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# A survey on categorification of Verma modules

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**Abstract**

We provide an introduction to the higher representation theory of Kac–Moody algebras and categorification of Verma modules.

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***Keywords:*** *Categorification, Quantum Groups, Categorical actions, Verma modules.*

## 1. Introduction

Higher Representation Theory studies actions of groups, algebras, ..., on categories. In HRT the usual basic structures of representation theory, like vector spaces and linear maps, are replaced by category theory analogs, like categories and functors. Opposite to vector spaces and linear maps, the world of categories is tremendously big, offering enough room for finding richer structures: for example, replacing linear maps by a functors always comes accompanied by a "higher structure" which is associated to

This survey is an extended version of the material covered in a three 90 minute lecture series on categorification of Verma modules, held as part of the Junior Hausdorff Trimester Program "Symplectic Geometry and Representation Theory" of the Hausdorff Research Institute in Bonn in November 2017. I'd like to aknowledge the organizers of the program. Special thanks to Daniel Tubbenhauer and Grégoire Naisse for comments on a preliminary version of these notes.

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natural transformations between them. This higher structure is invisible to traditional representation theory.

Categorical actions of Lie algebras were first developed by Chuang and Rouquier [1] to solve a conjecture on modular representation theory of the symmetric group called the Broué conjecture. There were parallel ideas being developed at that time by Frenkel, Khovanov and Stroppel [3] based on earlier work of Khovanov and collaborators. All these ideas were boosted by the categorification of quantum groups by Lauda [11], Khovanov–Lauda [8, 9, 10] and Rouquier [20] and converged to what is called nowadays *Higher Representation Theory*.

Besides it relations with representation theory, HRT has shown to share interesting connections with other subjects, like for example topology [13, 17, 19, 21]. A popular example is the construction of Khovanov's link homology in [21] (see also [22]), giving it a conceptual context in terms of HRT of $\mathfrak{sl}_2$.

*Overview:* This series of lectures consist of an introduction to my joint work with Grégoire Naisse on categorification of Verma modules for quantum Kac–Moody algebras [15, 18, 18].

- In Section 1, we first give the necessary background on representation theory of (quantum) $\mathfrak{sl}_2$ adjusting the exposition in [14] to the quantum case. We the give a somehow detailed overview on the categorification of the finite-dimensional irreducible representations of quantum $\mathfrak{sl}_2$ using categories of modules for cohomologies of finite-dimensional Grassmannians and partial flag varieties. This is due to Frenkel–Khovanov–Stroppel [3] and independently to Chuang–Rouquier [1], and is an example of how such categorifications arise naturally.
- In Section 2, by working with infinite Grassmannians and adding a bit more structure we are able to categorify Verma modules for quantum $\mathfrak{sl}_2$. One natural sub-product of the geometric approach to categorification of Verma modules is a certain superalgebra extending the well-known nilHecke algebra, one of the fundamental ingredients in the categorification of quantum $\mathfrak{sl}_2$.
- In Section 3 we explain the case of categorification of Verma modules for Kac–Moody algebras. This requires a generalization of KLR algebras, the latter being the main ingredient in the categorification of quantum groups by Khovanov–Lauda–Rouquier.

## 1. Background

### 1.1 $\mathfrak{sl}_2$-*actions*

#### 1.1.1 Quantum $\mathfrak{sl}_2$

Let $\Bbbk = \mathbb{C}(q)$. Quantum $\mathfrak{sl}_2$ is the associative $\Bbbk$-algebra $U$ generated by $e, f$ and $k^{\pm 1}$, modulo the relations

$$kf = q^{-2} fk, \quad ke = q^2 ek, \quad kk^{-1} = k^{-1}k = 1, \quad ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

It is a Hopf algebra with comultiplication $\Delta(e) = 1 \otimes e + \otimes k^{-1}$, $\Delta(f) = k \otimes f + f \otimes 1$, $\Delta(k^{\pm 1}) \otimes k^{\pm 1}$, antipode $S(k^{\pm 1}) = k^{\mp 1}$, $S(e) = -ek$ and $S(f) = -k^{-1}f$. This is a quantization of the universal enveloping algebra of $\mathfrak{sl}_2$.

#### 1.1.2 $\mathfrak{sl}_2$.-modules

We say that (quantum) $\mathfrak{sl}_2$ *acts* on the $\Bbbk$-vector space $M$ if we have operators

$$E, F, K^{\pm 1} \in End_{\Bbbk}(M)$$

such that, for every $m \in M$ we have

$$KF(m) = q^{-2} FK(m), \quad KE(m) = q^{-2} EK(m), \quad KK^{-1}(m) = m = K^{-1}K(m),$$

and

$$EF(m) = FE(m) + \frac{K(m) - K^{-1}(m)}{q - q^{-1}}.$$

We say that $\mathfrak{sl}_2$ acts on $M$ through the application $f \mapsto F$, $e \mapsto E$, $k^{\pm 1} \mapsto K^{\pm 1}$ or that $M$ is an $\mathfrak{sl}_2$-*module*, or even that $M$ is a *representation* of $\mathfrak{sl}_2$. Sometimes we denote $F(m) = F.m$, $E(m) = E.m$, etc ...

A subspace $n \subseteq M$ which is closed under the $\mathfrak{sl}_2$-action ($U.N \subseteq N$) is called a *submodule*. An $\mathfrak{sl}_2$-module is *irreducible* if it does not contain any proper submodule (i.e. different from $\{0\}$ and $M$).

#### 1.1.3 Integrable modules

We say that an $\mathfrak{sl}_2$-action on $M$ is *integrable* (or that $M$ is integrable) if for every $m \in M$ we have $E^{r_1}(m) = 0$ and $F^{r_2}(m) = 0$ for $r_1$, $r_2 \gg 0$. Note that $r_1$ and $r_2$ depend on $m$. In the case of weight modules (see 1.1.4 below) and $q = 1$, the name comes from the fact that one can "integrate these up to the group".

### 1.1.4 Weight modules

Fix a complex number $\xi$ and for $\alpha \in \xi + \mathbb{Z} \subset \mathbb{C}$ put $q^{\alpha} = \lambda q^{\alpha-\xi} \in \Bbbk[\lambda^{\pm 1}]$.

Suppose that $M$ has an eigenvector for $K$ that is, $M$ has a vector $m_{\mu}$, for some $\xi \in \mathbb{C}$ and some $\mu \in \xi + \mathbb{Z}$, such that $K(m_{\mu}) = q^{\mu} m_{\mu}$. We say that $m_{\mu}$ is a *weight vector* of *weight* $\mu$. Note that $M$ becomes a $\Bbbk[\lambda^{\pm 1}]$-vector space.

It is easy to show that in this case $F(m_{\mu})$ and $E(m_{\mu})$ are also weight vectors of weights $\mu - 2$ and $\mu + 2$ respectively. We see that the subvector space $U.m_{\mu} \subseteq M$ is a submodule which consists only of weight vectors. This is an example of a type of modules called *weight modules*. The weight space $M_{\mu} \subseteq M$ is the subspace consisting of weight vectors of weight $\mu$:

$$M_{\mu} = \{m \in M \mid K(m) = q^{\mu} m\}.$$

Define the *support* *supp*($M$) as the set of all its weights: *supp*($M$) = $\{m \in \mathbb{C} \mid M_{\mu} \neq 0\}$. Then, we have that

$$\bigoplus_{\mu \in supp(M)} M_{\mu} \subseteq M$$

is a submodule and, in general $M$ is a weight module if as a vector space, it is the direct-sum of all its weight spaces:

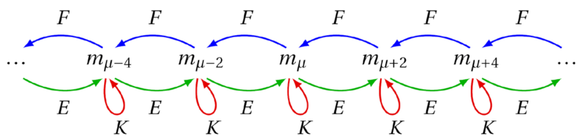
$$M = \bigoplus_{\mu \in supp(M)} M_{\mu}.$$

From now on we will only consider weight modules.

There are also non-weight modules, and these are necessarily infinite-dimensional.

### 1.1.5 Dense modules

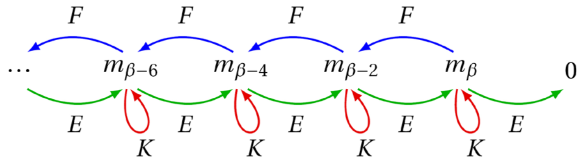
In the special case $M = U.m_{\mu}$ all weight spaces are 1-dimensional, and it is useful to depict $M$ as in the diagram below:




This is a collection of 1-dimensional $\Bbbk$-vector spaces fixed by $k^{\pm 1}$, while $F$ and $E$ allow moving between them. This is an example of a class of modules called *dense modules*. In this case $supp(M) = \mu + 2\mathbb{Z}$.

1.1.6 Verma modules

Suppose that in a dense module as the one above we have $E(m_\beta) = 0$ for some $\beta \in supp(M)$. Then $U.m_\beta \subset M$ is a submodule with support $\beta - 2\mathbb{N}_0$ which is called a *Verma module* and denoted $M(\beta)$.




The vector $m_\beta$ for which $E(m_\beta) = 0$ is a *highest weight vector* (of highest weight $\beta$) and $M(\beta)$ is said to be a *highest weight module*[1] (of highest weight $\beta$) (the terminology should be clear from the diagrams).

Verma modules are also called *standard modules*. They are defined as induced modules. Let $U(b) \subset U$ be the (Hopf) subalgebra generated by $k^{\pm 1}$ and $e$. It is an example of *Borel subalgebra* and this one is the standard Borel subalgebra.

Let $\mathbb{k}_\beta = \mathbb{k}v_\beta$ be a 1-dimensional representation of $U(b)$ generated by a weight vector $v_\beta$ of weight $\beta$:

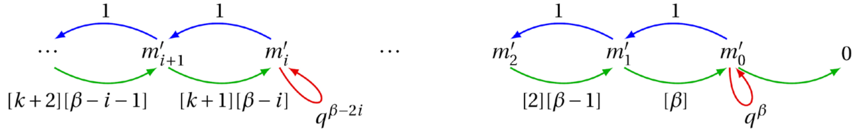
$$k.v_\beta = q^\beta v_\beta, \quad e.v_\beta = 0.$$

The Verma module $M(\beta)$ is the induced module

$$M(\beta) = U \otimes_{U(b)} \mathbb{k}_\beta.$$

It is easy to see that it is a highest weight module of highest weight $\beta$ with the vector $1 \otimes v_\beta$ the highest weight vector, and that all weight spaces are 1-dimensional, and therefore it agrees with the description above as a submodule of a dense module. Physicists like lowest-weight modules, as do Rouquier [20].

The description of $M(\beta)$ as an induced module has the advantage of giving immediately a basis, the *F basis*. From now on we find convenient to label the basis vectors $m'_0, m'_1, \ldots$ :

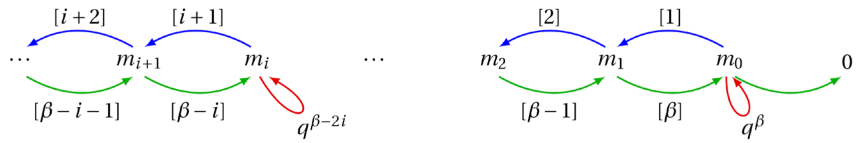



Here, for $\alpha \in \xi + \mathbb{Z}$ we put

[1] There is also the corresponding notion of *lowest weight module*.

$$[\alpha] := \frac{q^{\alpha} - q^{-\alpha}}{q - q^{-1}} \in \Bbbk(q)[\lambda^{\pm 1}].$$

There are other interesting bases: the *canonical basis* $\{m_0, m_1, ...\}$ :




1.1.7 The Shapovalov form

Verma modules come equipped with a bilinear form, called the *Shapovalov* form $(-,-)_\beta$. It is the bilinear form on $M(\beta)$ uniquely defined, for $m$, $m' \in M(\beta)$, $u \in U$, and $f \in \Bbbk$, by

- $(m_0, m_0)_\beta = 1$,
- $(um, m')_\beta = (m, \rho(u)m')_\beta$, where $\rho$ is the $q$-linear antiautomorphism $U \to U$ defined by $\rho(e) = q^{-1}k^{-1}f$, $\rho(f) = q^{-1}ke$ and $\rho(k) = k$,
- $f(m, m')_\beta = (fm, m')_\beta = (m, fm')_\beta$.

For example,

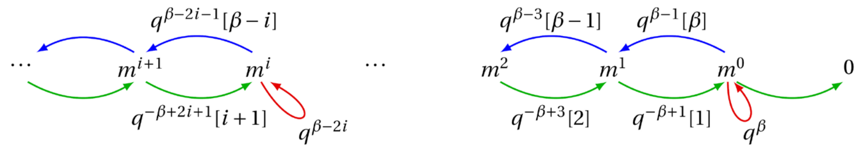
$$(m_i, m_i)_\beta = q^{i(\beta - i)} \frac{[\beta - i + 1][\beta - i + 2]\cdots[\beta]}{[i]!}.$$

Note that the canonical basis and the $F$ basis are both orthogonal w.r.t. the Shapovalov form and this will be important later

When $M(\beta)$ is irreducible, the Shapovalov form is nondegenerate (this is in fact a *iff* condition, since the radical of $\langle -, - \rangle$ is a submodule). This allows defining a *dual canonical basis* of $M(\beta)$, denoted $\{m^0, m^1, ...\}$, as $(m_i, m^j) = \delta_{i,j}$. This gives

$$m^i = q^{-i(\beta - i)} \frac{[i]!}{[\beta - i + 1][\beta - i + 2]\cdots[\beta]} m_i.$$

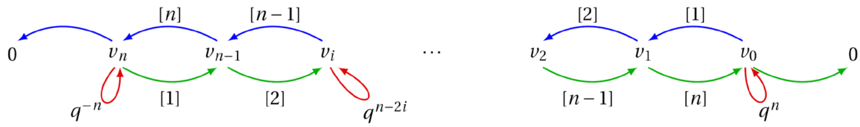
We need the $[\beta + i]$'s to be invertible in $\Bbbk$ (we can work for example in $\Bbbk(q^\beta)$. In this basis the $\mathfrak{sl}_2$-action is described in the diagram below.



