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SCHUR–WEYL DUALITY, VERMA MODULES, AND ROW QUOTIENTS OF ARIKI–KOIKE ALGEBRAS

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We prove a Schur–Weyl duality between the quantum enveloping algebra of $\mathfrak{gl}_m$ and certain quotient algebras of Ariki–Koike algebras, which we describe explicitly. This duality involves several algebraically independent parameters and the module underlying it is a tensor product of a parabolic universal Verma module and a tensor power of the standard representation of $\mathfrak{gl}_m$. We also give a new presentation by generators and relations of the generalized blob algebras of Martin and Woodcock as well as an interpretation in terms of Schur–Weyl duality by showing they occur as a special case of our algebras.

1. Introduction

Schur–Weyl duality is a celebrated theorem connecting the finite-dimensional modules over the general linear and the symmetric groups. It states that, over a field $\Bbbk$ that is algebraically closed, the actions of $GL_m(\Bbbk)$ and $\mathfrak{S}_n$ on $V = (\Bbbk^m)^{\otimes n}$ commute and form double centralizers. Several variants of (quantum) Schur–Weyl duality are known; see for example [Ariki et al. 1995; Bao et al. 2018; Balagović et al. 2020; Chari and Pressley 1996; Jimbo 1986; Sakamoto and Shoji 1999] for such variants related to our paper. One particular family of generalizations of interest for us uses a module akin to the one appearing in Schur–Weyl duality, but with an infinite-dimensional module instead of V. For example, [Iohara et al. 2018] establishes a Schur–Weyl duality between $\mathcal{U}_q(\mathfrak{sl}_2)$ and the blob algebra of Martin and Saleur [1994] with the underlying module being a tensor product of a projective Verma module with several copies of the standard representation of $\mathcal{U}_q(\mathfrak{sl}_2)$. We should warn the reader that in [Iohara et al. 2018] the blob algebra was called the Temperley–Lieb algebra of type B (see [Lacabanne et al. 2020] for further explanations).

1A. ***In this paper.*** We consider the tensor product of a parabolic universal Verma module with the m-folded tensor product of the standard representation for $\mathcal{U}_q(\mathfrak{gl}_m)$ to establish a Schur–Weyl duality with a quotient of Ariki–Koike algebras. Ariki–Koike algebras were first considered by Cherednik [1987] as a cyclotomic quotient

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of the affine Hecke algebra of type A. These algebras were later rediscovered and studied by Ariki and Koike [1994] from a representation theoretic point of view. Independently, Broué and Malle [1993] attached a Hecke algebra to certain complex reflection groups, and Ariki–Koike algebras turn out to be the Hecke algebras associated to the complex reflection groups $G(d,1,n)$.

Recall that the *Ariki–Koike algebra* $\mathcal{H}(d,n)$ with parameters $q \in \Bbbk^*$ and $\underline{u} = (u_1, \dots, u_d) \in \Bbbk^d$ is the $\Bbbk$-algebra with generators $T_0, T_1, \dots T_{n-1}$, where T_0 satisfies $T_0T_1T_0T_1 = T_1T_0T_1T_0$, $T_0T_i = T_iT_0$ for $i > 1$, and $\prod_{i=1}^d (T_0 - u_i) = 0$, and $T_1, \dots, T_{n-1}$ generate a finite-dimensional Hecke algebra of type A.

We consider the semisimple case, where the simple modules V_μ of $\mathcal{H}(d,n)$ are indexed by d-partitions of n.

Let $\underline{m} = (m_1, \dots, m_d)$ be a d-tuple of positive integers and $\mathcal{P}^n_{\underline{m}}$ be the set of all d-partitions $\mu = (\mu^{(1)}, \dots, \mu^{(d)})$ of n such that $l(\mu^{(i)}) \le m_i$ for all $1 \le i \le d$.

In this paper we introduce the *row-quotient* algebra $\mathcal{H}_{\underline{m}}(d,n)$, that depends on $\underline{m}$ as the quotient of $\mathcal{H}(d,n)$ by the kernel of the surjection

$$\mathcal{H}(d,n) \twoheadrightarrow \prod_{\mu \in \mathcal{P}^n_{\underline{m}}} \mathrm{End}_\Bbbk(V_\mu).$$

Let $M^{\mathfrak{p}}(\Lambda)$ be a parabolic Verma module and V the standard representation for $\mathcal{U}_q(\mathfrak{gl}_m)$. In our conventions, $\mathfrak{p}$ is standard and has Levi factor $\mathfrak{l} = \mathfrak{gl}_{m_1} \times \cdots \times \mathfrak{gl}_{m_d}$, with $m_i \ge 1$ and $m_1 + m_2 + \cdots + m_d = m$ and Λ depends on d algebraically independent parameters $\lambda_1, \dots, \lambda_d$ (see Section 3B for more details). Thanks to the braided structure on the category of integrable modules over $\mathcal{U}_q(\mathfrak{gl}_m)$, we define a left action of $\mathcal{H}(d,n)$ on $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$ in Section 4. Our main result is:

Theorem A (Theorem 4.2 and Lemma 4.1).

- *The actions of $\mathcal{U}_q(\mathfrak{gl}_m)$ and $\mathcal{H}(d,n)$ on $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$ commute with each other, which endow $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$ with a structure of $\mathcal{H}(d,n) \otimes \mathcal{U}_q(\mathfrak{gl}_m)$-module.*
- *The algebra morphism $\mathcal{H}(d,n) \to \mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n})$ is surjective and factors through an isomorphism*

$$\mathcal{H}_{\underline{m}}(d,n) \xrightarrow{\simeq} \mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}). \tag{1}$$

- *There is an isomorphism of $\mathcal{H}(d,n) \otimes \mathcal{U}_q(\mathfrak{gl}_m)$-modules*

$$M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n} \simeq \bigoplus_{\mu \in \mathcal{P}^n_{\underline{m}}} V_\mu \otimes M^{\mathfrak{p}}(\Lambda, \mu),$$

where $M^{\mathfrak{p}}(\Lambda, \mu)$ is a simple module (see Section 3B).

The isomorphism in equation (1) has several particular specializations (Corollaries 4.3–4.7), some of them recovering well-known algebras:

- If $\mathfrak{p} = \mathfrak{gl}_m$ and $m \geq n$, then $\mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n})$ is isomorphic to the Hecke algebra of type A.
- If $\mathfrak{p} = \mathfrak{gl}_m$ and $m = 2$, then $\mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n})$ is isomorphic to the Temperley–Lieb algebra of type A.
- If $\mathfrak{p}$ is such that $m \geq nd$, $m_i \geq n$ for all $1 \leq i \leq d$, then $\mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n})$ is isomorphic to the Ariki–Koike algebra $\mathcal{H}(d, n)$.
- If $\mathfrak{p}$ is such that $d = 2$ and $m_1, m_2 \geq n$, then $\mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n})$ is isomorphic to the Hecke algebra of type B with unequal and algebraically independent parameters (see [Geck and Jacon 2011, Example 5.2.2(c)]).
- If the parabolic subalgebra $\mathfrak{p}$ coincides with the standard Borel subalgebra of $\mathcal{U}_q(\mathfrak{gl}_m)$ then $\mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n})$ is isomorphic to Martin and Woodcock's [2003] generalized blob algebra $\mathcal{B}(d, n)$. This generalizes the case of $\mathcal{U}_q(\mathfrak{sl}_2)$ covered in [Iohara et al. 2018].

In the last case, this gives a new interpretation of the generalized blob algebras $\mathcal{B}(d, n)$ in terms of Schur–Weyl duality. We also give a new presentation of $\mathcal{B}(d, n)$ as a quotient of Ariki–Koike algebras:

Theorem B (Theorem 2.15). *Suppose that $\mathcal{H}(d, n)$ is semisimple and that for every i, j, k we have $(1 + q^{-2})u_k \neq u_i + u_j$. The generalized blob algebra $\mathcal{B}(d, n)$ is isomorphic to the quotient of $\mathcal{H}(d, n)$ by the two-sided ideal generated by the element*

$$\tau = \prod_{1 \leq i < j \leq d} \left[(T_1 - q) \left(T_0 - q \frac{u_i + u_j}{q + q^{-1}} \right) (T_1 - q) \right].$$

1B. ***Connection to other works.*** The idea of writing this note originated when we started thinking of possible extensions of our work in [Lacabanne et al. 2020] to more general Kac–Moody algebras and were not able to find the appropriate generalizations of [Iohara et al. 2018] in the literature. When we were finishing writing this note Peng Shan informed us about [Rouquier et al. 2016], whose results are far beyond the ambitions of this article. Nevertheless, we expect our results to be connected to [Rouquier et al. 2016, §8] using a braided equivalence of categories between a category of modules for the quantum group $\mathcal{U}_q(\mathfrak{gl}_m)$ and a category of modules over the affine Lie algebra $\widehat{\mathfrak{gl}}_m$, which is due to Kazhdan and Lusztig [1993; 1994]. However, the explicit description of the endomorphism algebra of $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$, which was our first motivation towards categorification later on, does not seem to appear anywhere in [Rouquier et al. 2016] except in the particular case of our Corollary 4.5.

