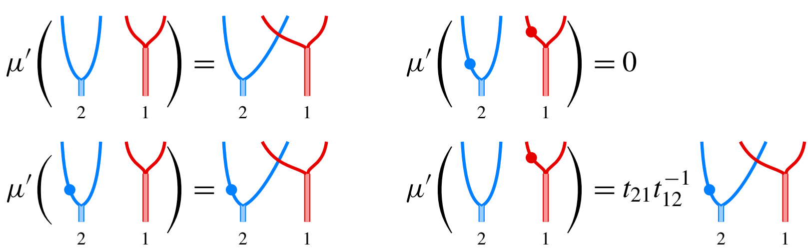


$p=0 \qquad p=1 \qquad p=1 \qquad p=0$

of $F_2^2F_1^2(2,2,0)$, to get



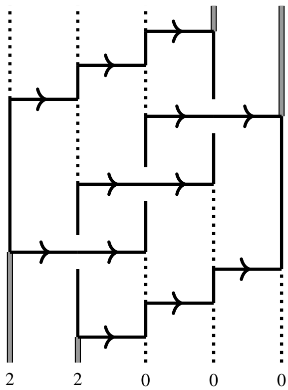
and



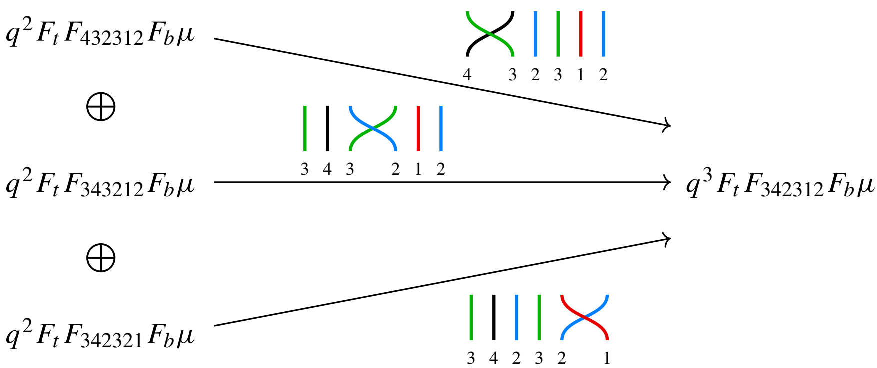
In this case we also have $\mu'\delta'=0$.

Contrary to the previous case, we have $p(\mu') = 1$ and $p(\delta') = 0$. The maps μ' and δ' can also be made to agree with [Ozsváth et al. 2013], but the parity is reversed (as with ι and ε above).

3D3. *A sample computation.* We now compute the homology of the left-handed trefoil T in its lowest and highest homological degrees. Consider the following presentation of T,



The computation of $H_0(T)$ is fairly simple: up to an overall degree shift it is the homology in degree 1 of the complex

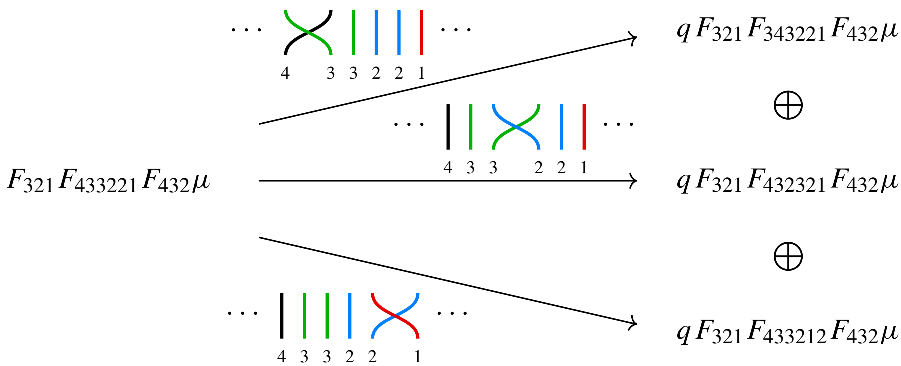
(27)


The three terms in homological degree zero are isomorphic to $F_{43^{(2)}2^{(2)}1}$. Composing the isomorphisms from $F_{43^{(2)}2^{(2)}1}$ to F_{432312}, F_{343212} and to F_{342321} with the corresponding maps above gives three maps that differ by a sign.