The Verma module $M(\beta)$ is irreducible unless $\beta \in \mathbb{N}_0$. If $\beta = n \in \mathbb{N}_0$ then $M(n)$ contains $M(-n-2)$ as a submodule.

### 1.1.8 Finite-dimensional modules

We denote $V(n)$ the quotient $M(n)/M(-n-2)$. It has a canonical basis $\{v_0, v_1, \ldots\}$ :




The basis $\{v_0, v_1, \ldots\}$ is a particular case of Lusztig-Kashiwara's canonical bases for finite-dimensional irreducible representations of quantum groups.

Note that $V(n)$ has several symmetries: it is invariant under the operation that switches $v_i \leftrightarrow v_{n-2i}$ for all $i \in \{0, \ldots, n\}$ and $E \leftrightarrow F$.

Note also that in this case

$$\frac{k-k^{-1}}{q-q^{-1}}v_i = [n-2i]v_i = (q^{n-2i-1} + q^{n-2i-3} + \ldots + q^{-n+2i+3} + q^{-n+2i+1})v_i$$

if $n - 2i \geq 0$ (and its negative if $n - 2i \leq 0$) is a finite sum, and therefore, the main $\mathfrak{sl}_2$-relation can be written

$$EF(v) - FE(v) = [\mu]v,$$

for $v$ in the weight space $V_\mu$. Note that, as defined above, $[\mu]$ is a polynomial in $q$. As long as we have the weight space decomposition and the representation is finite dimensional we don't really need $k^{\pm 1}$. actually this is more general: we can get rid of $k^{\pm 1}$ acting on a weight module $M$ whenever the support $supp\,(M) \subseteq (\mathbb{Z})$.

The Shapovalov form descends to an nondegenerate bilinear form $(-,-)_n$ on $V(n)$ (since we have modded out by its radical. This allows defining the *dual canonical basis* of $V(n)$in the same way as before yielding[2]:

$$v^i = q^{i(-n+1)}\frac{[i]!}{[n-i+1]!}v_i. \tag{1}$$

Either the canonical basis and the dual canonical basis are orthogonal bases w.r.t. to $(-,-)_{\underline{n}}$.

2 This is a convention which is different from [3]. The dual canonical basis elements there were defined as $(v_i, v_i) = q^{j(n-j)}\delta_{ij}$.

For each $n \in \mathbb{N}_0$ there is a unique isomorphism class of $n + 1$-dimensional irreducible representation for $\mathfrak{sl}_2$. Moreover, every finite-dimensional representation of $\mathfrak{sl}_2$ decomposes into a direct sum of $V(n)$'s for various $n$'s.

### 1.2 *Categorical* $\mathfrak{sl}_2$*-actions*

#### 1.2.1 What should a categorical $\mathfrak{sl}_2$-action be?

Roughly speaking, a categorical $\mathfrak{sl}_2$-action on a category $\mathcal{C}$ consists of functors $F$, $E$, $k^{\pm 1}$ on $\mathcal{C}$ that "satisfy the $\mathfrak{sl}_2$-relations".

There are several ways of defining what is to satisfy the $\mathfrak{sl}_2$-relations, and apparently we have to make a choice.

The *Grothendieck group* of a category $\mathcal{C}$ endowed with a class of distinguished triples (e.g. exact sequences in abelian categories, triangles in triangulated categories, direct-sum decompositions $A \cong B \oplus C$ in additive categories) is the abelian group $k_0(\mathcal{C})$, freely generated by symbols $[A]$ for objects $A$ of $\mathcal{C}$, subjected to relations $[B] = [A] + [C]$ for each distinguished triple $(A, B, C)$. Sometimes we can take different Grothendieck groups for the same category. We will then use the notation $G_0(-)$ in the case we take the Grothendieck group w.r.t. exact sequences.

Since we are assuming almost nothing about $\mathcal{C}$, at the time being it seems reasonable to ask that the functors F, E, $\mathrm{K}^{+1}$ induce an $\mathfrak{sl}_2$-action on the Grothendieck group[3] of $\mathcal{C}$. This means the assignment $f \mapsto [F]$, $e \mapsto [E]$, $k^{\pm 1} \mapsto [K^{\pm 1}]$ defines an $\mathfrak{sl}_2$-action on $k_0(\mathcal{C})$.

This seems to be the simplest definition, but it doesn't say much about the functors, nor about $\mathcal{C}$. The only information we can extract at this moment is that $\mathcal{C}$ is a *graded category* (in which $q$ corresponds to the grading shift $\{1\}$ via $q[A] = [A\{1\}]$). We also know that the functors $\{F, E, K^{+1}\}$ induce operators on the Grothendieck group. Assuming it exists, we call it a *naïve* categorical $\mathfrak{sl}_2$-action. This can be slightly improved by demanding that the functor F is isomorphic to a left adjoint of E :

- $\mathfrak{sl}_2$ *acts weakly on* $\mathcal{C}$ *if the functors* F, E, $\mathrm{K}^{+1}$ *induce an* $\mathfrak{sl}_2$*-action on the Grothendieck group of* $\mathcal{C}$*, and* F *is isomorphic to a left adjoint of* E .

We say that $(\mathcal{C}, F, E, k^{\pm 1})$ is a *weak categorification* of the $\mathfrak{sl}_2$-module $k_0(\mathcal{C})$.

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3 In other words, we well be only interested in *exact functors*, where exact means they preserve the triples (i.e. additive, exact or triangulated).

Note that (naïve, weak) categorical actions on $\mathcal{C}$ gives bilinear forms on $K_0(\mathcal{C})$ (depending on the type of category $\mathcal{C}$ is). For example, one can have

$$\langle [X],[Y]\rangle = \mathrm{gdim}(\mathrm{Hom}_{\mathcal{C}}(X,Y)).$$

To go any further we have to restrict the class of categories $\mathcal{C}$ on which we act.

1.2.2 Integrable categorical $\mathfrak{sl}_2$-actions

Suppose that $\mathfrak{sl}_2$ acts weakly on $\mathcal{C}$. Suppose also that $\mathcal{C}$ has a zero object and if for every object $X$ of $\mathcal{C}$ we have $E^{r_1}(X)=0$ and $F^{r_2}(X)=0$ for $r_1$, $r_2 \gg 0$, where $E^{r_1}$ (resp $F^{r_2}$) is the composite of E (resp. F) with itself $r_1$ (resp. $r_2$) times.

In this case, the Grothendieck group of $\mathcal{C}$ is a direct sum of finite-dimensional irreducible representations. Moreover, all weights occurring in $K_0(\mathcal{C})$ are integers: $supp(K_0(\mathcal{C})) \subseteq \mathbb{Z}$.

It seems reasonable to assume that $\mathcal{C}$ has finite coproducts (direct sums) and moreover to ask that it has a block decomposition (recall the orthogonality of the several bases w.r.t. the Shapovalov form)

$$\mathcal{C} = \bigoplus_{\mu\in\mathbb{Z}} \mathcal{C}_\mu,$$

where $\mathcal{C}_\mu \subseteq \mathcal{C}$ is the full subcategory generated by objects $M$ such that $[M] \in K_0(\mathcal{C})_\mu$.

In this case we can give a step further and ask the functors $\{E, F\}$ to satisfy the following isomorphisms (recall we don't need $k^{\pm 1}$ anymore):

$$\begin{aligned} EF(X) &\cong FE(X)\oplus Id^{[\mu]}(X), \quad \text{for } \mu \geq 0, \\ FE(X) &\cong EF(X)\oplus Id^{[-\mu]}(X), \quad \text{for } \mu \leq 0, \end{aligned}$$

for every object $X \in \mathcal{C}_i$. Here $Id^{[\mu]}(X) = X\{\mu-1\} \oplus X\{\mu-3\} \oplus \ldots X\{-\mu+3\} \oplus Z\{-\mu+1\}$. Looking further at the form of the direct sum decompositions above we see that we'd better work with categories that are at least *additive*.

We have a bit more information about the functors that realize the action (we have a bit more of knowledge about the higher structure).

We have made a **crucial choice**: we have imposed that the functors F and E satisfy a direct-sum decomposition realizing the $\mathfrak{sl}_2$-commutator. Later we will find important to reformulate this condition.

We can now summarize.

**Definition 1.1 :** An *integrable categorical* $\mathfrak{sl}_2$*-action* on an additive category $\mathcal{C} = \bigoplus_{\mu \in \mathbb{Z}} \mathcal{C}_\mu$ consists of functors F and E such that

- $F$ is isomorphic to a left adjoint of $E$ up to degree shift, and they satisfy

$$F^{r_1}(X) = 0, \quad E^{r_2}(X) = 0, \quad \text{for} \quad r_i = r_i(\mu) \gg 0 \quad (i = 1, 2),$$

and

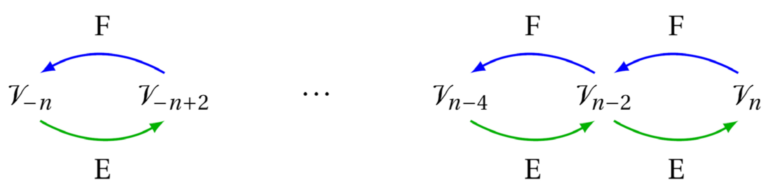
$$EF(X) \cong FE(X) \oplus Id^{[\mu]}(X), \quad \text{for} \quad \mu \geq 0,$$
$$FE(X) \cong EF(X) \oplus Id^{[-\mu]}(X), \quad \text{for} \quad \mu \leq 0,$$

for every object $X \in \mathcal{C}_\mu$.

Note we haven't given a notion of $\mathfrak{sl}_2$-action that is not naïve, nor weak, nor integrable. This will be done later in the context of categorification of Verma modules.

### 1.3 *Categorification of the finite-dimensional irreducibles: the CR–FKS approach*

The main idea is *to replace the weight spaces with categories*, on which $f$ and $e$ act via (exact) functors $F$ and $E$:




and ask these functors to be an adjoint pair (($F$, $E$) as always) and to satisfy the $\mathfrak{sl}_2$-relations:

$$EF(\mathcal{V}_{n-2k}) \cong FE(\mathcal{V}_{n-2k}) \oplus Id^{[n-2k]}_{\mathcal{V}_{n-2k}}, \quad n - 2k \geq 0,$$
$$FE(\mathcal{V}_{n-2k}) \cong EF(\mathcal{V}_{n-2k}) \oplus Id^{[-n+2k]}_{\mathcal{V}_{n-2k}}, \quad n - 2k \leq 0.$$

#### 1.3.1 *Categorification of the weight spaces: the cohomology of Grassmannians*

For $0 \leq k \leq n$, let $G_k(n)$ denote the variety of complex $k$-planes in $\mathbb{C}^N$. The cohomology ring of $G_k(n)$ has a natural structure of a $\mathbb{Z}$-graded $\mathbb{Q}$-algebra,

$$H^*(G_k(n), \mathbb{Q}) = \bigoplus_{0 \leq k \leq k(n-k)} H^k(G_k, \mathbb{Q}).$$

We write $H_k := H^*(G_k(n), \mathbb{Q})$.

The graded ring $H_k$ can be given an explicit description in terms of Chern classes. We have

$$H_k = Q[c_1,\dots,c_k,\overline{c}_1,\dots,\overline{c}_{n-k}]/I_{k,n},$$

where $\deg c_j = 2j = \deg\overline{c}_j$, and $I_{k,n}$ is the ideal generated by equating the terms in the equation

$$(1+c_1t+c_2t^2\cdots+c_kt^k)(1+\overline{c}_1t+\overline{c}_2t^2+\cdots+\overline{c}_{n-k}t^{n-k}) = 1$$

that are homogeneous in $t$. This is a neat way of encoding a large number of relations at once.

**Example 1.2 :** As a simple example, take $k = 1$. Then $G_1(n)$ is the complex projective space, and $H_1 \cong \mathbb{Q}[x]/(x^n)$ with $\deg(x) = 2$.

Let

- $H_k-\text{gmod}$ : the category of graded, finitely generated, projective $H_k$-modules, with degree-preserving maps, and set
- $\mathcal{V}_{n-2k} = H_k-\text{gmod}$.

The rings $H_k$ being graded local rings (they have an unique maximal left/right ideal) implies that their Grothendieck group is a free $Z[q, q^{-1}]$-module, generated by a unique indecomposable projective module, since objects satisfy the Krull-Schmidt property.