Another motivation for the results presented here resides in the potential applications to low-dimensional topology, as indicated in [Rose and Tubbenhauer 2019]. We find that it would be also interesting to investigate the use of several Verma modules in a tensor product as suggested in [Daugherty and Ram 2018].

2. Ariki–Koike algebras, row quotients and generalized blob algebras

We recall the definition of Ariki–Koike algebras and define some quotients which will appear as endomorphism algebras of modules over a quantum group. As a particular case we recover the generalized blob algebras of Martin and Woodcock [2003] and we obtain a presentation of these blob algebras that seems to be new.

2A. ***Reminders on Ariki–Koike algebras.*** Fix once and for all a field $\Bbbk$ and two positive integers d and n and choose elements $q \in \Bbbk^*$ and $u_1, \dots, u_d \in \Bbbk$. We recall the definition of the Ariki–Koike algebra introduced in [Ariki and Koike 1994], which we view as a quotient of the group algebra of the Artin–Tits braid group of type B.

Definition 2.1. The *Ariki–Koike algebra* $\mathcal{H}(d,n)$ with parameters $q \in \Bbbk^*$ and $\underline{u} = (u_1, \dots, u_d) \in \Bbbk^d$ is the $\Bbbk$-algebra with generators $T_0, T_1, \dots T_{n-1}$, the relation

$$(T_i - q)(T_i + q^{-1}) = 0,$$

the cyclotomic relation

$$\prod_{i=1}^{d}(T_0 - u_i) = 0,$$

and the braid relations

$$\begin{aligned} T_i T_j &= T_i T_j && \text{if } |i-j| > 1,\\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \le i \le n-2,\\ T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0. \end{aligned}$$

Remark 2.2. We use different conventions than [Ariki and Koike 1994]. In order to recover their definition, one should replace q by q^2, T_0 by a_1, and qT_{i-1} by a_i.

As in the type A Hecke algebra, for any $w \in \mathfrak{S}_n$ we can define unambiguously T_w by choosing any reduced expression of w.

It is shown in [Ariki and Koike 1994] that the algebra $\mathcal{H}(d,n)$ is of dimension $d^n n!$ and a basis is given in terms of Jucys–Murphy elements, which are recursively defined by $X_1 = T_0$ and $X_{i+1} = T_i X_i T_i$.

Theorem 2.3 [Ariki and Koike 1994, Theorems 3.10, 3.20]. *A basis of $\mathcal{H}(d,n)$ is given by the set*

$$\left\{X_1^{r_1} \dots X_d^{r_d} T_w \mid 0 \le r_i < d,\, w \in \mathfrak{S}_n\right\}.$$

Moreover, the center of $\mathcal{H}(d, n)$ is generated by the symmetric polynomials in $X_1, \dots, X_d$.

We end this section with a semisimplicity criterion due to Ariki [1994], which in our conventions takes the following form.

Theorem 2.4 [Ariki 1994, Main Theorem]. *The algebra $\mathcal{H}(d, n)$ is semisimple if and only if*

$$\Bigg(\prod_{\substack{-n<l<n\\ 1\le i<j\le d}} (q^{2l}u_i - u_j)\Bigg)\Bigg(\prod_{1\le i\le n} (1+q^2+q^4+\dots+q^{2(i-1)})\Bigg) \neq 0.$$

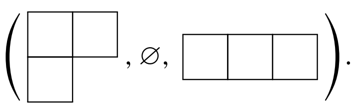
2B. ***Modules over Ariki–Koike algebras.*** In this section, we suppose that the algebra $\mathcal{H}(d, n)$ is semisimple. Ariki and Koike [1994] gave a construction of the simple $\mathcal{H}(d, n)$-modules, using the combinatorics of multipartitions.

2B1. *d-partitions and the Young lattice.* A partition μ of n of length $l(\mu) = k$ is a nonincreasing sequence $\mu_1 \ge \mu_2 \ge \dots \ge \mu_k > 0$ of integers summing to $|\mu| = n$. A d-partition of n is a d-tuple of partitions $\mu = (\mu^{(1)}, \dots, \mu^{(d)})$ such that $\sum_{i=1}^{d} |\mu^{(i)}| = n$. Given a d-partition μ its Young diagram is the set

$$[\mu] = \big\{(a, b, c) \in \mathbb{N} \times \mathbb{N} \times \{1, \dots, d\} \mid 1 \le a \le l(\mu), 1 \le b \le \mu_a^{(c)}\big\},$$

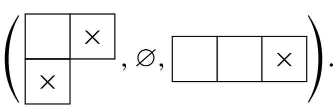
whose elements are called boxes. We usually represent a Young diagram as a d-tuple of sequences of left-aligned boxes, with $\mu_a^{(c)}$ boxes in the a-th row of the c-th component.

Example 2.5. The Young diagram of the 3-partition ((2, 1), ∅, (3)) of 6 is



A box γ of $[\mu]$ is said to be *removable* if $[\mu] \setminus \{\gamma\}$ is the Young diagram of a d-partition ν, and in this case the box γ is said to be *addable* to ν.

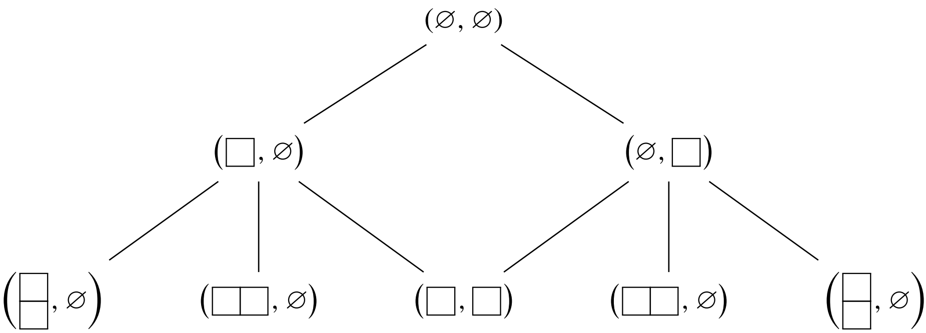
Example 2.6. The removable boxes of the 3-partition ((2, 1), ∅, (3)) below are depicted with a cross



With respect to the above definitions, we will also use the evident notions of adding a box to a Young diagram or removing a box from a Young diagram.

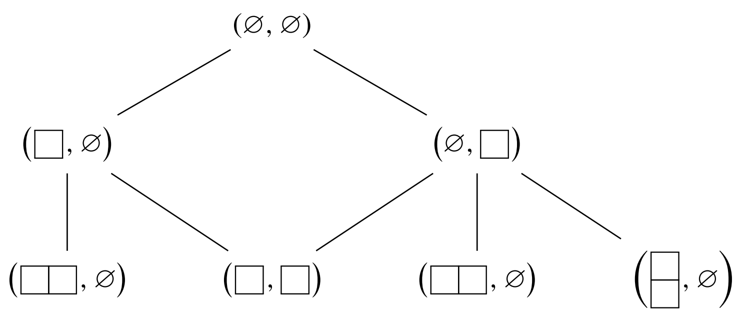
We consider the Young lattice for d-partitions and some sublattices. It is a graph with vertices consisting of d-partitions of any integers, and there is an edge between two d-partitions if and only if one can be obtained from the other by adding or removing a box.

Example 2.7. The beginning of the Young lattice for 2-partitions is the following:



If we fix $\underline{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, we then define $\mathcal{P}^n_{\underline{m}}$ as the set of d-partitions μ such that $l(\mu^{(i)}) \leq m_i$. We will also consider the corresponding sublattice of the Young lattice.