By inspection, one sees that up to a sign, these three maps are equal to the map δ from the case (a) in the previous subsection. The cokernel map in (27) is therefore two-dimensional. Adding the degree shifts one obtains

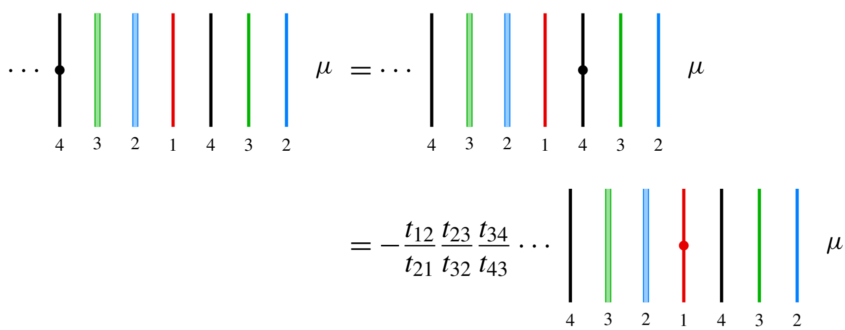
$$H_0(T) = q^{-1}\Bbbk \oplus q^{-3}\Bbbk.$$

We now compute $H_{-3}(H)$. Up to an overall degree shift it is computed as the homology in degree zero of the complex



Here $\mu = (2, 2, 0, 0, 0)$ and the factors F_{321} and F_{432} are the upper and lower closures of the diagram. We write F_t for F_{321} and F_b for F_{432} and sometimes we write $F_t F_{433221} F_b \mu$ instead of $F_{321} F_{433221} F_{432}\mu$, etc., and we only depict the pertinent part of the morphisms.

In the following we will use the identities

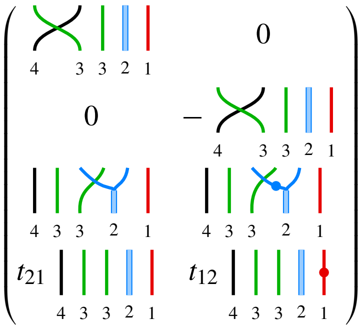
(28) 

The first equality follows from Lemma 2.4 after using (3) on the second strand labeled 4 to pull it to the left. The second equality can be checked by a applying (3) three times.

Coming back to $H_{-3}(T)$ we apply the isomorphisms

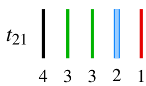
$$F_{433221} \simeq qF_{4332^{(2)}1} \oplus q^{-1}F_{4332^{(2)}1},$$
$$F_{343221} \simeq qF_{3432^{(2)}1} \oplus q^{-1}F_{3432^{(2)}1},$$
$$F_{433212} \simeq F_{4332^{(2)}1},$$

to obtain the isomorphic complex



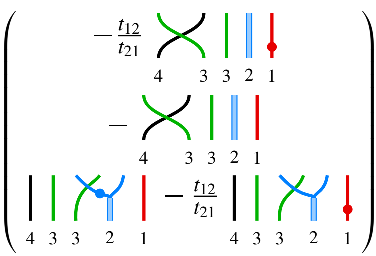
$$\begin{pmatrix} q\,F_t F_{4332^{(2)}1} F_b\mu \\ q^{-1} F_t F_{4332^{(2)}1} F_b\mu \end{pmatrix} \longrightarrow \begin{pmatrix} q^2 F_t F_{3432^{(2)}1} F_b\mu \\ F_t F_{3432^{(2)}1} F_b\mu \\ q\,F_t F_{432321} F_b\mu \\ q\,F_t F_{4332^{(2)}1} F_b\mu \end{pmatrix}.$$

By Gaussian elimination of the acyclic complex



$$q\,F_t F_{4332^{(2)}1} F_b\mu \longrightarrow q\,F_t F_{4332^{(2)}1} F_b\mu.$$

we obtain the homotopy equivalent complex

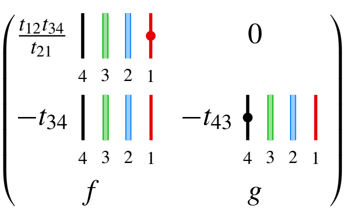


$$q^{-1} F_t F_{4332^{(2)}1} F_b\mu \longrightarrow \begin{pmatrix} q^2 F_t F_{3432^{(2)}1} F_b\mu \\ F_t F_{3432^{(2)}1} F_b\mu \\ q\,F_t F_{432321} F_b\mu \end{pmatrix}.$$