Hence,

$$K_0(\mathcal{V}_n)\otimes_{\mathbb{Z}[q,q^{-1}]}\mathbb{Q}(q) \cong \mathbb{Q}(q),$$

so that the category

$$\mathcal{V} = \bigoplus_{k=0}^{n}\mathcal{V}_{n-2k}$$

categorifies the irreducible representation $V(n)$ in the sense that

$$K_0(\mathcal{V}(n)) = \bigoplus_{k=0}^{n}K_0(\mathcal{V}_{n-2k})\otimes_{\mathbb{Z}[q^{\pm1}]}\mathbb{Q}(q)\cong V(n)$$

as $\mathbb{Q}(q)$-vector spaces.

At this point we have not yet defined a categorical $\mathfrak{sl}_2$-action...

### 1.3.2 Moving between the weight spaces: the categorical $\mathfrak{sl}_2$-action

Consider the partial flag variety

$$G_{k,k+1}(n) = \{(W_k,W_{k+1})\,|\,\dim_{\mathbb{C}}W_k = k,\ \dim_{\mathbb{C}}W_{k+1} = (k+1),\ 0\subset W_k\subset W_{k+1}\subset\mathbb{C}^n\}.$$

We write $H_{k,k+1} := H^*(G_{k,k+1}(n))$ for the cohomology ring of this variety. Again, this ring is simple to describe explicitly in terms of Chern classes: polynomial ring:

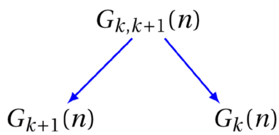
$$H_{k,k+1} := \mathbb{Q}[c_1, c_2, \ldots c_k; \xi; \overline{c}_1, \overline{c}_2, \ldots, \overline{c}_{n-k-1}] / I_{k,k+1,n}, \tag{2}$$

where $I_{k,k+1,n}$ is the ideal generated by equating the homogeneous terms in the equation

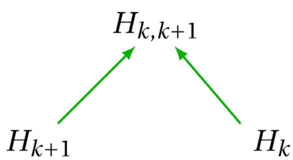
$$(1 + c_1 + c_2 t^2 + \ldots + c_k t^k)(1 + \xi t)(1 + \overline{c}_1 t + \overline{c}_2 t^2 + \ldots + \overline{c}_{n-k-1} t^{n-k-1}) = 1.$$

As before, everything is completely explicit. Here the generator $\xi$ has degree 2 and corresponds to the Chern class of the natural line bundle over $G_{k,k+1}(n))$ whose fibre over a point $0 \subset W_k \subset W_{k+1} \subset \mathbb{C}^n$ in $G_{k,k+1}(n))$ is the line $W_{k+1}/W_k$.

This variety has natural forgetful maps




inducing inclusion maps

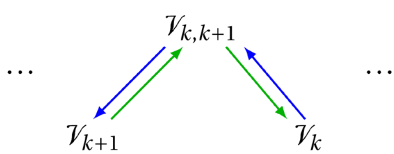



on cohomology.

These inclusions make $H_{k,k+1}$ an $(H_{k,k+1}, H_k)$-bimodule. Since these rings are commutative we can also think of $H_{k,k+1}$ as an $(H_k, H_{k+1})$-bimodule which we will denote by $H_{k+1,k}$.

Recall that we get functors between categories of modules by tensoring with a bimodule. We compose these functors by tensoring the corresponding bimodules.

The action of $e$ and $f$ is given by tensoring with the bimodules $H_{k+1,k}$. and $H_{k,k+1}$, respectively.



Define the functors

$$
\begin{aligned}
F_k &: \mathcal{V}_{n-2k} \to \mathcal{V}_{n-2k-2} \qquad \mathrm{Res}_{k+1}^{k,k+1}(H_{k,k+1} \otimes_k (-))\{1-n+k\},\\
E_k &: \mathcal{V}_{n-2kk+2} \to \mathcal{V}_{n-2k} \qquad \mathrm{Res}_{k}^{k,k+1}(H_{k,k+1} \otimes_{k+1} (-))\{-k\}.
\end{aligned}
$$

The grading shifts in the definition of E andF are necessary to ensure that these functors satisfy the $\mathfrak{sl}_2$-relations in Proposition 1.4 below.

**Proposition 1.3 :** The functors $F_k$, $E_k$ have both left and right adjoints and commute with the grading shift functor on graded modules.

This implies that they are exact, take projectives to projectives and therefore induce maps on the Grothendieck groups. As a matter of fact,

$$F_k(H_k) = \oplus_{[k]} H_{k+1}\{1-n+k\}, \quad E_k(H_{k+1}) = \oplus_{[n-k]} H_k\{-k\}.$$

**Proposition 1.4 :** *The functors* $\mathrm{F}_k$, $\mathrm{E}_k$ *satisfy the* $\mathfrak{sl}_2$*-relations*

$$
\begin{aligned}
\mathrm{E}_k\mathrm{F}_k &\cong \mathrm{F}_k\mathrm{E}_k \oplus Id_{\mathcal{V}_k}^{[n-2k]}, \qquad n-2k \geq 0,\\
\mathrm{F}_k\mathrm{E}_k &\cong \mathrm{E}_k\mathrm{F}_k \oplus Id_{\mathcal{V}_k}^{[n-2k]}, \qquad n-2k \leq 0.
\end{aligned}
$$

1.3.3 Categorification of $V(n)$

Put

$$\mathrm{F} = \bigoplus_{k\geq 0} \mathrm{F}_k \quad \text{and} \quad \mathrm{E} = \bigoplus_{k\geq 0} E_k.$$

**Theorem 1.5 : (Frenkel-Khovanov-Stroppel, Chuang-Rouquier)**

(1) *Functors* E *and* F *induce an action on the Grothendieck group* $K_0(\mathcal{V}(n))$.

(2) *With this action* $K_0(\mathcal{V}(n))$ *is isomorphic with* $V(n)$, *as* $\mathfrak{sl}_2$*-modules.*

(3) *The isomorphism sends classes of projective indecomposables to canonical basis elements.*

(4) $([M], [N])_n = \mathrm{gdim}\,\mathrm{Hom}_{\mathcal{V}(n)}(M, N)$.

Due to results of Chuang-Rouquier and Rouquier we know that $\mathcal{V}(n)$ is essentially unique. This will be our typical example of a strong categorical action, where the isomorphisms are fixed by the 2-morphisms. An important ingredient is an action of the nilHecke algebra.

1.3.4 Categorifying the dual canonical basis

In order to categorify the dual canonical basis we have to work with a bigger category. We consider

- $H_k$ – fmod: the category of graded, finitely generated, $H_k$-modules, with degree-preserving maps, and set [resume]
- $\tilde{\mathcal{V}}_{n-2k} \underset{n}{=} H_k - \mathrm{f\,mod}$,
- $\tilde{\mathcal{V}} = \bigoplus_{k=1} \tilde{\mathcal{V}}_{n-2k} = H_k - \mathrm{f\,mod}$.

The Grothendieck group $G_0(\mathcal{V}_{n-2k})$ is a free $\mathbb{Z}[q, q]$ -module generated by the unique simple module since objects in $\mathcal{V}_{n-2k}$ either have finite length (and the unique indecomposable projective module after tensoring with $\mathbb{Q}(q)$ over $\mathbb{Z}[q^{\pm 1}]$).

**Example 1.6 :** For example, for $H_1$ we have that the simple $S$ of $\mathbb{Q}[x]/x^n$ is the quotient of $\mathbb{Q}[x]/x^n$ by the (maximal) ideal generated by $x$, and its projective cover is the indecomposable $\mathbb{Q}[x]/x^n$. The projective indecomposable $\mathbb{Q}[x]/x^n$ has a composition series

$$0 \subseteq x^{n-1}\mathbb{Q}[x]/x^n \subseteq \cdots \subseteq x^2\mathbb{Q}[x]/x^n \subseteq x\mathbb{Q}[x]/x^n \subseteq \mathbb{Q}[x]/x^n,$$

where $x^m\mathbb{Q}[x]/x^n \subseteq x^{m-1}\mathbb{Q}[x]/x^n$ is the submodule generated by $x^m$. We have

$$\frac{x^i\mathbb{Q}[x]/x^n}{x^{i+1}\mathbb{Q}[x]/x^n} \cong S\{2i\},$$

and so, in the Grothendieck group we have

$$[\mathbb{Q}[x]/x^n] = \sum_{i=0}^{n-1} q^{2i}[S] = q^{n-1}[n]_q[S],$$

where we have written $[-]_q$ for quantum numbers to avoid confusion with the notation for the classes on the Grothendieck group (cf. (1) which gives $v_1 = q^{n-1}[n]v^1$).

For the Grothendieck group of $\tilde{\mathcal{V}}(n)$ we have an isomorphism

$$G_0(\tilde{\mathcal{V}}(n)) = \bigoplus_{k=0}^{n} G_0(\tilde{\mathcal{V}}_{n-2k}) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{Q}(q) \cong V(n),$$

as $\mathbb{Q}(q)$-vector spaces.

The categorical action is constructed as as before, and we have the following.

**Theorem 1.7 : (Frenkel-Khovanov-Stroppel, Chuang-Rouquier)**

(1) *Functors* E *and* F *induce an action on the Grothendieck group* $G_0(\mathcal{V}(n))$.

(2) *With this action* $G_0(\mathcal{V}(n))$ *is isomorphic with* $(\mathcal{V}(n)$, *as* $\mathfrak{sl}_2$*-modules.*

(3) *The isomorphism sends classes of projective indecomposables to canonical basis elements, and classes of irreducibles to dual canonical basis elements.*

The duality between a canonical basis vector and a dual canonical basis vector is categorified by a Hom-like form.

Within this construction it is not possible to categorify the "change-of basis", since the formulas expressing dual canonical basis vectors in terms of canonical basis vectors (and vice versa) involve denominators. One way to go around is to work with *completed Grothendieck groups* à la Achar-Stroppel. A different approach is to consider slightly different categories and work with topological Grothendieck groups, as we will see in §2.3.

Again in this case we have an action of the nilHecke algebra.

1.3.5. Unraveling the higher structure: the nilHecke algebra

We are interested in studying the natural transformations between various composites of the functors $\mathrm{F}_k$'s and $\mathrm{E}_k$'s. The presentation of $H_{k,k+1}$ we have makes it easy to explicitly construct bimodule homomorphisms and determine relations between them.

Up to a shift, the functor $\mathrm{F}^m$ decomposes into a direct sum of functors, each one involving tensoring (at the left) with a bimodule like

$$H_{r+m,r+m-1} \otimes_{r+m-1} H_{r+m-1,r+m-2} \otimes_{r+m-2} \cdots \otimes_{r+2} H_{r+2,r+1} \otimes_{r+1} H_{r+1,r}.$$

This bimodule is isomorphic to the bimodule

$$H_{r+m,\dots,r} \;=\; H^*(G_{r,r+1,\dots,r+m}(n),\mathbb{Q}),$$

where

$$G_{r,r+1,\dots,r+m}(n) = \{(W_r, W_{r+1},\dots,W_{r+m}) |\ \dim_{\mathbb{C}} W_j = j, 0 \subset W_r \subset \cdots \subset W_{r+m} \subset \mathbb{C}^n\}.$$

Once again, we can give an explicit description of this cohomology ring using Chern classes:

$$H_{r,\cdots,r+m} = \mathbb{Q}[c_1,\dots,c_r,\xi_1,\xi_2,\dots,\xi_m,\overline{c}_1,\dots,\overline{c}_{n-m}]/I_{r,\dots,r+m},$$

where $I_{r,\dots,r+m}$ is the ideal generated by the homogeneous terms in the equation

$$(1+x_1t+x_2t^2+\dots+x_rt^r)(1+\xi_1t)(1+\xi_2t)\cdots(1+\xi_mt)(1+y_1t+\dots+y_{n-m}t^{n-m}).$$

The degree two generators $\xi_j$ arise from the Chern classes of the line bundles $W_{r+1}/W_{r+j-1}$.

**Lemma 1.8 :** *The operators* $\xi_j$ *and* $\partial_j : H_{r,\cdots,r+m} \to H_{r,\cdots,r+m}$ *(*$1 \le i \le n$*,* $1 \le j \le n-1$*), defined on* $f \in H_{r,\cdots,r+m}$ *by*

$$\xi(f) = \xi_i f \quad \text{and} \quad \partial_j(f) = \frac{f - s_j(f)}{\xi_j - \xi_{j+1}},$$

*are* $(H_r, H_{r+m})$*-bimodule maps.*

The $\mathbb{Q}$-algebra generated by the operators $\xi_i(1 \le i \le n)$ and $\partial_i(1 \le i \le n-1)$ is called the *nilHecke algebra* and will be denoted $NH_m$. It can be defined over any associative unital ring $\Bbbk$ and has a presentation by the generators above and relations

$$\begin{gathered} \xi_i\xi_j = \xi_j\xi_i, \\ \partial_i\xi_j = \xi_j\partial_i \quad \text{if} \quad |i-j| > 1, \quad \partial_i\partial_j = \partial_j\partial_i \quad \text{if} \quad |i-j| > 1, \\ \partial_i\xi_i = \xi_{i+1}\partial_i + 1, \quad \partial_i^2 = 0, \\ \xi_i\partial_i = \partial_i\xi_{i+1} + 1, \quad \partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}. \end{gathered} \tag{3}$$

An immediate consequence of Lemma 1.8 is the following.

**Proposition 1.9 :** *The composite functors* $F^m$ *carry an action of the nilHecke algebra* $NH_m$*.*

By adjunction, the $E$'s also carry an action of the nilHecke algebra.