Example 2.8. For $m_1 = 1$ and $m_2 = 2$, the beginning of the Young lattice for 2-partitions μ with $l(\mu^{(1)}) \leq 1$ and $l(\mu^{(2)}) \leq 2$ is the following:



We end this subsection with the notion of a standard tableau of shape μ where μ is a d-partition of n. Such a standard tableau is a bijection $\mathfrak{t} : [\mu] \to \{1, \ldots, n\}$ such that for all boxes $\gamma = (a, b, c)$ and $\gamma' = (a', b', c)$ we have $\mathfrak{t}(\gamma) < \mathfrak{t}(\gamma')$ if $a = a'$ and $b < b'$ or $a < a'$ and $b = b'$. Giving a standard tableau of shape μ is equivalent to giving a path in the Young lattice from the empty d-partition to the d-partition μ.

Example 2.9. The standard tableau

$$\left(\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}, \varnothing, \begin{array}{|c|c|}\hline 2 & 3\\\hline\end{array}\right)$$

of shape $((1, 1), \varnothing, (2))$ correspond to the path

$$(\varnothing, \varnothing, \varnothing) \to (\square, \varnothing, \varnothing) \to (\square, \varnothing, \square) \to (\square, \varnothing, \square\square) \to \left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \varnothing, \square\square\right).$$

2B2. *Constructing the simple modules.* We present the construction of simple modules of the Ariki–Koike algebra following [Ariki and Koike 1994, Section 3]. This construction is similar to the classical construction of simple modules of the symmetric group, the Hecke algebra of type A or of the complex reflection group $G(d, 1, n)$. This construction describes explicitly the action of the Ariki–Koike algebra on a vector space. For $\mu = (\mu^{(1)}, \ldots, \mu^{(d)})$ a d-multipartition of n, we set

$$V_\mu = \bigoplus_{\mathfrak{t}} \Bbbk v_{\mathfrak{t}},$$

where the sum is over all the standard tableaux of shape μ. Ariki and Koike gave an explicit action of the generators on the basis of V_μ given by the standard tableaux. The action of T_0 is diagonal with respect to this basis:

$$T_0 v_{\mathfrak{t}} = u_c v_{\mathfrak{t}},$$

where c is such that $\mathfrak{t}(1, 1, c) = 1$. The action of T_i is more involved and depends on the relative positions of the numbers i and $i + 1$ in the tableau $\mathfrak{t}$:

(1) if i and $i + 1$ are in the same row of the standard tableau $\mathfrak{t}$, then $T_i v_{\mathfrak{t}} = q v_{\mathfrak{t}}$,

(2) if i and $i+1$ are in the same column of the standard tableau $\mathfrak{t}$, then $T_i v_{\mathfrak{t}} = -q^{-1} v_{\mathfrak{t}}$,

(3) if i and $i + 1$ neither appear in the same row nor the same column of the standard tableau $\mathfrak{t}$, then T_i will act on the two-dimensional subspace generated by $v_{\mathfrak{t}}$ and $v_{\mathfrak{s}}$, where $\mathfrak{s}$ is the standard tableau obtained from $\mathfrak{t}$ by permuting the entries i and $i + 1$. The explicit matrix is given in [Ariki and Koike 1994] and we will not need it.

Proposition 2.10 [Ariki and Koike 1994, Theorem 3.7]. *If μ is any d-multipartition of n, the space V_μ is a well-defined $\mathcal{H}(d, n)$-module and it is absolutely simple. A set of isomorphism classes of simple $\mathcal{H}(d, n)$-modules is moreover given by $\{V_\mu\}_\mu$, for μ running over the set of d-partitions of n.*

The action of the Jucys–Murphy elements is also diagonal in the basis of standard tableaux:

$$X_i v_{\mathfrak{t}} = u_c q^{2(b-a)} v_{\mathfrak{t}}, \tag{2}$$

where $\mathfrak{t}(a, b, c) = i$. A useful consequence of Proposition 2.10 is the following: if V is a simple $\mathcal{H}(d, n)$-module and $v \in V$ is a common eigenvector for $X_1, \ldots, X_d$ with eigenvalues as in (2) for some standard tableau $\mathfrak{t}$ of shape μ, then V is isomorphic to V_μ.

From the explicit description of the modules V_μ, using the standard inclusion $\mathcal{H}(n,d) \hookrightarrow \mathcal{H}(n+1,d)$, it is easy to see that for any d-partition of $n+1$ we have

$$\mathrm{Res}^{\mathcal{H}(n+1,d)}_{\mathcal{H}(n,d)}(V_\mu) \simeq \bigoplus_\nu V_\nu,$$

where the sum is over all d-partition ν of n whose Young diagram is obtained by deleting one removable box from the Young diagram of μ. The branching rule of the inclusions $\mathcal{H}(1,d) \subset \mathcal{H}(2,d) \subset \cdots \subset \mathcal{H}(n,d)$ is therefore governed by the Young lattice of d-partitions.

2C. ***Row quotients of $\mathcal{H}(d,n)$ and generalized blob algebras.*** We now define the row quotients of $\mathcal{H}(d,n)$ which will appear later as endomorphism algebras of a tensor product of modules for $\mathcal{U}_q(\mathfrak{gl}_m)$.

Definition 2.11. Let $\underline{m} = (m_1,\dots,m_d) \in \mathbb{N}^d$ and recall that the algebra $\mathcal{H}(d,n)$ is assumed to be semisimple, which implies that $\mathcal{H}(d,n) \simeq \prod_\mu \mathrm{End}_{\Bbbk}(V_\mu)$, the product being over all d-partitions of n. Recall also that $\mathcal{P}^n_{\underline{m}}$ is the set of d-partitions of n with i-th component of length at most m_i.

The *$\underline{m}$-row quotient of* $\mathcal{H}(d,n)$, denoted $\mathcal{H}_{\underline{m}}(d,n)$, is the quotient of $\mathcal{H}(d,n)$ by the kernel of the surjection

$$\mathcal{H}(d,n) \twoheadrightarrow \prod_{\mu\in\mathcal{P}^n_{\underline{m}}} \mathrm{End}_{\Bbbk}(V_\mu).$$

Remark 2.12. If $m_i \geq n$ for all $1 \leq i \leq d$ then $\mathcal{H}_{\underline{m}}(d,n) \simeq \mathcal{H}(d,n)$.

Similar to the case of $\mathcal{H}(d,n)$, we have inclusions $\mathcal{H}_{\underline{m}}(1,d) \subset \mathcal{H}_{\underline{m}}(2,d) \subset \cdots \subset \mathcal{H}_{\underline{m}}(n,d)$ and the branching rule is governed by the corresponding truncation of the Young lattice of d-partitions.

2C1. *Generalized blob algebras.* In the particular case where $m_i = 1$ for all $1 \leq i \leq d$, we recover the definition of the generalized blob algebras [Martin and Woodcock 2003, Equation (14)], which we denote by $\mathcal{B}(d,n)$. Under a mild hypothesis on the parameters, we give a presentation of $\mathcal{B}(d,n)$.

We consider the following element of $\mathcal{H}(d,n)$:

$$\tau = \prod_{1\leq i<j\leq d}\left[(T_1-q)\left(T_0 - q\frac{u_i+u_j}{q+q^{-1}}\right)(T_1-q)\right].$$

This element may look cumbersome, but can be better understood thanks to the following lemma:

Lemma 2.13. *The two-sided ideal of $\mathcal{H}(d,n)$ generated by τ is equal to the two-sided ideal generated by*

$$(T_1-q)\prod_{1\leq i<j\leq d}\big(X_1+X_2-(u_i+u_j)\big).$$

Proof. A simple computation in $\mathcal{H}(d,n)$ shows that

$$(T_1-q)\left(T_0-q\frac{u_i+u_j}{q+q^{-1}}\right)(T_1-q)=q\big(X_1+X_2-(u_i+u_j)\big)(T_1-q).$$

We therefore conclude using the fact that $(T_1-q)^2=-(q+q^{-1})(T_1-q)$ and that T_1 commutes with X_1+X_2. □

We now investigate which $\mathcal{H}(d,n)$-modules V_μ factor through the quotient by the two-sided ideal generated by τ.