Applying the isomorphisms

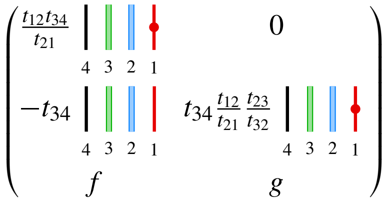
$$F_{4332^{(2)}1} \simeq q\,F_{43^{(2)}2^{(2)}1} \oplus q^{-1} F_{43^{(2)}2^{(2)}1} \tag{29}$$

and $F_{3432^{(2)}1} \simeq F_{43^{(2)}2^{(2)}1}$ gives the isomorphic complex



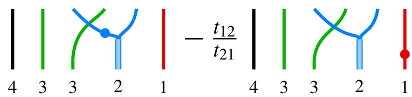
$$\begin{pmatrix} F_t F_{43^{(2)}2^{(2)}1} F_b\mu \\ q^{-2} F_t F_{43^{(2)}2^{(2)}1} F_b\mu \end{pmatrix} \longrightarrow \begin{pmatrix} q^2 F_t F_{43^{(2)}2^{(2)}1} F_b\mu \\ F_t F_{43^{(2)}2^{(2)}1} F_b\mu \\ q\,F_t F_{432321} F_b\mu \end{pmatrix},$$

or

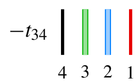


$$\begin{pmatrix} F_tF_{43^{(2)}2^{(2)}1}F_b\mu \\ q^{-2}F_tF_{43^{(2)}2^{(2)}1}F_b\mu \end{pmatrix} \longrightarrow \begin{pmatrix} q^2F_tF_{43^{(2)}2^{(2)}1}F_b\mu \\ F_tF_{43^{(2)}2^{(2)}1}F_b\mu \\ qF_tF_{432321}F_b\mu \end{pmatrix},$$

where f (resp. g) is the composite of the map from $F_{43^{(2)}2^{(2)}1}$ (resp. $q^{-2}F_{43^{(2)}2^{(2)}1}$) to $q^{-1}F_{4332^{(2)}1}$ in (29) and



Gaussian elimination of the acyclic complex

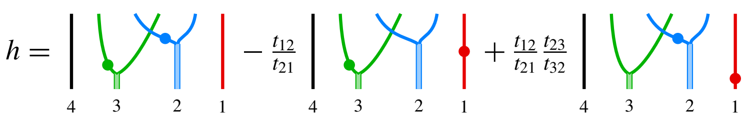


$$F_tF_{43^{(2)}2^{(2)}1}F_b\mu \longrightarrow F_tF_{43^{(2)}2^{(2)}1}F_b\mu,$$

yields the homotopy equivalent complex

$$q^{-2}F_tF_{43^{(2)}2^{(2)}1}F_b\mu \xrightarrow{\begin{pmatrix} 0 \\ h \end{pmatrix}} \begin{pmatrix} q^2F_tF_{43^{(2)}2^{(2)}1}F_b\mu \\ qF_tF_{432321}F_b\mu \end{pmatrix},$$

where



Since we are only interested in the lowest homological degree we restrict to considering the complex

$$q^{-2}F_tF_{43^{(2)}2^{(2)}1}F_b\mu \xrightarrow{h} qF_tF_{432321}F_b\mu.$$

Finally, applying the isomorphism $F_tF_{432321}F_b \simeq F_tF_{4332^{(2)}1}F_b$ results in the isomorphic complex

$$q^{-2}F_tF_{43^{(2)}2^{(2)}1}F_b\mu \xrightarrow{0} qF_tF_{432321}F_b\mu.$$

Adding the shift corresponding to the normalization (26), and using the fact that $F_t F_{43^{(2)}2^{(2)}1} F_b \mu$ is a $\Bbbk$-supervector space of graded dimension $q + q^{-1}$, yields

$$H_{-3}(T) = q^{-7}\Bbbk \oplus q^{-9}\Bbbk,$$

which agrees with the odd Khovanov homology of T.

4. Further properties of $\mathfrak{R}$

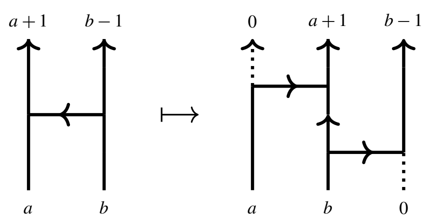
In this section we sketch several of its higher representation theory properties of $\mathfrak{R}$, some of them we have used in the previous section.