The nilHecke algebra is $\mathbb{Z}$-graded with $\deg(\xi) = 2$ and $\deg(\partial_i) = 2$. Later we will call this grading the $q$-grading.

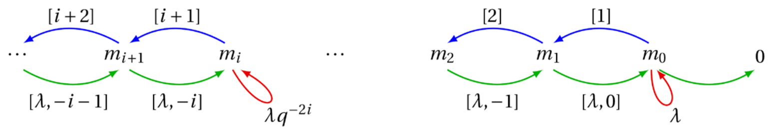
Note also that the category $\mathcal{V}(n)$ is abelian. In order for it to have a nilHecke action is enough it is $\Bbbk$-linear. For example we can work with projective objects and $k_0$.

## 2. Categorification of Verma modules : $\mathfrak{sl}_2$

### 2.1 (*Universal) Verma modules revisited*

Recall that we put $\lambda = q^\beta$ and treat $\lambda$ as a formal parameter. This gives the universal Verma module which, abusing notation, we write $M(\lambda)$. It is universal in the sense that any Verma module can be obtained from this one by specializing $\lambda$: there is an "evaluation map" $M(\lambda) \to M(\beta)$ $(\lambda \mapsto q^\beta)$. Note that our ground field now contains $\mathbb{C}(q, \lambda)$. We prefer to work over $\mathbb{C}((q, \lambda))$, the field of formal Laurent series for technical reasons related to the fact that we interpret denominators in $\mathbb{C}(q, \lambda)$ as Laurent series.

In terms of the *canonical basis*, the universal Verma module $M(\lambda)$ has the form:




where

$$[\lambda, r] = \frac{\lambda q^r - \lambda^{-1} q^{-r}}{q - q^{-1}}$$

(recall this is $[\beta + r]$).

## 2.2 *Towards a categorification of a Verma module: initial constraints*

Just by looking at the diagram above for $M(\lambda)$ one sees that one needs to categorify "multiplication by $[\lambda, k]$". We see that

- we need (at least) a *bigraded category*.

  Note also that we cannot write $[\lambda, k]$ as a finite sum. One way forward is to [resume]

- interpret the denominator as a power sum

$$\frac{1}{q - q^{-1}} = -q^{-1}(1 + q + q^2 + \cdots),$$

  and ask our categories to have (controlled) *infinite coproducts*.

  But we also have the minus signs!. One possibility to deal with them is to [resume]

- work with *supercategories*[4].

  Notation: For an object $X$ in such a category we denote by $X\langle r, s\rangle$ its shift up by $r$ units in the first grading (the "$q$") and by $s$ units in the second (the "$\lambda$") and by $\Pi X$ its shift in the $\mathbb{Z}/2\mathbb{Z}$-degree, called the *parity*.

  To categorify multiplication by $\frac{1}{q-q^{-1}}$ one can (and we will!) consider the infinite coproduct

$$\coprod_{i \in \mathbb{N}_0} \Pi(-)\langle 2i - 1, 0\rangle.$$

[4] See for example [2.6] or [7] for the basic background on super structures.

We have made some choices, and they seem reasonable at this point! Altogether, if we are to categorify Verma modules we can work with (at least) *additive*, *bigraded*, *supercategories*, which have *infinite coproducts*.

In this context a *weak categorification of the Verma module* $M(\lambda)$ should consist of an additive, bigraded, supercategory, with infinite coproducts

$$\mathcal{M}(\lambda) = \bigoplus_{\mu \in \sup p(M(\lambda))} \mathcal{M}(\lambda)_{\mu},$$

and functors $\mathrm{F}, \mathrm{E}, \mathrm{K}^{\pm 1} \in \mathrm{Fun}(\mathcal{M}(\lambda))$ such that F is a left adjoint of E, and they satisfy a categorical version of the $\mathfrak{sl}_2$-relations.

Let's look at what we mean by $\mathfrak{sl}_2$-relations in this context. There is a point whose importance is fundamental: F is a left adjoint *but not a right adjoint* of E (the pair (F, E) is not an adjoint pair), otherwise, there would be an $r \in \mathbb{N}$ such that $\mathrm{F}^{r} = 0$ (see the remarks right after the definition of strong integrable 2-representation).

This means that F, E and $\mathrm{K}^{+1}$ *cannot be connected through a direct-sum decomposition* as in the case of categorification of integrable representations. Otherwise, that maps realizing the decomposition could be used to imply that (F, E) is a biadjoint pair.

Recall that we want this EF-relation to imply the $\mathfrak{sl}_2$-commutator on the Grothendieck group. Therefore, the next type of relation one can think of is to ask that $M(\lambda)$ admits exact sequences and the composite functors EF and FE be related through an exact sequence...

One of the principal features of the universal Verma module is that it projects to the irreducible representation $V(n)$ (for any n). It seems reasonable to impose that

- *a weak categorification of* $M(\lambda)$ *comes equipped with a categorical projection onto a categorification of* $V(n)$.

Let's sketch a provisional definition.

**Definition 2.1 :** (Provisional definition)**.** A *weak categorification of the Verma module* $M(\lambda)$ is an additive, bigraded, supercategory $M(\lambda)$, with infinite coproducts and admitting exact sequences,

$$\mathcal{M}(\lambda) = \bigoplus_{\mu \in \sup p(M(\lambda))} \mathcal{M}(\lambda)_{\mu},$$

and functors $\mathrm{F}, \mathrm{E}, \mathrm{K}^{\pm 1} \in \mathrm{Fun}(\mathcal{M}(\lambda))$, which commute with grading shifts, and descend to operators on a Grothendieck group of $M(\lambda)$ and satisfy:

(1) $\mathrm{F}(\mathcal{M}(\lambda)_{\mu}) \subseteq \mathcal{M}(\lambda)_{\mu-2}$, $\mathrm{E}(\mathcal{M}(\lambda)_{\mu}) \subseteq \mathcal{M}(\lambda)_{\mu+2}$, $\mathrm{K}(\mathcal{M}(\lambda)_{\mu}) \subseteq \mathcal{M}(\lambda)_{\mu}$, for all $\mu \in$ supp $(\mathcal{M}(\lambda))$,

(2) F is isomorphic to a left adjoint of E (up to a shift),

(3) $\mathrm{KF} \cong \mathrm{FK}\langle -2, 0\rangle$, $\mathrm{KE} \cong \mathrm{EK}\langle 2, 0\rangle$, $\mathrm{KF}^{\pm 1} \cong \mathrm{F}^{\pm 1}\mathrm{K} \cong \mathrm{ID}_{\mathcal{M}(\lambda)}$,

(4) F, E $\mathrm{K}^{+1}$ and commute with grading shifts and with the parity change $\Pi$,

(5) there is a (non-split) exact sequence

$$0 \to \mathrm{EF} \to \mathrm{FE} \to \mathrm{QK} \oplus \Pi\,\mathrm{QK}^{-1},$$

where Q is the infinite coproduct $\mathrm{Q}(-) = \coprod_{i \in \mathbb{N}_0} (\Pi(-))\langle 2i-1, 0\rangle$,

(6) For any $n \in \mathbb{N}_0$ there is a "projection" from $M(\lambda)$ to a categorification of $V(n)$.

2.3 *Topological Grothendieck groups*

The fact that we are to work with *infinite coproducts* impose **severe restrictions** on the categories we will work with. Recall that we in order to categorify Verma modules we need the Grothendieck group of each block $M(\lambda)_{\mu}$ ($\mu \in$ supp$(M(\lambda))$ to be finite dimensional (and non-zero).

We will work with (bigraded, super, locally additive[5] categories $\mathcal{C}$ whose enriched $A \in \mathcal{C}$ spaces

$$\mathrm{HOM}(M,N) \;=\; \bigoplus_{\ell,r \in \mathbb{Z}} \mathrm{HOM}^{\ell,r}(M,N) = \bigoplus_{\ell,r \in \mathbb{Z}} \mathrm{Hom}(M, N\langle \ell, r\rangle)$$

are finite-dimensional in each degree. Moreover we demand that

- pairs $(\ell, r) \in \mathbb{Z}^2$ for which $\mathrm{HOM}^{\ell,r}(M,N) \neq 0$, lie inside a cone $C \subset \mathbb{Z}^2$ compatible with an order in $\mathbb{Z}^2$,
- $\mathrm{HOM}^{\ell,r}(M,N) = 0$ for $r \ll 0$ or $r \gg 0$,

5 We say that an additive, strictly $\mathbb{Z}$-graded category $\mathcal{C}(A\{r\} \not\cong A$ for all $A \in \mathcal{C}$) is locally additive if all its locally finite coproducts are biproducts.)

We also require that $\mathcal{C}$ has:

- local Krull-Schmidt property: every object decomposes into a locally finite direct sum of small[6] objects having local endomorphisms rings (see [15, §5.1]),

  or

- local Jordan-Hölder property (if $\mathcal{C}$ is abelian: every object has locally finitely many composition factors, plus a stability condition) (see [15, §5.2]), or both.

**Remark 2.2 :** A locally Krull-Schmidt category is idempotent complete. Moreover, an object with local endomorphism ring is indecomposable, and have only 0 and 1 as idempotents.

For these type of categories we can define the

- *Topological Grothendieck group* $\mathbf{G}_0(\mathcal{C})$ : this is the free $\mathbb{Z}((q, \lambda))$-module generated by the classes of simple objects (up to shifts).
- *Topological split Grothendieck group* $\mathbf{F}_0(\mathcal{C})$ : this is the free $\mathbb{Z}((q, \lambda))$-module generated by the classes of indecomposables (up to shifts).

Both Grothendieck groups above are modules over $\mathbb{Z}_\pi = \mathbb{Z}[\pi]/\pi^2-1$. When specializing the parameter $\pi = -1$ and extending the scalars to $\mathbb{Q}$, we write

$$\widetilde{\mathbf{G}_0}(\mathcal{C}) = \mathbf{G}_0(\mathcal{C}))\otimes_{\mathbb{Z}_\pi} \mathbb{Q}[\pi]/(\pi+1),$$

and the same for $\widetilde{\mathbf{K}}_0(\mathcal{C})$.

**Remark 2.3 :** The conditions above are mainly technical and are necessary to be able to define Grothendieck groups with the correct properties for the sake of categorification. A complete description of these categories and the details its Grothendieck groups can be found in [15, §5].

### 2.4 *Extending CR–FKS to Verma modules*

In the following it seems more natural to categorify $M(\lambda q^{-1})$ and this will be clear very soon. We want to find nice bigraded (super)rings $\Omega_k$ with 1-dimensional Grothendieck groups and $(\Omega_k, \Omega_{k+1})$-bimodules (denoted $\Omega_{k,k+1}$), such that

---

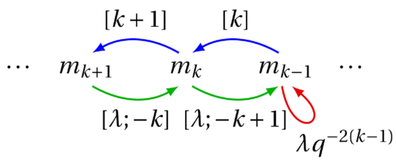
6 An object $A$ in a category $\mathcal{C}$ is small if every map $f : A \to \coprod_{i\subset I} B_i$ factors through $\coprod_{j\in J} B_j$ for a finite subset $J \subset I$.

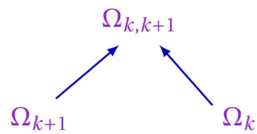
- $\Omega_{k,k+1}$ is a free $\Omega_{k+1}$-left module of graded rank $[k+1]$,
- $\Omega_{k,k+1}$ is a free $\Omega_k$-left module of graded rank $[\lambda; -k]$,
- The $(\Omega_k, \Omega_k)$-bimodules

$$\Omega_{k(k+k)k} := \Omega_{k,k+1} \otimes_{k+1} \Omega_{k+1,k}, \quad \Omega_{k(k-k)k} := \Omega_{k,k-1} \otimes_{k-1} \Omega_{k-1,k}$$

are related through a short exact sequence

$$0 \longrightarrow \Omega_{k(k-1)k} \to \Omega_{k(k+1)k} \to \Omega_k[\xi]\langle 2k+2,-1\rangle \oplus \Pi\Omega_k[\xi]\langle -2k,1\rangle.$$




2.4.1 Categorification of the weight spaces of $M(\lambda q^{-1})$ : $H^*(G(n))$ and $H^*(G(n))^!$

Let $G_k$ be the Grassmannian variety of $k$-planes in $\mathbb{C}^\infty$. Its (rational) cohomology ring is just a polynomial ring generated by the Chern classes

$$H(G_k) \cong \mathbb{Q}[x_1, \ldots, x_k], \quad \deg(x_k) = 2k,$$

The Ext-algebra $\mathrm{Ext}_{H(G_k)}(\mathbb{Q}, \mathbb{Q})$ is an exterior algebra

$$\mathrm{Ext}_{H(G_k)}(\mathbb{Q}, \mathbb{Q}) = \wedge^\bullet(\omega_1, \ldots, \omega_k),$$

with $\deg(\omega_k) = (q\deg(\omega_k), \lambda\deg(\omega_i) = (-2k, 2)$.

We form

$$\Omega_k = H(G_k) \otimes \mathrm{Ext}_{H(G_k)}(\mathbb{Q}, \mathbb{Q}),$$

which we regard as a *bigraded superring*. Here the $x_i$'s are *even* while the $\omega_i$'s are *odd*.

Put

$$\mathcal{M}_{\lambda q^{-1-2k}} = \Omega_k - \mathrm{mod}_{lf}.$$

The latter being bigraded, left (super) $\Omega_k$-modules that are finite dimensional on each degree and cone bounded.