Proposition 2.14. *The element τ acts by zero on V_μ if and only if $l(\mu^{(k)})\leq 1$ for every k such that $(1+q^{-2})u_k\neq u_i+u_j$ for all i, j.*

Proof. Suppose that μ and k are such that $l(\mu^{(k)})\geq 2$ with $(1+q^{-2})u_k\neq u_i+u_j$ for all i, j. Then there exists a tableau $\mathfrak{t}$ of shape μ such that 1 and 2 are in the first two columns of the k-th component of the Young diagram of μ. By definition of V_μ, the generator T_1 acts on $v_\mathfrak{t}$ by multiplication by $-q^{-1}$. The Jucys–Murphy element X_1 acts on $v_\mathfrak{t}$ by multiplication by u_k whereas the Jucys–Murphy element X_2 acts on $v_\mathfrak{t}$ by multiplication by $q^{-2}u_k$. Therefore, thanks to Lemma 2.13, τ does not act by zero on V_μ.

It remains to check that τ acts by zero on V_μ with $l(\mu^{(k)})\leq 1$ whenever $(1+q^{-2})u_k\neq u_i+u_j$ for all i, j. Let $\mathfrak{t}$ be a standard tableau of shape μ. If 1 and 2 are in the same component of the tableau $\mathfrak{t}$, then either 1 and 2 are in the same row and T_1 acts on $v_\mathfrak{t}$ by multiplication by q, either 1 and 2 are in the same column and X_1+X_2 acts on $\mathfrak{t}$ by multiplication by $(1+q^{-2})u_k$. The second case is possible only if there exists i, j such that $(1+q^{-2})u_k=u_i+u_j$ and then τ acts by zero. If 1 and 2 are in two different Young diagrams and X_1+X_2 acts on $\mathfrak{t}$ by u_k+u_l, where k (resp. l) is such that $\mathfrak{t}(1,1,k)=1$ (resp. $\mathfrak{t}(1,1,l)=2$). In both cases, τ acts by zero. □

Theorem 2.15. *Suppose that $\mathcal{H}(d,n)$ is semisimple and that for every i, j, k we have $(1+q^{-2})u_k\neq u_i+u_j$. The generalized blob algebra $\mathcal{B}(d,n)$ is isomorphic to the quotient of $\mathcal{H}(d,n)$ by the two-sided ideal generated by τ.*

Proof. Recall that we suppose that $m_1=\cdots=m_d=1$. Thanks to Proposition 2.14, the element τ is in the kernel of the surjection

$$\mathcal{H}(d,n)\twoheadrightarrow\prod_{\mu\in\mathcal{P}^n_{\underline{m}}}\mathrm{End}_{\Bbbk}(V_\mu).$$

Therefore, we have a surjection $\mathcal{H}(d,n)/\mathcal{H}(d,n)\tau\mathcal{H}(d,n) \twoheadrightarrow \mathcal{B}(d,n)$. Once again, thanks to Proposition 2.14, the simple modules of $\mathcal{H}(d,n)/\mathcal{H}(d,n)\tau\mathcal{H}(d,n)$ are exactly the V_μ with $\mu \in \mathcal{P}^n_{\underline{m}}$ which shows that the above surjection is an isomorphism. □

3. Quantum $\mathfrak{gl}_m$, parabolic Verma modules and tensor products

We recall the definition of the quantum enveloping algebra of $\mathfrak{gl}_m$, and we also recall some basic properties of its modules, e.g., concerning parabolic Verma modules.

3A. ***The quantum enveloping algebra of $\mathfrak{gl}_m$.*** Let q be an indeterminate. The following definition of $\mathcal{U}_q(\mathfrak{gl}_m)$ is over the field $\mathbb{Q}(q)$, but, via scalar extension, we will also consider it over a field containing $\mathbb{Q}(q)$ without further notice.

Definition 3.1. The quantum enveloping algebra $\mathcal{U}_q(\mathfrak{gl}_m)$ is the $\mathbb{Q}(q)$-algebra with generators $L_i^{\pm 1}$, E_j and F_j, for $1 \le i \le m$ and $1 \le j \le m-1$ with the relations

$$L_i^{\pm 1}L_i^{\mp 1} = 1, \qquad L_iL_j = L_jL_i, \qquad L_iE_j = q^{\delta_{i,j}-\delta_{i,j+1}}E_jL_i,$$
$$L_iF_j = q^{-\delta_{i,j}+\delta_{i,j+1}}F_jL_i, \qquad [E_i, F_j] = \delta_{i,j}\frac{L_iL_{i+1}^{-1} - L_i^{-1}L_{i+1}}{q-q^{-1}},$$

and the quantum Serre relations

$$E_iE_j = E_jE_i \text{ if } |i-j|>1, \quad E_i^2E_{i\pm 1} - (q+q^{-1})E_iE_{i\pm 1}E_i + E_{i\pm 1}E_i^2 = 0,$$
$$F_iF_j = F_jF_i \text{ if } |i-j|>1, \quad F_i^2F_{i\pm 1} - (q+q^{-1})F_iF_{i\pm 1}F_i + F_{i\pm 1}F_i^2 = 0.$$

We endow it with a structure of a Hopf algebra, with comultiplication Δ, counit ε and antipode S given on generators by the following:

$$\begin{aligned}
\Delta(L_i) &= L_i \otimes L_i, & \varepsilon(L_i) &= 1, & S(L_i) &= L_i^{-1},\\
\Delta(E_i) &= E_i \otimes 1 + L_iL_{i+1}^{-1} \otimes E_i, & \varepsilon(E_i) &= 0, & S(E_i) &= -L_i^{-1}L_{i+1}E_i,\\
\Delta(F_i) &= F_i \otimes L_i^{-1}L_{i+1} + 1 \otimes F_i, & \varepsilon(F_i) &= 0, & S(F_i) &= -F_iL_iL_{i+1}^{-1}.
\end{aligned}$$

Set $\mathcal{U}_q(\mathfrak{gl}_m)^0$ as the subalgebra generated by $(L_i)_{1\le i\le m}$, and $\mathcal{U}_q(\mathfrak{gl}_m)^{\ge 0}$ as the subalgebra generated by $(L_i, E_j)_{1\le i\le m,\, 1\le j\le m-1}$.

We denote by $P = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i$ the weight lattice of $\mathfrak{gl}_m$ with $\mathbb{Z}$-basis given by the fundamental weights $(\varpi_i)_{1\le i\le m}$ where $\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i$. We denote by Q the root lattice with $\mathbb{Z}$-basis given by the simple roots $(\alpha_i)_{1\le i\le d-1}$ where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Denote by Φ^+ the set of positive roots, by P^+ the set of dominant weights for $\mathfrak{gl}_m$, that is $\mu = \sum_{i=1}^m \mu_i\varepsilon_i$ with $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_m$. We also endow P with the standard nondegenerate bilinear form: $\langle \varepsilon_i, \varepsilon_j\rangle = \delta_{i,j}$. The symmetric group $\mathfrak{S}_m$ acts on P by permuting the coordinates and leaves the bilinear form $\langle\cdot,\cdot\rangle$ invariant. Finally, let ρ be the half-sum of the positive roots.

We will often work with extensions $\mathbb{Z}[\beta_1, \dots, \beta_k] \otimes_{\mathbb{Z}} P$, where the β_i's are indeterminates and we also extend the bilinear form $\langle \cdot, \cdot \rangle$ to $\mathbb{Z}[\beta_1, \dots, \beta_k] \otimes_{\mathbb{Z}} P$.

3B. ***Weights and parabolic Verma modules.*** Suppose that our field $\Bbbk$ contains the field $\mathbb{Q}(q)$ and let M be an $\mathcal{U}_q(\mathfrak{gl}_m)$-module over the ground field $\Bbbk$. An element $v \in M$ is said to be a weight vector if $L_i v = \varphi(\varepsilon_i)v$, where $\varphi : P \to \Bbbk$ is the corresponding weight. The module M is said to be a weight module if the action of the elements $L_1, \dots, L_m$ is simultaneously diagonalizable. A highest weight module is a weight module M such that $M = \mathcal{U}_q(\mathfrak{gl}_m)v$, where v is a weight vector such that $E_i v = 0$ for $1 \le i \le m-1$.

It is well-known that finite-dimensional weight $\mathcal{U}_q(\mathfrak{gl}_m)$-modules of type 1 are parametrized by the set P^+ of dominant weights.