4A. ***Supercategorical action on $\mathcal{R}^\Lambda(k, d)$.*** Given a $\mathfrak{gl}_n$-weight $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$ we write $\overline{\Lambda} = (\Lambda_1 - \Lambda_2, \ldots, \Lambda_{n-1} - \Lambda_n)$ for the corresponding $\mathfrak{sl}_n$-weight. The superalgebra $\overline{\mathfrak{R}}^\Lambda(\nu)$ for $\mathfrak{gl}_k$ is defined to be the same as the superalgebra $\overline{\mathfrak{R}}^{\overline{\Lambda}}(\nu)$ for $\mathfrak{sl}_k$.

We now explain how the bifunctor $\Phi\colon \mathfrak{R} \times \mathfrak{R}^\Lambda \to \mathfrak{R}^\Lambda$ in (18). gives rise to an action of $\mathfrak{gl}_k$ on $\mathcal{R}^\Lambda(k, d)$ for Λ a dominant integrable $\mathfrak{gl}_k$-weight of level 2 with $\Lambda_1 + \cdots \Lambda_n = d$. A diagram D in $\mathcal{R}^\Lambda(k, d)$ with leftmost region labeled μ defines a web W_D with bottom boundary labeled Λ and with top boundary labeled μ. We denote $f_i, e_i \in U_q(\mathfrak{gl}_k)$ the Chevalley generators.

Behind Tubbenhauer's construction in [Tubbenhauer 2014] there is the observation that the transformation

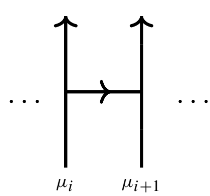
(30)



turns any web into a web with all horizontal edges pointing to the right. This goes through the obvious embedding of $\mathfrak{gl}_k$ into $\mathfrak{gl}_{k+1}$.

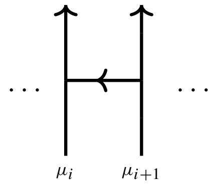
- The generator f_i acts by stacking the web

(31)

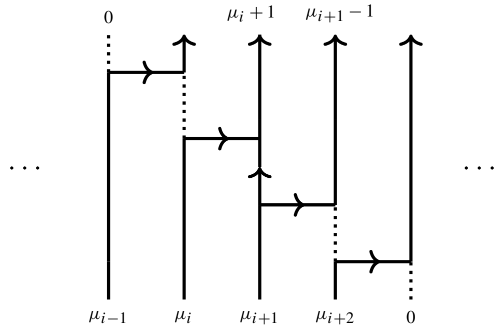


on the top of W_D. This means that f_i acts on $\mathcal{R}^\Lambda(n, d)$ as the functor that adds a strand labeled i to the left of D.

- To define the action of e_i we stack the web



on the top of W_D, then we use Tubbenhauer's trick (30) to put in a form that uses only F's. The transformation in (30) is not local and in order to be well defined one needs to keep trace of the indices before and after acting with an e_i. Tubbenhauer's trick gives

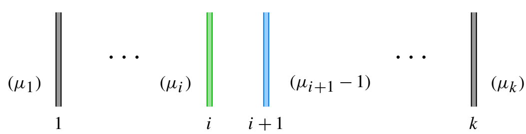


Every time we act with an e_i we embed $U_q(\mathfrak{gl}_k) \hookrightarrow U_q(\mathfrak{gl}_{k+1})$ and set

$$e_i(W_D) = f_{1^{(\mu_1)}\dots i-1^{(\mu_{i-1})}} f_i^{(\mu_i)} f_{i+1}^{(\mu_{i+1}-1)} f_{i+2^{(\mu_{i+2})}\dots k^{(\mu_k)}}(\mu, 0)(W_D).$$

After being acted with an e_j, f_i acts on W_D through the web corresponding to $f_{i+1}(\mu, 0)$.

We define the action of e_i on $\mathcal{R}^\Lambda(k, d)$ as the superfunctor that adds



to the left of D (here (μ_1), etc., are the thicknesses) that is, we act with the identity 2-morphism of $F_{1^{(\mu_1)}\dots i-1^{(\mu_{i-1})}} F_i^{(\mu_i)} F_{i+1}^{(\mu_{i+1}-1)} F_{i+2^{(\mu_{i+2})}\dots k^{(\mu_k)}}(\mu, 0)$.