### 2.4.2 Moving between the weight spaces: the categorical $\mathfrak{sl}_2$-action

Let $G_{k,k+1}$ be the infinite 1-step partial flag variety

$$\{0 \subset W_k \subset W_{k+1} \subset \mathbb{C}^\infty \mid \dim_{\mathbb{C}}(W_j) = j\}.$$

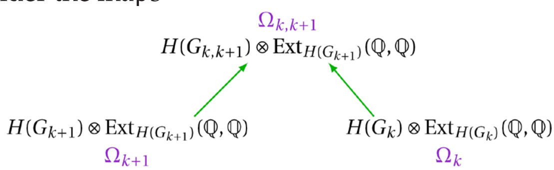
We have

$$H(G_{k,k+1}) \;\cong\; \mathbb{Q}[y_1,\dots,y_k,\xi], \quad \deg(y_j) = 2j, \deg(\xi) = 2.$$

Put

$$\Omega_{k,k+1} \;=\; H(G_{k,k+1}) \otimes Ext_{H(G_{k+1})}(\mathbb{Q},\mathbb{Q}).$$

We consider the maps




Explicitly, these maps are

$$\phi_k^* : \Omega_k \to \Omega_{k,k+1}, \quad \begin{cases} x_j \mapsto y_j, \\ \omega_j \mapsto \omega_j + \xi\omega_{j+1}, \end{cases}$$

and

$$\psi_{k+1}^* : \Omega_{k+1} \to \Omega_{k,k+1}, \quad \begin{cases} x_j \mapsto y_j + \xi j_{j-1}, \\ \omega_j \mapsto \omega_j. \end{cases}$$

with $y_0 = 1$ and $y_{i+1} = 0$.

Define the functors

$$\mathrm{F}_k : \mathcal{M}_{\lambda q^{-1-2k}} \to \mathcal{M}_{\lambda q^{-1-2(k+1)}} \qquad \mathrm{Res}_{k+1}^{k,k+1}(\Omega_{k,k+1} \otimes_k (-))\langle -k, 0\rangle,$$

$$\mathrm{E}_k : \mathcal{M}_{\lambda q^{-1-2(k+1)}} \to \mathcal{M}_{\lambda q^{-1-2k}} \qquad \mathrm{Res}_{k}^{k,k+1}(\Omega_{k,k+1} \otimes_{k+1} (-))\langle k+2, -1\rangle,$$

$$\mathrm{K}_k : \mathcal{M}_{\lambda q^{-1-2k}} \to \mathcal{M}_{\lambda q^{-1-2k}} \qquad (-)\langle 2k, 1\rangle,$$

and $Q_k : \mathcal{M}_{\lambda q^{-1-2k}} \to \mathcal{M}_{\lambda q^{-1-2k}}$, defined for all $k \geq 0$ by $\mathbb{Q}(-) = \Pi(-) \otimes \mathbb{Q}[\xi]\langle 1, 0\rangle$ and put

$$\mathcal{M}(\lambda q^{-1}) \;=\; \bigoplus_{k\geq 0} \mathcal{M}_{\lambda q^{-1-2k}},$$

and

$$\mathrm{F} = \bigoplus_{k\geq 0} \mathrm{F}_k, \qquad \mathrm{E} = \bigoplus_{k\geq 0} \mathrm{E}_k, \qquad \mathrm{K} = \bigoplus_{k\geq 0} \mathrm{K}_k.$$

**Proposition 2.4 :** *Functors* F, E *are exact and* E *is isomorphic to a right adjoint of* F (*they are not biadjoint!). Moreover, there is an action of the nilHecke algebra on* $F^m$ (*and on* $E^m$).

Actually, there is a bigger (super)algebra acting on $F^m$ and $E^m$. It can be computed through bimodule homomorphisms the same way we did $NH_m$. We will find this algebra again in § 3.1.2.

**Theorem 2.5 :** *We have natural isomorphisms*

$$KK^{-1} \cong Id \cong K^{-1}K,$$
$$KF \cong FK\langle -2,0\rangle, \quad KE \cong EK\langle 2,0\rangle,$$

*and a natural exact sequence*

$$0 \longrightarrow FE \longrightarrow EF \longrightarrow KQ \oplus \Pi K^{-1}Q \longrightarrow 0.$$

2.4.3. The categorification theorem

**Theorem 2.6 :**

(1) *The functors* F, E *and* K *induce an action of quantum* $\mathfrak{sl}_2$ *on the Grothendieck group of* $\mathcal{M}(\lambda q^{-1})$. *With this action* $\mathbf{K}_0(\mathcal{M})$ *is isomorphic with the Verma module* $\mathcal{M}(\lambda q^{-1})$ *after specializing the action of* $[\Pi]$ *to –1.*

(2) *The isomorphism from* $\mathbf{K}_0(\mathcal{M}(\lambda q^{-1}))$ *sends classes of projective indecomposable objects to canonical basis elements, and classes of irreducibles to dual canonical basis elements.*

2.5 *DGAs and the recovering of CK–FKS's categorification of* $V(n)$

2.5.1 DG rings

For $n \in \mathbb{N}_0$ and for each $k$ we turn $\Omega_k$ and $\Omega_{k,k+1}$ into DG rings by introducing differentials $d_n^k$ and $d_n^{k,k+1}$, both with degrees $\langle n, -2\rangle$ and $\mathbb{Z}/2\mathbb{Z}$-degree 1.

These act trivially on $H(G_k)$ and $H(G_{k,k+1})$ and send the generators of $\mathrm{Ext}_{H(G_i)}(\mathbb{Q},\mathbb{Q})$ to elements of $H(G_k)$ (and $H(G_{k,k+1})$ respectively). Moreover, these differentials commute with the canonical maps that give $\Omega_{k,k+1}$ the structure of a $(\Omega_k, \Omega_{k+1})$-bimodule, so that it becomes a DG bimodule.

**Proposition 2.7 :** *The DG rings* $(\Omega_k, d_n^k)$ *and* $(\Omega_{k,k+1}, d_n^k)$ *are formal. Moreover we have quasi-isomorphisms*

$$(\Omega_k, d_n^k) \cong_{q.i.} (H(G_k(n)), 0),$$
$$(\Omega_{k,k+1}, d_n^k) \cong_{q.i.} (H(G_{k,k+1}(n)), 0).$$

### 2.5.2 The snake lemma and consequences

Recall that in the case of $\mathcal{V}(n)$ we had a direct sum decomposition for the functors EF and FE. We will now see that this comes naturally as a consequence of our exact sequence and differentials.

We can equip $\Omega_k[\xi] \oplus \Pi\Omega_k[\xi]\langle -2k-2\rangle$ with a differential $d_n$ such that it becomes a DG bimodule over $(\Omega_k, d_n^k)$. This is not a direct sum of two DG bimodules, since $d_n$ mixes terms of both summands. This differential is compatible with the maps in the SES and we have a short exact sequence of DG $((\Omega_k, d_n), (\Omega_k, d_n))$-bimodules

$$0 \to (\Omega_{k(k-1)k}, d_n) \to (\Omega_{k(k+1)k}, d_n) \to \\ (\Omega_k[\xi]\langle 2k,-1\rangle \oplus \Pi\Omega_k[\xi]\langle -2k-2,1\rangle, d_n) \to 0.$$

By the snake lemma, it descends to a long exact sequence of $H(\Omega_k, d_n) \cong H(G_{k;n})$ -bimodules

$$\dots \longrightarrow H^1(\Omega_{k(k+1)k}, d_n) \longrightarrow H^1(\Omega_k[\xi]\langle 2k,-1\rangle \oplus \Pi\Omega_k[\xi]\langle -2k-2,1\rangle, d_n) \longrightarrow$$
$$H^0(\Omega_{k(k-1)k}, d_n) \longrightarrow H^0(\Omega_{k(k+1)k}, d_n) \longrightarrow H^0(\Omega_k[\xi]\langle 2k,-1\rangle \oplus \Pi\Omega_k[\xi]\langle -2k-2,1\rangle, d_n) \longrightarrow$$
$$H^1(\Omega_{k(k-1)k}, d_n) \longrightarrow \dots$$

We know that the homology of $(\Omega_{k(k+1)k}, d_n)$ is concentrated in parity 0 and thus we have a long exact sequence

$$0 \longrightarrow H^1(\Omega_k[\xi]\langle 2k,-1\rangle \oplus \Pi\Omega_k[\xi]\langle -2k-2,1\rangle, d_n) \longrightarrow$$
$$H^0(\Omega_{k(k-1)k}, d_n) \longrightarrow H^0(\Omega_{k(k+1)k}, d_n) \longrightarrow H^0(\Omega_k[\xi]\langle 2k,-1\rangle \oplus \Pi\Omega_k[\xi]\langle -2k-2,1\rangle, d_n) \longrightarrow$$
$$0.$$

Since we have explicit formulas and nice decompositions we can easily compute the homologies:

(1) For $n-2k \geq 0$ the homology of $(\Omega_k[\xi]\langle 2k,-1\rangle \oplus \Pi\Omega_k[\xi]\langle -2k-2,1\rangle, d_n)$ is concentrated in parity 0 and given by

$$\bigoplus_{\{n-2k\}} q^{2k} H(G_{k;n}).$$

Therefore we get the following short exact sequence

$$0 \to H(G_{k,k-1;n}) \otimes_{H(G_{k-1;n})} H(G_{k-1,k;n})$$
$$\hookrightarrow H(G_{k,k+1;n}) \otimes_{H(G_{k+1;n})} H(G_{k+1,k;n}) \twoheadrightarrow \bigoplus_{\{n-2k\}} q^{2k} H(G_{k;n}) \to 0.$$

(2) For $n - 2k \leq 0$ the homology is concentrated in parity 1 and it is isomorphic to

$$\bigoplus_{\{2k-n\}} q^{-2k-2} \lambda^2 \Pi H(G_{k;n}).$$

After shifting by the degree of the connecting homomorphism, it yields the short exact sequence

$$0 \to \bigoplus_{[2k-n]} q^{2k} H(G_{k;n}) \hookrightarrow H(G_{k,k-1;n}) \otimes_{H(G_{k-1;n})} H(G_{k-1,k;n})$$
$$\twoheadrightarrow H(G_{k,k+1;n}) \otimes_{H(G_{k+1;n})} H(G_{k+1,k;n}) \to 0.$$

**Proposition 2.8 :** *Both exact sequences split, and recover the well-known $\mathfrak{sl}_2$ categorical action of CR–FKS [1, 3] using cohomology of the finite Grassmannians and 1-step flag varieties.*

Define the DG $((\Omega_{k+1}, d_n), (\Omega_k, d_n))$-bimodule

$$(\hat{\Omega}_{k+1,k}, d_n) = (\Omega_{k+1,k}, d_n)\langle 0,0 \rangle,$$

and the *DG* $((\Omega_k, d_n), (\Omega_{k+1}, d_n))$ − bimodule

$$(\hat{\Omega}_{k,k+1}, d_n) = (\Omega_{k,k+1}, d_n)\langle -n, 1 \rangle.$$

**Proposition 2.9 :** *We have quasi-isomorphisms of bigraded DG* $((\Omega_k, d_n), (\Omega_k, d_n))$*-bimodules*

$$(\hat{\Omega}_{k,k+1} \otimes_{k+1} \hat{\Omega}_{k+1,k}, d_n) \cong (\hat{\Omega}_{k,k-1} \otimes_{k-1} \hat{\Omega}_{k-1,k}, d_n) \oplus_{[n-2k]} (\Omega_k, d_n), \text{ if } n-2k \geq 0,$$
$$(\hat{\Omega}_{k,k-1} \otimes_{k-1} \hat{\Omega}_{k-1,k}, d_n) \cong (\hat{\Omega}_{k,k+1} \otimes_{k+1} \hat{\Omega}_{k+1,k}, d_n) \oplus_{[2k-n]} (\Omega_k, d_n), \text{ if } n-2k \leq 0.$$

### 2.5.3 Derived equivalences

Let $\mathcal{D}^c(\hat{\Omega}_k, d_n)$ and $\mathcal{D}^c(\hat{\Omega}_{k,k+1}, d_n)$ be respectively the derived category of bigraded, left, compact $(\hat{\Omega}_k, d_n)$ modules and the derived category of bigraded, left, compact $(\hat{\Omega}_{k+1,k}, d_n)$-modules.

**Proposition 2.10 :** *There are equivalences of triangulated categories between*

$$\mathcal{D}^c(\hat{\Omega}_k, d_n) \cong \mathcal{D}^b(H(G_k(n)) - g\,\mathrm{mod}) = \mathcal{D}^b(\mathcal{V}_{k+1,k})$$
$$\mathcal{D}^c(\hat{\Omega}_{k+1,1}, d_n) \cong \mathcal{D}^b(H(G_{k,k+1}(n)) - g\,\mathrm{mod}) = \mathcal{D}^b(\mathcal{V}_{k+1,k}),$$

*where $\mathcal{M}^b(-)$ is the bounded derived category.*

Recall that $\mathcal{V}_k$ and $\mathcal{V}_{k,k+1}$ are the categories used in CR–FKS.