In this paper, we will be interested in modules over the field $\mathbb{Q}(q, \lambda_1, \dots, \lambda_k)$, where $\lambda_i = q^{\beta_i}$ is an indeterminate (recall that q is formal and so q^{β_i} is also formal). Moreover, we only consider type 1 modules, where the weights are of the form

$$\varphi(\nu) = q^{\langle \mu, \nu \rangle},$$

for some $\mu \in \mathbb{Z}[\beta_1, \dots, \beta_k] \otimes_{\mathbb{Z}} P$ and for all $\nu \in P$.

We now turn to parabolic Verma modules. Let $\mathfrak{p}$ be a standard parabolic subalgebra of $\mathfrak{gl}_m$ with Levi factor $\mathfrak{l} = \mathfrak{gl}_{m_1} \times \cdots \times \mathfrak{gl}_{m_d}$, where $m_i \ge 1$ and $\sum_{i=1}^d m_i = m$. Denote by I the set $\{\widetilde{m}_i \mid 1 \le i \le d-1\}$, where $\widetilde{m}_i = m_1 + \ldots + m_i$, so that $\mathcal{U}_q(\mathfrak{l})$ is generated by L_i, E_j and F_j for $1 \le i \le m$ and $j \notin I$ and $\mathcal{U}_q(\mathfrak{p})$ is generated by L_i, E_j and F_k for $1 \le i \le m$, $1 \le j \le m-1$ and $k \notin I$. Denote by P_i^+ the set of dominant weights for $\mathfrak{gl}_{m_i}$. We identify the set $P_1^+ \times \cdots \times P_d^+$ with the dominant weights $P_{\mathfrak{l}}^+$ of $\mathfrak{l}$ by the map

$$(\mu^{(1)}, \dots, \mu^{(d)}) \to \sum_{i=1}^{d} \Bigl(\sum_{j=1}^{m_i} \mu_j^{(i)} \varepsilon_{\widetilde{m}_{i-1}+j} \Bigr).$$

For a dominant weight $\mu \in P_{\mathfrak{l}}^+$, we have an simple integrable finite-dimensional $\mathcal{U}_q(\mathfrak{l})$-module $V^{\mathfrak{l}}(\Lambda, \mu)$ of highest weight

$$\Lambda_\mu = \sum_{i=1}^{d} \Bigl(\sum_{j=1}^{m_i} (\beta_i + \mu_j^{(i)}) \varepsilon_{\widetilde{m}_{i-1}+j} \Bigr).$$

Indeed, one can check that $\langle \Lambda_\mu, \alpha_i \rangle \in \mathbb{N}$ for any $i \notin I$. We turn this $\mathcal{U}_q(\mathfrak{l})$-module into a $\mathcal{U}_q(\mathfrak{p})$-module by setting $E_i V^{\mathfrak{l}}(\Lambda, \mu) = 0$ for all $i \in I$. Then the parabolic Verma module $M^{\mathfrak{p}}(\Lambda, \mu)$ is

$$M^{\mathfrak{p}}(\Lambda, \mu) = \mathcal{U}_q(\mathfrak{gl}_m) \otimes_{\mathcal{U}_q(\mathfrak{p})} V^{\mathfrak{l}}(\Lambda, \mu).$$

It is a highest weight module of highest weight Λ_μ. If $\mu = 0$, then we will simply denote this module by $M^{\mathfrak{p}}(\Lambda)$ and its highest weight by Λ.

Lemma 3.2. *For any $\mu \in P_{\mathfrak{l}}^+$, the parabolic Verma module $M^{\mathfrak{p}}(\Lambda, \mu)$ is simple.*

Proof. Since for any $i \in I$ the scalar product $\langle \Lambda_\mu, \alpha_i \rangle$ is not an integer, as one easily checks, the claim follows. □

Remark 3.3. If the parabolic subalgebra $\mathfrak{p}$ is the Borel subalgebra $\mathfrak{b}$ of upper triangular matrices, we have $\mathcal{U}_q(\mathfrak{p}) = \mathcal{U}_q(\mathfrak{gl}_m)^{\geq 0}$ and the parabolic Verma module $M^{\mathfrak{b}}(\Lambda)$ is the universal Verma module. The adjective universal means that any parabolic Verma module can be obtained from $M^{\mathfrak{b}}(\Lambda)$ by specialization of the parameters.

In the rest of this article, all dominant weights $\mu \in P_{\mathfrak{l}}^+$ will satisfy $\mu_{m_i}^{(i)} \geq 0$ for all $1 \leq i \leq d$, and it will be convenient to identify such a weight μ with the corresponding d-partition in $\mathcal{P}_{\underline{m}}^n$. We will use the same notation μ to denote the d-partition or the corresponding dominant weight.

We also denote by V the standard representation of $\mathfrak{gl}_m$ of dimension m. Explicitly, this is a highest weight module with highest weight ε_1, it has as a basis $v_1, \ldots, v_m$ and the action of $\mathcal{U}_q(\mathfrak{gl}_m)$ is given by

$$L_i \cdot v_j = q^{\delta_{i,j}} v_j, \quad E_i \cdot v_j = \delta_{i+1,j} v_{j-1} \quad \text{and} \quad F_i \cdot v_j = \delta_{i,j} v_{j+1}.$$

3C. *Tensor products and branching rule.* As $\mathcal{U}_q(gl_m)$ is a Hopf algebra, its category of modules can be endowed with a tensor product. Explicitly, given M and N two modules over a ground ring R, the action of the generators on $M \otimes_R N$ is given using the comultiplication: for all $v \in M$ and $w \in N$, we have

$$\begin{aligned} L_i \cdot (v \otimes w) &= L_i \cdot v \otimes L_i \cdot w, \\ E_i \cdot (v \otimes w) &= E_i \cdot v \otimes w + L_i L_{i+1}^{-1} \cdot v \otimes E_i \cdot w, \\ F_i \cdot (v \otimes w) &= F_i \cdot v \otimes L_i^{-1} L_{i+1}^{-1} \cdot w + v \otimes F_i \cdot w. \end{aligned}$$

We will write $\otimes$ instead of $\otimes_R$ to simplify the notation. Since we will be interested in the endomorphism algebra of $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$, we start by understanding the decomposition of this module.

Proposition 3.4. *For any $\mu \in \mathcal{P}_{\mathfrak{l}}^n$, there is an isomorphism of $\mathcal{U}_q(\mathfrak{gl}_m)$-modules*

$$M^{\mathfrak{p}}(\Lambda, \mu) \otimes V \simeq \bigoplus_{\nu \in \mathcal{P}_{\mathfrak{l}}^{n+1}} M^{\mathfrak{p}}(\Lambda, \nu),$$

where the sum is over all $\nu \in \mathcal{P}_{\mathfrak{l}}^{n+1}$ whose Young diagram is obtained from the Young diagram of μ by adding one addable box.

Proof. We start by showing that $M^{\mathfrak{p}}(\Lambda, \mu) \otimes V$ has a filtration given by the $M^{\mathfrak{p}}(\Lambda, \nu)$ as in the statement. First, we have the following tensor identity:

$$\big(\mathcal{U}_q(\mathfrak{gl}_m) \otimes_{\mathcal{U}_q(\mathfrak{p})} V^{\mathfrak{l}}(\Lambda, \mu)\big) \otimes V \simeq \mathcal{U}_q(\mathfrak{gl}_m) \otimes_{\mathcal{U}_q(\mathfrak{p})} (V^{\mathfrak{l}}(\Lambda, \mu) \otimes V).$$

Noticing that $L \mapsto \mathcal{U}_q(\mathfrak{gl}_m) \otimes_{\mathcal{U}_q(\mathfrak{p})} L$ is an exact functor from the category of finite-dimensional $\mathcal{U}_q(\mathfrak{p})$-modules to the category of $\mathcal{U}_q(\mathfrak{gl}_m)$-modules, it remains to show that

$$V^{\mathfrak{l}}(\Lambda, \mu) \otimes V \simeq \bigoplus_{\nu \in \mathcal{P}_{\mathfrak{l}}^{n+1}} V^{\mathfrak{l}}(\Lambda, \nu),$$

where the sum is over all $\nu \in \mathcal{P}_{\mathfrak{l}}^{n+1}$ whose Young diagram is obtained from the Young diagram of μ by adding one addable box. This follows from the usual branching rule for $\mathcal{U}_q(\mathfrak{gl}_{m_i})$-modules.