Denote $\Phi(e_i)$ and $\Phi(f_i)$ the morphisms in $\mathfrak{R}^\Lambda$ that act as endofunctors of $\mathcal{R}^\Lambda(n, d)$ through the action above. It is clear that $\Phi(uv) = \Phi(u)\Phi(v)$ for u, $v \in U_q(\mathfrak{gl}_k)$. Note that $\Phi(1)(\mu)$ is a canonical element $F_{\mathrm{can}}(\mu)$ as introduced in (23).

Lemma 4.1. *We have natural isomorphisms*

$$\Phi(e_i)\Phi(f_i)(\lambda) \simeq \Phi(f_i)\Phi(e_i)(\lambda) \oplus \Phi(1)^{\bigoplus[\bar{\lambda}_i]}(\lambda) \qquad \text{if } \bar{\lambda}_i \geq 0,$$
$$\Phi(f_i)\Phi(e_i)(\lambda) \simeq \Phi(e_i)\Phi(f_i)(\lambda) \oplus \Phi(1)^{\bigoplus[-\bar{\lambda}_i]}(\lambda) \quad \text{if } \bar{\lambda}_i \leq 0.$$

Proof. These are instances of the categorified higher Serre relations. Denote $F_u = F_{1^{(\lambda_1)}\dots i-1^{(\lambda_{i-1})}}$ and $F_d = F_{i+2^{(\lambda_{i+2})}\dots k^{(\lambda_k)}}$. We have

$$\begin{aligned}\Phi(e_i)\Phi(f_i)(\lambda) &= F_u F_i^{(\lambda_i-1)} F_{i+1}^{(\lambda_{i+1})} F_d F_i(\lambda, 0)\\ &\simeq F_u F_i^{(\lambda_i-1)} F_{i+1}^{(\lambda_{i+1})} F_i(\dots, \lambda_i, \lambda_{i+1}, 0, \lambda_{i+2}, \dots) F_d, (\lambda, 0),\end{aligned}$$

and $\quad \Phi(f_i)\Phi(e_i)(\lambda) = F_t F_{i+1} F_i^{(\lambda_i)} F_{i+1}^{(\lambda_{i+1}-1)} F_b(\lambda, 0),$

and therefore, it is enough to check that the relations above are satisfied by the superfunctors $F_i^{(\lambda_i-1)} F_{i+1}^{(\lambda_{i+1})} F_i(\lambda_i, \lambda_{i+1}, 0)$ and $F_{i+1} F_i^{(\lambda_i)} F_{i+1}^{(\lambda_{i+1}-1)}(\lambda_i, \lambda_{i+1}, 0)$. Suppose $\lambda_i \geq \lambda_{i+1}$. Then we have $\lambda_i \in \{1, 2\}$ and $\lambda_{i+1} \in \{0, 1\}$. The computations involved are rather simple and we can check the four cases separately.

(1) $(\lambda_i, \lambda_{i+1}) = (1, 0)$:

$$\begin{aligned}\Phi(e_i)\Phi(f_i)(\lambda) &= F_i^{(\lambda_i-1)} F_{i+1}^{(\lambda_{i+1})} F_i(\lambda_i, \lambda_{i+1}) = F_i(1, 0) = 0 \oplus F_{\text{can}}(1, 0),\\ &= \Phi(f_i)\Phi(e_i)(\lambda) \oplus \Phi(1)(\lambda).\end{aligned}$$

(2) $(\lambda_i, \lambda_{i+1}) = (1, 1)$:

$$\Phi(e_i)\Phi(f_i)(\lambda) = F_i F_{i+1}(1, 1, 0) = \Phi(f_i)\Phi(e_i)(\lambda).$$

(3) $(\lambda_i, \lambda_{i+1}) = (2, 0)$:

$$\begin{aligned}\Phi(e_i)\Phi(f_i)(\lambda) &= F_i F_i(2, 0, 0)\\ &\simeq q F_i^{(2)}(2, 0, 0) + q^{-1} F_i^{(2)}(2, 0, 0) = \Phi(1)^{\oplus[2]}(\lambda).\end{aligned}$$