The induction (derived) functor $\mathrm{Ind}_k^{k+1,k}$ is the derived tensor functor associated with the DG bimodule $(\hat{\Omega}_{k+1,k}, d_n)$ :

$$\mathrm{Ind}_k^{k+1,k} = (\hat{\Omega}_{k+1,k}, d_n) \otimes_k^L (-) : \mathcal{D}^c(\hat{\Omega}_k, d_n) \to \mathcal{D}^c(\hat{\Omega}_{k+1,k}, d_n)$$

and the restriction functor $\mathrm{Res}_k^{k+1,k}$ coincides with the (derived) functor

$$\mathrm{Res}_k^{k+1,k} = \mathbf{R}\mathrm{Hom}_{(\Omega_k, d_n)}((\hat{\Omega}_{k+1,k}, d_n), -) : \mathcal{D}^c(\hat{\Omega}_{k+1,k}, d_n) \to \mathcal{D}^c(\hat{\Omega}_k, d_n).$$

Analogously, we define

$$\mathrm{Ind}_{k+1}^{k+1,k} : \mathcal{D}^c(\hat{\Omega}_{k+1}, d_n) \to \mathcal{D}^c(\hat{\Omega}_{k+1,k}, d_n),$$
$$\mathrm{Res}_{k+1}^{k+1,k} : \mathcal{D}^c(\hat{\Omega}_{k+1,k}, d_n) \to \mathcal{D}^c(\hat{\Omega}_{k+1}, d_n).$$

For each $k \geq 0$ define the functors

$$F_k(-) = \mathrm{Res}_{k+1}^{k,k+1} \circ ((\hat{\Omega}_{k+1,k}, d_n) \otimes_k^{\mathbf{L}} (-)),$$

and

$$E_k(-) = \mathrm{Res}_k^{k,k+1} \circ ((\hat{\Omega}_{k,k+1}, d_n) \otimes_{k+1}^{\mathbf{L}} (-)),$$

where $(\hat{\Omega}_{k+1,k}, d_n)$ is seen as a DG $((\Omega_{k+1,k}, d_n), (\Omega_k, d_n))$-bimodule and $(\hat{\Omega}_{k,k+1}, d_n)$ as a DG $((\Omega_k, d_n), (\Omega_{k,k+1}, d_n))$-bimodule.

**Corollary 2.11 :** *The functors* $\mathrm{F}_k$ *and* $\mathrm{E}_k$ *are biadjoint up to a shift. Moreover we have natural isomorphisms*

$$\mathrm{E}_k \circ \mathrm{F}_k \cong \mathrm{F}_{k-1} \circ \mathrm{E}_{k-1} \oplus_{[n-2k]} \mathrm{Id}_k, \qquad \text{if } n - 2k \geq 0,$$

and

$$\mathrm{F}_{k-1} \circ \mathrm{E}_{k-1} \cong \mathrm{E}_k \circ \mathrm{F}_k \oplus_{[2k-n]} \mathrm{Id}_k, \qquad \text{if } n - 2k \leq 0.$$

**Corollary 2.12 :** *Define the category* $\mathcal{W}(n) = \bigoplus_{k\geq 0} \mathcal{D}^c(\hat{\Omega}_k, d_n)$. *We have a* $\mathbb{Z}[q, q^{-1}]$*-linear isomorphism of* $U_q(\mathfrak{sl}_2)$*-modules,* $K_0(\mathcal{W}(n)) \cong V(n)$, *for all* $n \geq 0$.

2.5.4 nilHecke action

By taking tensor products we can form the DG $((\Omega_k, d_n), (\Omega_{k+m}, d_n)$-bimodule $(\hat{\Omega}_{k,\ldots,k+m}, d_n)$ and the DG $(\Omega_{k+m}, d_n), (\Omega_k, d_n))_n$-bimodule $(\hat{\Omega}_{k+m,\ldots,k}, d_n)$.

**Proposition 2.13 :** *The nilHecke algebra* $\mathrm{NH}_m$ *acts as endomorphisms of the DG bimodules*

$$(\hat{\Omega}_{k,\ldots,k+m}, d_n) \quad \text{and} \quad (\hat{\Omega}_{k+m,\ldots,k}, d_n).$$

**Corollary 2.14 :** *The nilHecke algebra* $\mathrm{NH}_s$ *acts as endomorphisms of* $\mathrm{E}^s$ *and of* $\mathrm{F}^s$.

This action coincides with the one from Lauda and Chuang-Rouquier.

## 3. Algebraic categorification of Vermas for symmetrizable $\mathfrak{g}$

### 3.1 *Towards 2-Verma modules for* $\mathfrak{sl}_2$

Recall that the nilHecke algebra $\mathrm{NH}_m$ was obtained by studying natural transformations of the CR–FKS functors $\mathrm{F}^m$ (and $\mathrm{E}^m$). We have also seen that the categorification of $V(n)$ obtained from $\mathcal{D}(\mathcal{M}(\lambda q^{-1}), d_n)$ is canonically isomorphic to the one using $\mathcal{D}^b(\mathcal{V}(n), 0)$ (recall $\mathcal{V}(n)$ is CR–FKS's). The action of the nilHecke algebra on $\mathcal{M}(\lambda q^{-1})$ descends to an action on $\mathcal{D}(\mathcal{M}(\lambda q^{-1}), d_n)$ that coincides with the action on $\mathcal{V}(n)$. But we can say a bit more:

- *There is a (bigraded, super-) algebra* $A_m$, *that can be seen as an extension of the nilHecke algebra* $\mathrm{NH}_m$ *and acts on* $\mathrm{F}^m$ *by natural transformations (and therefore on* $\mathrm{E}^m$*).*

Here is the main idea: the composite functor $\mathcal{F}^n$ acting on $\mathcal{M}(\lambda q^{-1})$ decomposes into functors associated with bimodules of the form

$$\Omega_{k,\ldots,k+n} := \Omega_{k,k+1} \otimes_{k+1} \Omega_{k+1,k+2} \otimes_{k+2} \cdots \otimes_{k+n-1} \Omega_{k+n-1,k+n}.$$

One can compute that

$$\Omega_{k,\ldots,k+n} \cong \mathbb{Q}[x_1,\ldots,x_k,\xi_1,\ldots,\xi_m] \otimes \wedge^\bullet(\sigma_1,\ldots,\sigma_k,\omega_1,\ldots,\omega_n),$$

where $\deg(x_i) = (2i, 0)$, $\deg(\xi_i) = (2, 0)$, $\deg(\sigma_i) = (-2i, 2)$ and $\deg(\omega_i) = (-2(k+i), 2)$.

One can verify that action of $NH_n$ on $\mathbb{Q}[\xi_1,\ldots,\xi_n]$ extend to maps of $(\Omega_k, \Omega_{k+n})$ bimodules iff $\partial_i(\omega_j) = -\delta_{ij}\omega_{i+1}$ (the sign is just a convention). Note that $\partial_i$ is an even operator of degree $\deg(\partial_i) = (-2, 0)$.

As we did above we can define a bigraded, (super)algebra $A_{k,n}$ as the algebra of operators on $\mathbb{Q}[\xi_1,\ldots,\xi_m]\otimes\wedge^{\bullet}(\omega_1,\ldots,\omega_n)$ generated by $\partial_i$ ($i = 1, \ldots, n-1$) and multiplication by $\xi_j$ and by $\omega_i$ ($j = 1, \ldots, n$).

In the sequel we will consider the case $k = 0$ and the superalgebra $A_n = A_{0,n}$.

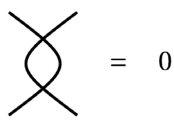
### 3.1.1 Cyclotomic nilHecke algebra: categorification of $V(n)$ using NH

The nilHecke algebra can be given a diagrammatic presentation as follows.

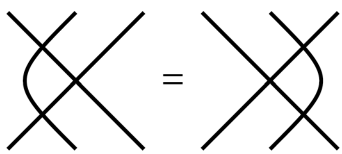
*Generators*: The following $n$-strand diagrams (the $q$-degree is indicated under the diagram).


Let $\Bbbk$ be a commutative unital ring (we can take $\Bbbk = \mathbb{Z}$).

**Definition 3.1 :** Let $NH_n$ be the $\Bbbk$-algebra generated by isotopy classes of the diagrams described above with multiplication given by gluing diagrams on top of each other. We read diagrams from bottom to top by convention and so $ab$ means we stack $a$ atop of $b$. The diagrams are subjected to the local relations below.



(4)

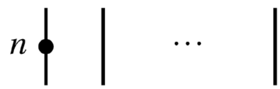


(5)

(6)

**Definition 3.2 :** $\mathrm{NH} = \bigoplus_{m\geq 0} \mathrm{NH}_m$.

Fix $n \in \mathbb{N}$. Define the *cyclotomic ideal* $I^n \subset \mathrm{NH}$ as the 2-sided ideal generated by all diagrams having $n$ dots on the leftmost strand:




**Definition 3.3 :** The *cyclotomic nilHecke algebra* $\mathrm{NH}^n$ is the quotient $\mathrm{NH}^n = \mathrm{NH}^n / I^n$. We have

$$\mathrm{NH}^n = \bigoplus_{k\geq 0} \mathrm{NH}^n_k.$$

**Proposition 3.4 :** *There is an isomorphism* $\mathrm{NH}^n_k \cong \mathrm{Mat}(k!, H_k)$.

This implies immediately that the Grothendieck group of $\mathrm{NH}_k$ is 1-dimensional, since Morita equivalent rings have the same Grothendieck groups. This also implies the following.

**Corollary 3.5 :** *There is an isomorphism of* $\mathbb{Q}$*-vector spaces*

$$\bigoplus_{k\geq 0} K_0(\mathrm{NH}^n_k - \mathrm{gmod}) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{Q}(q) \cong V(n).$$

It is natural to take $\mathrm{NH}^n_k$ as a categorification of the weight space $V_{n-2k}$. The $\mathfrak{sl}_2$-action on $\mathrm{NH}^n$ – *gmod* follows a familiar scheme using induction and restriction functors for the inclusion $\mathrm{NH}^n_k \to \mathrm{NH}^n_{k+1}$ that add a vertical strand at the right of a diagram from $\mathrm{NH}^n_k$. We will see this in detail in §3.1.2 and §3.1.7.

3.1.2. The superalgebras $A_n$

- *Generators*: The following $n$-strand diagrams (a triple $(q, \lambda, \pi)$ below each diagram indicates its $q$-degree, its $\lambda$-degree and its parity). The nilHecke generators,


$(2,0,0)$ $\qquad\qquad$ $(-2,0,0)$

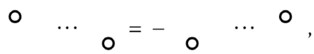
and the floatig dots,

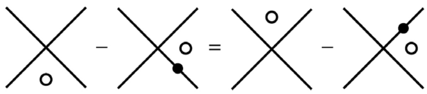
$(-2\ell, 2, 1)$

Here, there are $\ell \geq 1$ strands to the left of the floating dot. Note the degree of a floating dot is not defined locally.

We say an isotopy is *admissible* if it doesn't change the relative height of floating dots (we are assuming that diagrams are equipped with a height function).

**Definition 3.6 :** Let $A_n$ be the $\Bbbk$-(super)algebra generated by admissible isotopy classes of the diagrams described above with multiplication defined as in $\mathrm{NH}_m$. The diagrams are subjected to the nilHecke relations of Definition 3.1, together with the local relations (7) and (8) below.

 (7)

 (8)

**Definition 3.7 :** $A = \bigoplus_{n \geq 0} A_n$.

3.1.3. Bases for $A_n$ (optional)

From the defining relations one can see immediately that one can write a diagram of $A_n$ as a $\Bbbk$-linear combination of diagrams containing three regions:

(1) A region consisting of $n$ vertical strands and only floating dots,

(2) A region consisting of $n$ vertical strands and only (nilHecke) dots,

(3) A region consisting only of crossings.

The six ways of placing these regions give basis of $A_n$. For example,

**Proposition 3.8 :** *The superalgebra $A_n$ is a free $\Bbbk$-module. The sets*

$$\{x_1^{k_1} \cdots x_n^{k_n} T_\vartheta \omega_1^{\ell_1} \cdots \omega_n^{\ell_n} : k_i \in \mathbb{N}_0, \ell_i \in \{0,1\}, \vartheta \in S_n\},$$

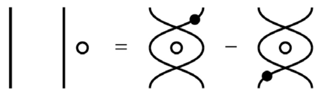
*and*

$$\{T_\vartheta x_1^{k_1} \cdots x_n^{k_n} \omega_1^{\ell_1} \cdots \omega_n^{\ell_n} : k_i \in \mathbb{N}_0, \ell_i \in \{0,1\}, \vartheta \in S_n\},$$

*being basis.*

There is another basis which turns out to be useful, defined in terms of a special type of floating dot. The following is now an easy consequence of the defining relations of $A_n$.

**Lemma 3.9 :** *We have the following relation in $A_n$ for any pair of consecutive strands:*

 (9)

**Definition 3.10 :** We say that a floating dot is *tight* if it is placed directly to the right of the leftmost strand.

For example, the floating dot in the diagram below is tight:

Using Lemma 3.9 recursively we can write any diagram of $A_n$ as a linear combination of diagrams involving only nilHecke generators and tight floating dots. Moreover we are able to give a basis for $A_n$ in terms of tight floating dots[7].