To show that the sum is direct, we use arguments from the infinite-dimensional representation theory of Lie algebras. We consider the usual category $\mathcal{O}$ for $\mathcal{U}_q(\mathfrak{gl}_m)$ [Mazorchuk 2012, Chapter 4]. We then show that each $M^{\mathfrak{p}}(\Lambda, \nu)$ lie in a different block of the category $\mathcal{O}$, which then implies that the sum is direct.

First, as $M^{\mathfrak{p}}(\Lambda, \nu)$ is a quotient of the universal Verma module $M^{\mathfrak{b}}(\Lambda_\nu)$, these two modules share the same central character. Therefore $M^{\mathfrak{p}}(\Lambda, \nu)$ and $M^{\mathfrak{p}}(\Lambda, \nu')$ are in the same block if and only if the central characters afforded by $M^{\mathfrak{b}}(\Lambda_\nu)$ and $M^{\mathfrak{b}}(\Lambda_{\nu'})$ are the same. But these central characters are equal if and only if Λ_ν and $\Lambda_{\nu'}$ are in the same orbit for the dot action of the symmetric group, which is the usual action of the symmetric group shifted by the sum of simple roots ρ.

We obtain that $M^{\mathfrak{p}}(\Lambda, \nu)$ and $M^{\mathfrak{p}}(\Lambda, \nu')$ are in the same block if and only if there exists $w \in \mathfrak{S}_m$ such that

$$w \cdot \Lambda_\nu = \Lambda_{\nu'}.$$

Now, suppose that $M^{\mathfrak{p}}(\Lambda, \nu)$ and $M^{\mathfrak{p}}(\Lambda, \nu')$ are in the same block. Since the dot action satisfies $w \cdot (\eta + \gamma) = w \cdot \eta + w(\gamma)$, we deduce that $w(\Lambda) = \Lambda$ so that w lies in $\mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_d}$. Then, writing $w = (w_1, \ldots, w_d)$, we find that $w_i \cdot \nu^{(i)} = \nu'^{(i)}$ for every $1 \leq i \leq d$. Since both $\nu^{(i)}$ and $\nu'^{(i)}$ are dominant weights, we deduce that $\nu^{(i)} = \nu'^{(i)}$ for every $1 \leq i \leq d$. Indeed, each orbit for the dot action contains a unique dominant weight.

Hence if $\nu \neq \nu'$, the parabolic Verma modules $M^{\mathfrak{p}}(\Lambda, \nu)$ and $M^{\mathfrak{p}}(\Lambda, \nu')$ are in different blocks of the category $\mathcal{O}$. □

Using the previous proposition and induction, one shows the following corollary.

Corollary 3.5. *There is an isomorphism*

$$M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n} \simeq \bigoplus_{\mu \in \mathcal{P}_{\mathfrak{l}}^{n}} M(\Lambda, \mu)^{n_\mu},$$

where n_μ is the number of paths from the empty d-partition to μ in the Young lattice of d-multipartitions.

3D. ***Braiding and an action of the Artin–Tits group of type B.*** The quantized enveloping algebra (or rather a completion of the tensor product with itself) contains an element, called the quasi-R-matrix, which is a crucial tool in defining a braiding on a subcategory of the $\mathcal{U}_q(\mathfrak{gl}_m)$-modules. Since there are several possible braidings, we make our choice explicit and refer to [Chari and Pressley 1994, 10.1.D] for more details.

In a completion of $\mathcal{U}_q(\mathfrak{gl}_m) \otimes \mathcal{U}_q(\mathfrak{gl}_m)$, we define an element Θ by

$$\Theta = \prod_{\alpha\in\Phi^+} \left(\sum_{n=0}^{+\infty} q^{\frac{1}{2}(n(n-1))} \frac{(q-q^{-1})^n}{[n]!} E_\alpha^n \otimes F_\alpha^n \right),$$

where

$$[n]! = \prod_{i=1}^{n} \frac{q^i - q^{-i}}{q-q^{-1}}$$

and E_α, F_α are the root vectors associated to a positive root α. If M and N are two $\mathcal{U}_q(\mathfrak{gl}_m)$ type 1 weight modules over the ground ring $\mathbb{Q}(q, \lambda_1, \dots, \lambda_{d-1})$ where $\mathcal{U}_q(\mathfrak{gl}_m)^{>0}$ act locally nilpotently, Θ induces an isomorphism of vector spaces $\Theta_{M,N} : M \otimes N \to M \otimes N$. We then define a morphism of $\mathcal{U}_q(\mathfrak{gl}_m)$-modules

$$c_{M,N} : M \otimes N \to N \otimes M,$$

by

$$c_{M,N} = \tau \circ f \circ \Theta_{M,N},$$

where τ is the flip $v \otimes w \mapsto w \otimes v$ and f is the map $v \otimes w \mapsto q^{\langle \mu,\nu \rangle} v \otimes w$ if v and w are of respective weights μ and ν. This endows the category of type 1 weight modules on which $\mathcal{U}_q(\mathfrak{gl}_m)^{>0}$ acts locally nilpotently with a braiding. In particular, we have the hexagon equation:

$$c_{L\otimes M,N} = (c_{L,N} \otimes \mathrm{Id}_M) \circ (\mathrm{Id}_L \otimes c_{M,N}) \quad \text{and} \quad c_{L,M\otimes N} = (\mathrm{Id}_M \otimes c_{L,N}) \circ (c_{L,M} \otimes \mathrm{Id}_N).$$

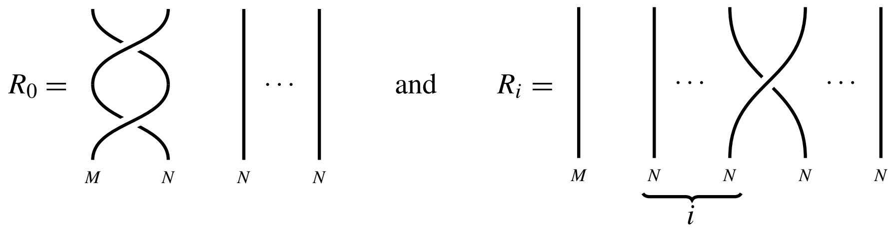
Let $\mathcal{B}_n$ be the Artin–Tits braid group of type B_n. It has the following presentation in terms of generators and relations:

$$\mathcal{B}_n = \left\langle \tau_0, \tau_1, \dots, \tau_{n-1} \;\middle|\; \begin{array}{ll} \tau_0\tau_1\tau_0\tau_1 = \tau_1\tau_0\tau_1\tau_0, & \\ \tau_i\tau_j = \tau_j\tau_i & \text{if } |i-j|>1, \\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1} & \text{for } 1 \le i \le n-2 \end{array} \right\rangle.$$

Using the braiding, we define the following endomorphisms of $M \otimes N^{\otimes n}$:

$$\begin{aligned} R_0 &= (c_{N,M} \circ c_{M,N}) \otimes \mathrm{Id}_{N^{\otimes n-1}}, \\ R_i &= \mathrm{Id}_{M\otimes N^{\otimes i-1}} \otimes c_{N,N} \otimes \mathrm{Id}_{N^{\otimes n-i-1}} \quad \text{for } 1 \le i \le n-1. \end{aligned}$$

Pictorially, one can represent these endomorphisms as



Proposition 3.6. *The assignment $\tau_i \mapsto R_i$ defines an action of $\mathcal{B}_n$ on the module $M \otimes N^{\otimes n}$ which commutes with the $\mathcal{U}_q(\mathfrak{gl}_m)$ action.*

Proof. The fact that R_i is a $\mathcal{U}_q(\mathfrak{gl}_m)$-morphism follows by definition of R_i. The fact that the defining relations of $\mathcal{B}_n$ are satisfied follows from the embedding of the braid group of type B_n into the braid group of type A_{n+1} [Iohara et al. 2018, Lemma 2.1]. □

Finally, we end this section with a lemma due to Drinfeld [1990, Proposition 5.1 and Remark 4) below] computing the action of the double braiding on highest weight modules, which is related with the action of the ribbon element.