(4) $(\lambda_i, \lambda_{i+1}) = (2, 1)$:

$$\begin{aligned}\Phi(e_i)\Phi(f_i)(\lambda) &= F_i F_{i+1} F_i(2, 1, 0)\\ &\simeq 0 \oplus F_i^{(2)} F_{i+1}(2, 1, 0) = \Phi(f_i)\Phi(e_i)(\lambda) \oplus \Phi(1)(\lambda).\end{aligned}$$

An this proves the first isomorphism in the statement. The second isomorphism can be checked using the same method. □

The proof of Lemma 4.1 uses several supernatural transformations between the various compositions of $\Phi(f_i)(\lambda)$ and $\Phi(e_i)(\lambda)$ and $\Phi(1)(\lambda)$ that can be given a presentation in terms of the diagrams from $\mathfrak{R}$. We act with such diagrams by stacking them on the top of the diagrams for the image of Φ. On the weight space $(1, 1)$ these maps coincide with the maps used to define the chain complex for a tangle diagram in the previous section. In the general case these maps are units and counits of adjunctions in the following.

Lemma 4.2. *Up to degree shifts, the functor $\Phi(e_i)$ is left and right adjoint to $\Phi(f_i)$.*

Lemma 4.3. *We have the following natural isomorphisms:*

$$\Phi(e_j)\Phi(f_i)(\lambda) \simeq \Phi(f_i)\Phi(e_j)(\lambda) \qquad \textit{for } i \neq j,$$
$$\Phi(f_i)\Phi(f_{i\pm 1})\Phi(f_i)(\lambda) \simeq \Phi(f_i^{(2)})\Phi(f_{i\pm 1})(\lambda) \oplus \Phi(f_{i\pm 1})\Phi(f_i^{(2)})(\lambda),$$
$$\Phi(e_i)\Phi(e_{i\pm 1})(\lambda)\Phi(e_i) \simeq \Phi(e_i^{(2)})\Phi(e_{i\pm 1})(\lambda) \oplus \Phi(e_{i\pm 1})\Phi(e_i^{(2)})(\lambda).$$

Proof. The proof consists of a case-by-case computation. We illustrate the proof with the case of $\Phi(e_i)\Phi(f_{i+1})(\lambda) \simeq \Phi(f_{i+1})\Phi(e_i)(\lambda)$ and leave the rest to the reader. We have

$$\Phi(e_i)\Phi(f_{i+1})(\lambda) = F_i^{(\lambda_i)} F_{i+1}^{(\lambda_{i+1}-2)} F_{i+2}^{(\lambda_{i+2}+1)} F_{i+1}(\lambda),$$
$$\text{and} \quad \Phi(f_{i+1})\Phi(e_i)(\lambda) = F_i^{(\lambda_i)} F_{i+2} F_{i+1}^{(\lambda_{i+1}-1)} F_{i+2}^{(\lambda_{i+2})}(\lambda),$$

which are zero unless $\lambda_{i+1} = 2$ and $\lambda_{i+2} \in \{0, 1\}$. If $\lambda_{i+1} = 2$ these can be written

$$\Phi(e_i)\Phi(f_{i+1})(\lambda) = F_i^{(\lambda_i)} F_{i+2}^{(\lambda_{i+2}+1)} F_{i+1}(\lambda),$$
$$\text{and} \quad \Phi(f_{i+1})\Phi(e_i)(\lambda) = F_i^{(\lambda_i)} F_{i+2} F_{i+1} F_{i+2}^{(\lambda_{i+2})}(\lambda).$$

The case $\lambda_{i+2} = 0$ is immmediate and the case $\lambda_{i+2} = 1$ follows from the Serre relation (8)–(9). □

As explained in [Brundan and Ellis 2017, Sections 1.5 and 6] the Grothendieck group of a ($\mathbb{Z}$-graded) monoidal supercategory is a $\mathbb{Z}[q^{\pm 1}, \pi]/(\pi^2-1)$-algebra. Nontrivial parity shifts will occur when applying Tubbenhauer's trick. All the above can be used to prove the following.

Theorem 4.4. *The assignment above defines an action of $U_q(\mathfrak{gl}_k)$ on $\mathcal{R}^\Lambda(k, d)$. With this action we have an isomorphism of $K_0(\mathcal{R}^\Lambda(k, d))$ with the irreducible, finite-dimensional, $U_q(\mathfrak{gl}_k)$-representation of highest weight Λ at $\pi = 1$.*

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