**Proposition 3.11 :** *There is a basis of $A_n$ defined in terms of generators of the nilHecke algebra and tight floating dots.*

3.1.4. The algebra $A_n$ is isomorphic to a matrix algebra

From the action of $\partial_i$ on the supercommutative ring $R = \Bbbk[\xi_1, \dots, \xi_n] \otimes \bigwedge^\bullet(\omega_1, \dots, \omega_n)$ explained above on can see that the symmetric group $S_n$ acts (from the left) on $R$: it acts via the permutation action on $\Bbbk[\xi_1, \dots, \xi_n]$ while the simple transposition $s_i = (i\ i + 1)$ acts on the $\omega_j$'s as

$$s_i(\omega_j) = \omega_j + \delta_{i,j}(\xi_i - \xi_{i+1})\omega_{i+1},$$

together with $s_i(fg) = s_i(f)s_i(g)$.

This action respects the bigrading as well as the parity, as one easily checks. On can easily check as well that the action of $A_n$ on $R$ corresponds with the diagrammatic presentation given above.

Denote $R^{S_n} \subset R$ be the subring of $S_n$-invariants. We have the following.

---

7 This basis is defined combinatorially (see 16, §2.2, §2.7 ) and its particular form is not important for this lectures.

**Proposition 3.12 :**

(1) *The supercenter of $A_n$ is isomorphic to* $R^{S_n}$.

(2) *There are isomorphisms*

$$A_n \cong \mathrm{End}_{R^{S_n}}(R) \cong \mathrm{Mat}(n!, R^{S_n})$$

*of bigraded superalgebras.*

As with $\mathrm{NH}_n$ this allows computing Grothendieck groups of $A_n$ very easily.

3.1.5 Categorical $\mathfrak{sl}_2$-action and a new categorification of $M(\lambda q^{-1})$

The inclusion of algebras $A_n \rightarrowtail A_{n+1}$ that adds a vertical strand to the right of a diagram gives rise to an induction functor[8]

$$\mathrm{Ind}_n^{n+1} : A_n - \mathrm{smod} \to A_{n+1} - \mathrm{smod}.$$

In terms of bimodules, it can be viewed as tensoring on the left with the ($A_{n+1}$, $A_k$)-bimodule $A_{n+1} \otimes_{A_n} -$. Taking its right adjoint gives a restriction functor

$$\mathrm{Res}_n^{n+1} : A_{n+1} - \mathrm{smod} \to A_n - \mathrm{smod},$$

which is given by tensoring with the ($A_n$, $A_{n+1}$)-bimodule $A_n \otimes_{A_{n+1}} -$.

**Proposition 3.13 :** *We have*

$$\mathrm{gdim}^s A_n = (F^n v_{\lambda q^{-1}}, F^n v_{\lambda q^{-1}})_{\lambda q^{-1}},$$

where $(-,-)_{\lambda q^m}$ is the universal Shapovalov form.

We shift these functors by the right amount to get an $\mathfrak{sl}_2$ commutator relation: we define the functors

$$\mathrm{F}_n := \mathrm{Ind}_n^{n+1}, \quad \mathrm{E}_n := \mathrm{Res}_n^{n+1}\langle 2n, -1\rangle, \quad Q_n := \Pi(-) \otimes \Bbbk[\xi]\langle 1, 0\rangle.$$

**Theorem 3.14 :** *There is a short exact sequence of functors*

$$0 \to F_{n-1}E_{n-1} \to E_n F_n \to Q_{n+1}\langle m-2n, 1\rangle \oplus \Pi Q_{n+1}\langle 2n-m, -1\rangle \to 0.$$

**Proposition 3.15 :** *Functors* $\mathrm{F}_n$ *and* $\mathrm{E}_n$ *are exact and send projectives to projectives.*

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8 As with $A_n$ we drop the prefix "super" from our terminology.

3.1.6 The categorification theorem

We now restrict to the case where $\Bbbk$ is a field of characteristic zero. In the following,

- $\mathbb{Z}_\pi$ is $\mathbb{Z}[\pi]/(\pi^2-1)$,
- $\mathbb{Z}_\pi((q,\lambda))$ is the ring of formal Laurent series in the variables $q$ and $\lambda$, given by the order $0 \prec q \prec \lambda$,
- $A_n$ – smod$_{lf}$ is the category of $A_n$-supermodules which have cone bounded (see §2.3), locally finite dimension over $\Bbbk$, with morphisms preserving the degrees.

The superalgebra $A_n$ has the following properties:

(1) It admits a unique indecomposable projective module $P_{(n)}$, up to isomorphism and grading shifts. This projective module is of locally finite dimension contained in a cone compatible with $\prec$.
(2) Its (topological) Grothendieck group is a $\mathbb{Z}_\pi((q,\lambda))$-module freely generated by the unique simple module.
(3) This simple module admits a projective cover given by $P_{(n)}$.
(4) Taking a (an infinite) projective resolution of $S$, it is not hard to see the Grothendieck group is also generated by the unique indecomposable projective module.

**Theorem 3.16 :** *The functors* $\mathrm{F} = \oplus_{n\geq 0}\mathrm{F}_n$ *and* $\mathrm{E} = \oplus_{n\geq 0}\mathrm{E}_n$ *induce an action of quantum* $\mathfrak{sl}_2$ *on the Grothendieck group* $\mathbf{G}_0(A-smod_{lf})$*. With this action there is an isomorphism*

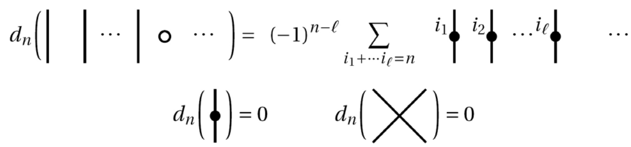
$$\mathbf{G}_0(A-\mathrm{smod}_{lf})\otimes_{\mathbb{Z}_\pi}\mathbb{Q}_\pi/(\pi+1)\cong M(\lambda q^m)$$

*of* $U_q(\mathfrak{sl}_2)$*-modules. This isomorphism sends classes of projective indecomposables to the canonical basis elements and classes of simples to dual canonical elements.*

3.1.7 Derived equivalences

It is also not hard to see that $d_n$ from §2.5 induces a differential on $A_m$ and turns it into a (bigraded) DGA.

In terms of the usual generators it is given by



**Proposition 3.17 :** $(A\ \ d_n)$ *is quasi-isomorphic to* $(\mathrm{NH}^n, 0)$.

The story with the snake lemma repeats again here, almost vertim ...

3.2. (*Quantum, symmetrizable*) *Kac–Moody algebras and Verma modules*

3.2.1 Quantum Kac--Moody algebras

Let $(I, \cdot)$ be a *Cartan datum*:

- $I$ is a finite set equipped with a symmetric bilinear form
$$- \cdot - : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z},$$
such that
(1) $i \cdot i \in \{2, 4, \ldots\}$, and
(2) $i \cdot j \in \{0, -1, -2, \ldots\}$ for all $i, j \in I$ with $i \neq j$.

Elements of $I$ are called *simple roots*. To such a Cartan datum we assign a graph $\Gamma$ with vertices given by $I$ and we put an edge between $i$ and $j$ whenever $i \cdot j \neq 0$.

A *root datum* of type $(I, \cdot)$ is given by two freely generated abelian groups $X$, $Y$, both containing $I$, and a perfect pairing $\langle -,- \rangle : Y \times X \to \mathbb{Z}$ such that $\langle i, j_X \rangle = 2\frac{i \cdot j}{i \cdot i}$ for all $i \in I \subset Y$ and . We call $Y$ the *weight lattice* and $X$ the *dual weight lattice*.

The *quantum Kac-Moody algebra* $\mathfrak{g}$ associated to the root datum $(I, \cdot)$ is the unital associative $\mathbb{Q}(q)$-algebra generated by $E_i$, $F_i$ and $K_\gamma$ for $i \in I$ and $\gamma \in Y$, with relations for all $\gamma, \gamma' \in Y$ and $i \in I$:
$$K_0 = 1, \quad K_\gamma K_{\gamma'} = K_{\gamma+\gamma'},$$
$$K_\gamma E_i = q^{\langle \gamma, i_X \rangle} E_i K_\gamma, \quad K_\gamma F_i = q^{-\langle \gamma, i_X \rangle} F_i K_\gamma,$$

and with the $\mathfrak{sl}_2$-commutator relation for all $i, j \in I$:
$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_{(i\cdot i/2)i} - K_{-(i\cdot i/2)i}}{q_i - q_i^{-1}},$$

where $q_i = q^{(i\cdot i)/2}$, and the quantum Serre relations for $i \neq j \in I$:
$$\sum_{a+b=d_{ij}+1} (-1)^a \begin{bmatrix} d_{ij}+1 \\ a \end{bmatrix}_{q_i} E_i^a E_j E_i^b = 0, \quad \sum_{a+b=d_{ij}+1} (-1)^a \begin{bmatrix} d_{ij}+1 \\ a \end{bmatrix}_{q_i} F_i^a F_j F_i^b = 0,$$

where $d_{ij} = -\langle i, j_X \rangle$ and $\begin{bmatrix} a \\ b \end{bmatrix}_{q_i}$ is the quantum binomial in the variable $q_i$.

Given a sequence $\mathbf{i} = i_1 \ldots i_m$ we write $F_{\mathbf{i}} = F_{i_1} \ldots F_{i_m}$ and the same for $E_{\mathbf{i}} = E_{i_1} \ldots E_{i_m}$.

3.2.2 Universal Verma modules

The (standard) *Borel subalgebra* $\mathfrak{b}$ of $\mathfrak{b}$ is the subalgebra generated by $K_\gamma$ and $E_i$ for all $i \in I$ and $\gamma \in Y$.

Let $\beta = \{\beta_i\}_{i\in I} \in \mathbb{C}^{|I|}$ and $\mathbb{C}_\beta = \mathbb{Q}((q,\beta))v_\beta$ be the $U_q(\mathfrak{b})$-module defined by

$$E_i v_\beta = 0, \qquad K_\gamma v_\beta = q^{\langle \beta,\gamma\rangle} v_\beta,$$

for all $i \in I$ and $\gamma \in Y$. The *universal Verma module* is the induced representation

$$M(\beta) \;=\; \mathfrak{g} \otimes_{\mathfrak{b}} \mathbb{C}_\beta.$$

It is an infinite dimensional $U_q(\mathfrak{g})$-weight module with highest weight $\beta$.

**Remark 3.18 :** Whenever $\beta_i \notin \mathbb{Z}$ we denote $\lambda_i = q^{\beta_i}$ and treat it as a formal parameter.

**Remark 3.19 :** The notation $\mathbb{Q}((q, \lambda))$ means the field of formal Laurent series in the variables $q$ and $\lambda_i$'s (if any). It is given by formal series with degrees contained in cones compatible with some fixed additive order $\prec$ on $\mathbb{Z}^{1+|I|}$. For the means of categorification, and to agree with common conventions, we will require that this order is given by $0 \prec q$ and $0 \prec \lambda_i$ for all $i \in I$, so that

$$\frac{1}{q-q^{-1}} \;=\; -q\frac{1}{1-q^2} = -q(1+q^2+q^4+\ldots).$$

We will also demand $0 \prec \lambda_i$ so that

$$\frac{1}{1-q_i^{-2}\lambda_i^2} \;=\; (1+q_i^{-2}\lambda_i^2+q_i^{-4}\lambda_i^4+\ldots).$$

Other choices could be possible and everything should work the same way with minor modifications.

3.2.3 (Parabolic) Verma modules and finite-dimensional irreducibles

- *A* (*standard*) parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is a subalgebra that contains $\mathfrak{b}$.

It is generated by $K_\gamma$, $E_i$ and $F_j$ for all $\gamma \in Y$, $i \in I$ and $j \in I_f$ for some fixed subset $I_f \subset I$.

(1) The part given by $K_\gamma$ and $E_j$, $F_j$ for $\gamma \in Y$ and $j \in I_f$ is called the *Levi factor* and written $\mathfrak{l}$.

(2) The part generated by $E_i$ for $i \in I_r = I \setminus I_f$ is called the *nilpotent radical* and denoted $\mathfrak{u}$.

(3) There is a decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$.

Fix a parabolic subalgebra $\mathfrak{p}$ and choose $\beta = \{\beta_i\}_{i\in I} \in \mathbb{C}^{|I|}$ such that $\beta_j = n_j \in \mathbb{N}$ for all $j \in I_f$. We write $N = \{n_j\}_{j\in I_f}$.

Let $V(\beta, N)$ be the unique, finite dimensional, irreducible representation of $\mathfrak{l}$ with highest weight $\beta$, over $\mathbb{Q}((q, \beta))$. We extend it to a representation of $\mathfrak{p}$ by setting $\mathfrak{u}.V(\beta, N) = 0$.

**Definition 3.20 :** The *parabolic Verma module* of highest weight $\beta$ associated to $\mathfrak{p} \subseteq \mathfrak{g}$ is the induced module

$$M^{\mathfrak{p}}(\beta, N) = \mathfrak{g} \otimes_{\mathfrak{p}} V(\beta, N).$$

The parabolic Verma module $M^{\mathfrak{p}}(\beta, N)$ is a weight module with highest weight $\beta$, infinite dimensional whenever $\mathfrak{p} \neq \mathfrak{g}$. We denote the highest weight vector $v_\beta$ by abuse of notation.