Lemma 3.7. *Let L, M and N be highest weight modules of respective highest weight λ, μ and ν such that $L \subset M \otimes N$. Then the double braiding $c_{N,M} \circ c_{M,N}$ restricted to N acts by multiplication by the scalar*

$$q^{\langle \lambda, \lambda+2\rho\rangle - \langle \mu, \mu+2\rho\rangle - \langle \nu, \nu+2\rho\rangle}.$$

4. The endomorphism algebra of $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$

The aim of this section is to prove the main result of this paper. We first explain why $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$ inherits an action of the Ariki–Koike algebra from the action of the braid group of type B_n. It is a classical result that the eigenvalues of R_i are q and $-q^{-1}$: the action of the braiding on $V \otimes V$ is

$$v_i \otimes v_j \mapsto \begin{cases} q v_j \otimes v_i & \text{if } i = j, \\ v_j \otimes v_i & \text{if } i > j, \\ v_j \otimes v_i + (q - q^{-1}) v_i \otimes v_j & \text{if } i < j. \end{cases}$$

Further, using Lemma 3.7, we easily compute the eigenvalues of the endomorphism R_0 in order to show that the action of $\mathcal{B}_n$ factors through the Ariki–Koike algebra.

Lemma 4.1. *The eigenvalues $u_1, \dots, u_d$ of R_0 on $M^{\mathfrak{p}}(\Lambda) \otimes V$ are equal to*

$$u_i = (\lambda_i q^{-\widetilde{m}_{i-1}})^2.$$

Proof. Let Λ be the highest weight of $M^{\mathfrak{p}}(\Lambda)$. The decomposition of $M^{\mathfrak{p}}(\Lambda)\otimes V$ is given in Proposition 3.4:

$$M^{\mathfrak{p}}(\Lambda)\otimes V \simeq \bigoplus_{i=1}^{d} M^{\mathfrak{p}}(\Lambda,\mu_i),$$

where μ_i is the d-partition of 1 whose only nonzero component is the i-th one and is equal to (1). The highest weight of $M^{\mathfrak{p}}(\Lambda,\mu_i)$ being $\Lambda+\varepsilon_{\widetilde{m}_{i-1}+1}$, the action of R_0 on $M^{\mathfrak{p}}(\Lambda,\mu_i)$ is given by

$$q^{\langle \Lambda+\varepsilon_{\widetilde{m}_{i-1}+1},\Lambda+\varepsilon_{\widetilde{m}_{i-1}+1}+2\rho\rangle-\langle\Lambda,\Lambda+2\rho\rangle-\langle\varepsilon_1,\varepsilon_1+2\rho\rangle},$$

and we check that

$$\langle \Lambda+\varepsilon_{\widetilde{m}_{i-1}+1},\Lambda+\varepsilon_{\widetilde{m}_{i-1}+1}+2\rho\rangle-\langle\Lambda,\Lambda+2\rho\rangle-\langle\varepsilon_1,\varepsilon_1+2\rho\rangle=2(\beta_i-\widetilde{m}_{i-1}). \quad \square$$

By the definition of the Ariki–Koike algebra, Proposition 3.6 and the previous lemma we thus get an action of the Ariki–Koike algebra for the parameters $u_i=(\lambda_i q^{-\widetilde{m}_{i-1}})^2$ on $M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n}$. Therefore, the assignment $T_i\mapsto R_i$ defines a morphism of algebras

$$\mathcal{H}(d,n)\to \operatorname{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n}).$$

Theorem 4.2.

- *The algebra morphism* $\mathcal{H}(d,n)\to \operatorname{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n})$ *is surjective and factors through an isomorphism*

$$\mathcal{H}_{\underline{m}}(d,n)\xrightarrow{\simeq}\operatorname{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n}).$$

- *There is an isomorphism of* $\mathcal{H}(d,n)\otimes\mathcal{U}_q(\mathfrak{gl}_m)$*-module*

$$M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n}\simeq\bigoplus_{\mu\in\mathcal{P}^n_{\underline{m}}} V_\mu\otimes M^{\mathfrak{p}}(\Lambda,\mu).$$

Proof. The first part of the theorem follows immediately from the second part and the definition of the row-quotient $\mathcal{H}_{\underline{m}}(d,n)$.

Using Corollary 3.5 and the fact that $\mathcal{H}(d,n)$ acts on $M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n}$ by $\mathcal{U}_q(\mathfrak{gl}_m)$-linear endomorphisms, we see that

$$M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n}\simeq\bigoplus_{\mu\in\mathcal{P}^n_{\mathfrak{l}}} \widetilde{V}_\mu\otimes M^{\mathfrak{p}}(\Lambda,\mu),$$

for some $\mathcal{H}(d,n)$-modules $\widetilde{V}_\mu$. Since the multiplicity of $M^{\mathfrak{p}}(\Lambda,\mu)$ in $M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n}$ is given by the number of paths in the Young lattice from the empty d-partition to the d-partition μ, we have $\dim(\widetilde{V}_\mu)=\dim(V_\mu)$. Showing that V_μ is a submodule of $\widetilde{V}_\mu$ will end the proof of the second part of the theorem.

Let $\mathfrak{t}$ be a standard Young tableau of shape μ and denote by $(a_i, b_i, c_i) = \mathfrak{t}^{-1}(i)$. Denote by $\mu[i]$ the d-partition of i obtained by adding the boxes labeled by 1 to i in the chosen standard tableau $\mathfrak{t}$ to the empty d-partition. We now choose a highest weight vector $v \in M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$ of weight Λ_μ such that for all $1 \leq i \leq n$ we have

$$v \in M^{\mathfrak{p}}(\Lambda, \mu[i]) \otimes V^{\otimes(n-i)} \subset M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}.$$

Using the branching rule, one see that such a vector exists and is unique up to a scalar. Let us show that this vector v is a common eigenvector of the Jucys–Murphy elements X_i. It is easy to see that the action of the Jucys–Murphy element X_i on $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$ is given by the double braiding

$$(c_{V, M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes(i-1)}} \circ c_{M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes(i-1)}, V}) \otimes \mathrm{Id}_{V^{\otimes(n-i)}}.$$

By Lemma 3.7, we obtain that X_i acts on v by multiplication by

$$q^{\langle \Lambda_{\mu[i]}, \Lambda_{\mu[i]} + 2\rho \rangle - \langle \Lambda_{\mu[i-1]}, \Lambda_{\mu[i-1]} + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle}.$$

Indeed, v lies in the summand

$$M^{\mathfrak{p}}(\Lambda, \mu[i]) \otimes V^{\otimes(n-i)} \subset M^{\mathfrak{p}}(\Lambda, \mu[i-1]) \otimes V \otimes V^{\otimes(n-i)}$$

of $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$. But $\Lambda_{\mu[i]} = \Lambda_{\mu[i-1]} + \varepsilon_{k_i}$, where $k_i = \widetilde{m}_{c_{i-1}} + a_i$ so that

$$\begin{aligned}\langle \Lambda_{\mu[i]}, \Lambda_{\mu[i]} + 2\rho \rangle - \langle \Lambda_{\mu[i-1]}, \Lambda_{\mu[i-1]} + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle \\ = 2\langle \Lambda_{\mu[i-1]}, \varepsilon_{k_i} \rangle + 2(1 - k_i) = 2(\beta_{c_i} + b_i - k_i),\end{aligned}$$

since the component of $\Lambda_{\mu[i-1]}$ on ε_{k_i} is $\beta_{c_i} + (b_i - 1)$. Therefore, X_i acts on v by multiplication by

$$(\lambda_{c_i} q^{b_i - k_i})^2 = u_{c_i} q^{2(b_i - a_i)}.$$

Therefore, the $\mathcal{H}(d, n)$ submodule spanned by v is isomorphic to V_μ and then V_μ is a submodule of $\widetilde{V}_\mu$. □

4A. *Some particular cases.* We finish by giving some special cases of Theorem 4.2 in order to recover various well-known algebras. The two first special cases involve the well-known situation without a parabolic Verma module: it suffices to note that, if $\mathfrak{p} = \mathfrak{gl}_m$, then $M^{\mathfrak{p}}(\Lambda)$ is the trivial module.

Corollary 4.3. *If the parabolic subalgebra $\mathfrak{p}$ is $\mathfrak{gl}_m$ and $m \geq n$, then the endomorphism algebra of $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$ is isomorphic to Hecke algebra of type A.*

Corollary 4.4. *If the parabolic subalgebra $\mathfrak{p}$ is $\mathfrak{gl}_m$ and $m = 2$, then the endomorphism algebra of $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$ is isomorphic to Temperley–Lieb algebra of type A.*

We now turn to special cases where $\mathfrak{p}$ is a strict subalgebra of $\mathfrak{gl}_m$. The following corollary follows from Remark 2.12.

Corollary 4.5. *For $\mathfrak{p}$ such that $m \geq nd$ and $m_i \geq n$ for all $1 \leq i \leq d$, the endomorphism algebra of $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$ is isomorphic to the Ariki–Koike algebra $\mathcal{H}(d, n)$.*

The Hecke algebra of type B with unequal parameters appears when we work with a standard parabolic subalgebra $\mathfrak{p}$ with Levi factor $\mathfrak{gl}_{m_1} \times \mathfrak{gl}_{m_2}$.

Corollary 4.6. *If the parabolic subalgebra $\mathfrak{p}$ is such that $d = 2$, $m_1 \geq n$ and $m_2 \geq n$, then the endomorphism algebra of $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$ is isomorphic to the Hecke algebra of type B with unequal and algebraically independent parameters.*

Finally, the last special case is a generalization of the $\mathfrak{gl}_2$ case of [Iohara et al. 2018], where we recover the generalized blob algebra.

Corollary 4.7. *If the parabolic subalgebra $\mathfrak{p}$ is the standard Borel subalgebra $\mathfrak{b}$ of $\mathfrak{gl}_m$, that is $d = m$ and $m_i = 1$ for $1 \leq i \leq d$, then the endomorphism algebra of $M(\Lambda) \otimes V^{\otimes n}$ is isomorphic to the generalized blob algebra $\mathcal{B}(d, n)$.*

5. Some remarks on the nonsemisimple case

This paper deals with the semisimple case, where the decomposition of $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$ as the sum of simple modules is a crucial tool to compute its endomorphism algebra. Nonsemisimple situations appear if q is no longer an indeterminate in the base field $\Bbbk$ but a root of unity. If q and the parameters $\lambda_1, \dots, \lambda_d$ appearing in the highest weight of $M^{\mathfrak{p}}(\Lambda)$ are no longer algebraically independent, a nonsemisimple situation may also appear. Indeed, the parabolic Verma module might not be simple anymore as it is readily seen from the case of $\mathfrak{gl}_2$. It is then natural to ask whether it is possible to extend the Schur–Weyl duality to the nonsemisimple case. Let us remark that if q is not a root of unity and if $\lambda_i \lambda_j^{-1} \notin \mathbb{Z}$ for all $1 \leq i, j \leq d$ then the behavior is similar to the one described in the previous sections.

In order to define the action, we use an "integral version" of the algebras $\mathcal{U}_q(\mathfrak{gl}_m)$ and $\mathcal{H}(d, n)$ and of the module $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$, compatible with the specialization at a root of unity.

We start with the Ariki–Koike algebra. The definition given in Section 2A is valid for any field $\Bbbk$ and any choice of parameters. Concerning the algebra $\mathcal{U}_q(\mathfrak{gl}_m)$, we consider Lusztig's integral from $\mathcal{U}_q^{\mathrm{res}}(\mathfrak{gl}_m)$ over $\mathbb{Z}[q, q^{-1}]$; see [Chari and Pressley 1994, Section 9.3]. It is also known that the quasi-R-matrix Θ is an element of (a completion of) $\mathcal{U}_q^{\mathrm{res}}(\mathfrak{gl}_m) \otimes \mathcal{U}_q^{\mathrm{res}}(\mathfrak{gl}_m)$. Then for a base field $\Bbbk$ and any $\xi \in \Bbbk^*$, the quantum group $\mathcal{U}_\xi(\mathfrak{gl}_m)$ is defined as $\Bbbk \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{U}_q^{\mathrm{res}}(\mathfrak{gl}_m)$, where we see $\Bbbk$ as a $\mathbb{Z}[q, q^{-1}]$-module via the morphism sending q to ξ.

The parabolic Verma module $M^{\mathfrak{p}}(\Lambda)$ is a highest weight module and we choose v_Λ a highest weight vector. We then have at our disposal an integral version, which is the submodule generated over $\mathcal{U}_q^{\mathrm{res}}(\mathfrak{gl}_m)$ by the highest weight v_Λ. Its specialization at $q=\xi$ will still be denoted $M^{\mathfrak{p}}(\Lambda)$. Similarly, we have a version at $q=\xi$ of the standard module V, which has a well-known integral form.

Since the quasi-R-matrix Θ lies in the Lusztig's integral form of the quantum group, we can similarly use the braiding to define the endomorphisms $R_0, R_1, \dots, R_{n-1}$ of the $\mathcal{U}_\xi(\mathfrak{gl}_m)$-module $M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n}$. As in the semisimple case, we have:

Proposition 5.1. *Let $\Bbbk$ be a field, $q\in\Bbbk^*$ and $\lambda_1,\dots,\lambda_d\in\Bbbk$. Then the assignment $T_i\mapsto R_i$ is a morphism of algebras from $\mathcal{H}(d,n)$ to $\mathrm{End}_{\mathcal{U}_\xi(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n})$. The parameters u_i of the Ariki–Koike algebra are still given by Lemma 4.1.*

It is more difficult to understand the image of map

$$\mathcal{H}(d,n)\to \mathrm{End}_{\mathcal{U}_\xi(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n}),$$

and even harder to describe the image and the kernel of the map. Iohara, Lehrer and Zhang [Iohara et al. 2018] studied the particular case of $\mathfrak{gl}_2$ and $\mathfrak{p}=\mathfrak{b}$ (this corresponds to $m=2$ and $d=2$) and proved that if q is an indeterminate in $\Bbbk$ and that $\lambda_1\lambda_2^{-1}=q^l$ for $l\in\mathbb{Z}$, $l\geq -1$, then the map $\mathcal{H}(d,n)\to\mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n})$ is surjective [Iohara et al. 2018, Proposition 5.11].

In order to extend the Schur–Weyl duality form the semisimple case to a non-semisimple case, a classical strategy [Doty 2009; Andersen et al. 2018] is to argue that the dimensions of the various algebras, such as $\mathrm{End}_{\mathcal{U}_\xi(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n})$ or $\mathcal{H}(d,n)$, are independent of the base field $\Bbbk$.

Following the arguments of [Andersen et al. 2018], a first step would be to determine whether the parabolic Verma module $M^{\mathfrak{p}}(\Lambda)$ is tilting in an appropriate category $\mathcal{O}$ of infinite-dimensional $\mathcal{U}_q(\mathfrak{gl}_m)$-modules. Since V is tilting and the tensor product of tilting modules is tilting, having $M^{\mathfrak{p}}(\Lambda)$ being tilting would mean that $M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n}$ is. Since the space of endomorphisms of a tilting module is flat, its dimension does not depend on the base field $\Bbbk$.

Concerning $\mathcal{H}(d,n)$, its definition is valid over the ring $\mathbb{Z}[q^{\pm 1},u_1,\dots,u_d]$ and it is known that the basis given in Theorem 2.3 is a basis over this ring. This implies that the dimension of the algebra $\mathcal{H}(d,n)$ is independent of the field $\Bbbk$ and the choice of $q\in\Bbbk^*$ and of $u_1,\dots,u_d\in\Bbbk$.

Therefore, if $M^{\mathfrak{p}}(\Lambda)$ is tilting in an appropriate category $\mathcal{O}$ of infinite-dimensional $\mathcal{U}_q(\mathfrak{gl}_m)$-modules, the map $\mathcal{H}(d,n)\to\mathrm{End}_{\mathcal{U}_\xi(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n})$ would be surjective for any base field $\Bbbk$.

If we want to consider the row-quotients $\mathcal{H}_{\underline{m}}(d,n)$ of $\mathcal{H}(d,n)$, one must first give a definition which does not rely on the semisimplicity of the algebra $\mathcal{H}(d,n)$ so that the map $\mathcal{H}(d,n)\to\mathrm{End}_{\mathcal{U}_\xi(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda)\otimes V^{\otimes n})$ factors through $\mathcal{H}_{\underline{m}}(d,n)$ and then study the existence of an integral basis of $\mathcal{H}_{\underline{m}}(d,n)$.

Let us stress that these arguments depend heavily on $M^{\mathfrak{p}}(\Lambda)$ being tilting and on the existence of an integral basis of $\mathcal{H}_{\underline{m}}(d,n)$. One may need some extra assumptions on the field $\Bbbk$, as for example being infinite, or on the parameters of the parabolic Verma module. This nonsemisimple behavior deserves further study, which was outside the scope of this paper.

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