- *We think of a parabolic Verma module intuitively as a "mixture of a finite-dimensional representation and a Verma module", in the sense that there are simple roots for which we have a Verma module (the subset $I_r \subseteq I$), and others for which we have a finite-dimensional representation (the subset $I_f \subseteq I$).*

**Theorem 3.21 :**

(1) *If $\beta_i \notin \mathbb{N}$ for all $i \in I_r$, then $M^{\mathfrak{p}}(\beta, N)$ is irreducible.*

(2) *If $\beta_i = n_i \in \mathbb{N}_0$ for some $i \in I_r$, then there is a short exact sequence of $\mathfrak{g}$-modules*

$$0 \to M^{\mathfrak{p}}(\beta \setminus \{q_i^{n_i}\} \cup \{q_i^{-n_i-2}\}, N) \to M^{\mathfrak{p}}(\beta, N) \to M^{\mathfrak{p}\oplus F_i}(\beta \setminus \{q_i^{n_i}\}, N \cup \{n_i\}) \to 0,$$

*where $\mathfrak{p} \oplus F_i$ is the parabolic subalgebra given by $I_f \cup \{i\}$.*

(3) *Given a parabolic Verma module $M^{\mathfrak{p}}(\beta, N)$ with $\beta_i \notin \mathbb{N}_0$ for some $i \in I_r$, then for any $n_i \in \mathbb{Z}$ there is a surjective map*

$$\mathrm{ev}_{n_i} : M^{\mathfrak{p}}(\beta, N) \twoheadrightarrow M^{\mathfrak{p}}(\beta \setminus \{\lambda_i\} \cup \{q_i^{n_i}\}, N),$$

*given by evaluating $\lambda_i = q_i^{n_i}$.*

We say that there is an arrow from irreducible $M^{\mathfrak{p}}(\beta,N)$ to $M^{\mathfrak{p}'}(\beta',N')$ if there is an evaluation map $\mathrm{ev}_{n_i}$ yielding an exact sequence

$$0 \to M^{\mathfrak{p}}(\beta \setminus \{\lambda_i\} \cup \{q_i^{-n_i-2}\}, N) \to ev_{n_i}\left(M^{\mathfrak{p}}(\beta,N)\right) \to M^{\mathfrak{p}'}(\beta',N') \to 0.$$

This allows us to define a partial order on the irreducible parabolic Verma modules, saying that $M^{\mathfrak{p}}(\beta, N)$ is greater than $M^{\mathfrak{p}'}(\beta', N')$ if there is a sequence of arrows from $M^{\mathfrak{p}}(\beta, N)$ to $M^{\mathfrak{p}'}(\beta', N')$. There are maximal elements given by the universal Verma module $M(\beta)$ and its shifts $M(\beta q^{\mu})$, and a collection of minimal elements given by all the finite dimensional irreducible modules $V(N)$.

3.2.4 The Shapovalov form

We consider some (anti-)automorphisms of $\mathfrak{g}$. First let $\overline{\phantom{x}}$ be the $\mathbb{C}(q)$-linear involution mapping $q$ to $q^{-1}$. Then let $\rho : \mathfrak{g} \to \mathfrak{g}^{op}$ be the $\mathbb{C}(q)$-linear algebra anti-involution defined by

$$\rho(E_i) = q_i^{-1} K_{-(i\cdot i/2)i} F_i, \quad \rho(F_i) = q_i^{-1} K_{(i\cdot i/2)i} E_i, \quad \rho(K_\gamma) = K_\gamma,$$

for all $i \in I$ and $\gamma \in Y$. Let also $\tau : \mathfrak{g} \to \mathfrak{g}^{op}$ be the $\mathbb{Q}(q)$-linear anti-automorphism given by

$$\tau(E_i) = q_i^{-1} K_{-(i\cdot i/2)i} F_i, \quad \tau(F_i) = q_i^{-1} K_{(i\cdot i/2)i} E_i, \quad \tau(K_\gamma) = K_{-\gamma},$$

for all $i \in I$ and $\gamma \in Y$, and

$$\tau(pW) = \overline{p}\tau(W), \ \tau(WW') = \tau(W')\tau(W),$$

for $W, W' \in U_q(\mathfrak{g})$ and $p \in \mathbb{C}(q)$.

**Definition 3.22 :** The *universal Shapovalov form*

$$(-,-) : M^{\mathfrak{p}}(\beta,N) \times M^{\mathfrak{p}}(\beta,N) \to \mathbb{Q}((q,\beta))$$

is the bilinear form defined by

- $(v_\beta, v_\beta) = 1$,
- $(uv, v') = (v, \rho(u)v')$, with $u \in \mathfrak{g}$ and $v, v' \in M^{\mathfrak{p}}(\beta,N)$,
- $f(v,v') = (fv,v') = (v, fv')$, with $f \in \mathbb{Q}((q,\beta))$.

The $\mathbb{C}(q)$-linear involution $\overline{\phantom{x}}$ extends to $\mathbb{Q}(q, \beta)$ (but not to $\mathbb{Q}((q, \beta))$ !) by sending $\lambda_i$ to $\lambda_i^{-1}$ for all $i \in I^r$.

3.2.5 Bases for $M^{\mathfrak{p}}(\beta, N)$

Any parabolic Verma module admits at least one natural basis (the $F$'s basis) $\{m_\mu\}_{\mu\in\mathrm{supp}(M^{\mathfrak{p}}(\beta,N))}$ generated by the action of the Chevalley generators $\{F_i\}_{i\in I}$ on the highest weight vector. Namely each element can be written as a $\mathbb{Q}((q,\beta))$-linear combination of the various $F_{i_r}^{b_r}\dots F_{i_1}^{b_1}v_\beta$ for some $i_1,\dots,i_r\in I$ and $b_1,\dots,b_r\in\mathbb{N}$.

Of course we do not have all possible combinations of words in the $F_i$'s because of the Serre relations and the fact that for some $i\in I_f$ the $F_i$'s act nilpotently.

Replacing each $F_i^b$ by the divided power $F_i^{(b)}=F_i^b/([b]_{q_i}!)$ gives an another useful basis denoted $\{m'_\mu\}_{\mu\in\mathrm{supp}(M^{\mathfrak{p}}(\beta,N))}$ that we refer to as *divided power basis*. For each such basis there is a dual basis $\{m^\mu\}_{\mu\in\mathrm{supp}(M^{\mathfrak{p}}(\beta,N))}$ and $\{m'^\mu\}_{\mu\in\mathrm{supp}(M^{\mathfrak{p}}(\beta,N))}$ defined respectively by the relations $(m_\mu,m^\nu)=\delta_{\mu,\nu}$ and $(m'_\mu,m'^\nu)=\delta_{\mu,\nu}$.

3.3 $\mathfrak{p}$-*KLR algebras*

Let $\Bbbk$ be a commutative unital ring (later we will need it to be a field of characteristic 0).

Fix a Cartan datum $(I,\cdot)$, a root datum and a parabolic subalgebra $\mathfrak{p}$ given by $I_f\subset I$. Using the notations from Khovanov and Lauda, we write for $\nu\in\mathbb{N}[I]$:

$$\nu = \sum_{i\in I}\nu_i\cdot i,\quad \nu_i\in\mathbb{N},$$

with $|\nu|=\sum_i\nu_i$ and $\mathrm{Supp}(\nu)=\{i\mid\nu_i\neq0\}$. We put $d_{ij}=-2\frac{i\cdot j}{i\cdot i}=-\langle i,j_X\rangle\in\mathbb{N}$.

We also fix a choice of scalars $\mathcal{Q}$ as introduced by Rouquier. Following the conventions in Cautis-Lauda, the set $\mathcal{Q}$ consists of:

- $t_{ij}\in\Bbbk^\times$ for all $i,j\in I$,
- $s_{ij}^{tv}\in\Bbbk$ for $i\neq j$, $0\le t<d_{ij}$ and $0\le v<d_{ji}$,
- $r_i\in\Bbbk^\times$ for all $i\in I$,

respecting

- $t_{ii}=1$,
- $t_{ij}=t_{ji}$ whenever $d_{ij}=0$,
- $s_{ij}^{tv}=s_{ji}^{vt}$.

For $t(i\cdot i)+v(j\cdot j)\neq-2(i\cdot j)$, or $t<0$, or $v<0$, we put $s_{ij}^{tv}=0$. Thus we have $s_{ij}^{pq}=0$ for $p>d_{ij}$ or $q>d_{ji}$. We will also write $s_{ij}^{d_{ij}0}=t_{ij}$ and $s_{ij}^{0d_{ji}}=t_{ji}$. Hence if $i\cdot j=0$ we get $s_{ij}^{00}=s_{ji}^{00}=t_{ij}=t_{ji}$.

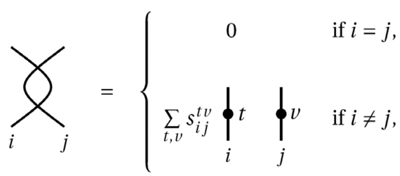
**Remark 3.23 :** A usual choice is given by $r_i = \pm 1$, $t_{ij} = 1$ and $s_{ij}^{tv} = 0$ for $t \neq d_{ij}$ or $v \neq d_{ji}$.

3.3.1 KLR algebras and their cyclotomic quotients

Consider the collection of braid-like diagrams on the plane connecting $|\nu|$ points on the horizontal axis $\mathbb{R} \times \{0\}$ to $|\nu|$ points on the horizontal line $\mathbb{R} \times \{1\}$, admitting no critical point when projected onto the $y$-axis, so that a strand can never turn around.

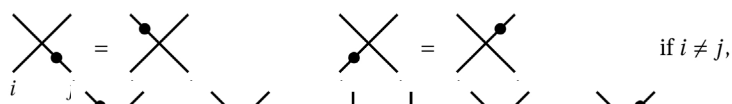
- We allow strands to intersect each other without triple intersection points.
- Each strand is labeled by a simple root, with $\nu_i$ strands labeled $i$, and they can carry dots.
- A non-negative integer $k \in \mathbb{N}$ next to a dot means there are $k$ consecutive dots on the strand.
- These diagrams are taken up to regular isotopy which does not create critical points.

**Definition 3.24 :** Let $R(\nu)$ be the $\Bbbk$-algebra generated by the diagrams described above with multiplication given by gluing diagrams on top of each other whenever the labels of the strands agree, and zero otherwise. We read diagrams from bottom to top by convention and so $ab$ means we stack $a$ atop of $b$. The diagrams are subjected to the local relations (10) to (13) below.




(10)

where the sum is restricted to the finite number of pairs $t, v \in \mathbb{N}$ such that $t(i \cdot i) + v(j \cdot j) = -2(i \cdot j)$,

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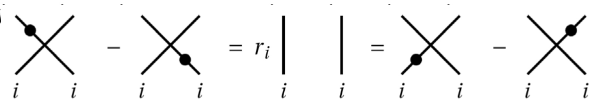


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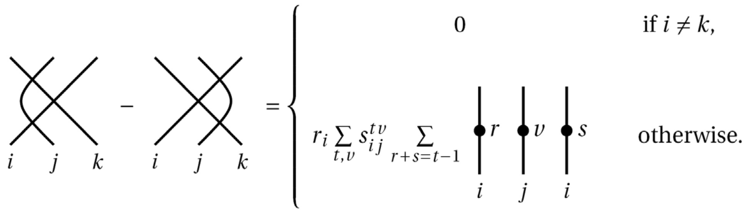
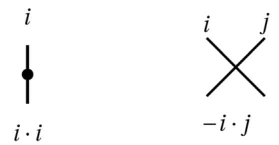



(13)

**Remark 3.25 :** We also remark that, whenever $i \cdot j = 0$ we have $t = v = 0$ and the sums in in (10) give $t_{ij}$. Also in this case, the sums over $t, v$ in (13) vanish since we must have $v = 0$.

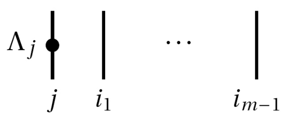
The algebra $R(v)$ is graded and generated by (we indicate the grading below the diagrams)




**Definition 3.26 :** $R = \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu)$.

3.3.2 Cyclotomic KLR algebras

Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be an integral dominant weight. Define the *cyclotomic ideal* $I^\Lambda(\nu) \subset R(\nu)$ as the 2-sided ideal generated by the diagrams




with $ji_1 \ldots i_{m-1} \in Seq(\nu)$ and $j \in I$.

**Definition 3.27 :** The *cyclotomic KLR algebra* $R^\Lambda(\nu)$ is the quotient

$$R^\Lambda(\nu) \;=\; R(\nu) / I^\Lambda(\nu).$$

We put

$$R^\Lambda \;=\; \bigoplus_{\nu \in \mathbb{N}[I]} R^\Lambda(\nu).$$

- The cyclotomic KLR algebra $R^\Lambda$ categorifies the irreducible $V(\Lambda)$ in the sense that there is a $\mathfrak{g}$-action on categories of modules of $R^\Lambda$ that descend to the Grothendieck groups, yielding isomorphisms of $\mathfrak{g}$-modules

$$G_0(R^\Lambda - \text{gmod}) \cong V(\Lambda), \quad K_0(R^\Lambda - \text{pmod}) \cong V(\Lambda).$$

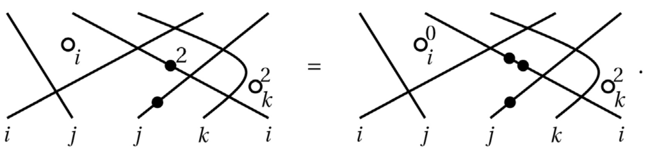
### 3.3.3 Extended KLR superalgebras: $\mathfrak{p}$-KLR superalgebras

We introduce below KLR-like super algebras associated to a pair $(\mathfrak{p}, \mathfrak{g})$, where $\mathfrak{g}$ is a quantum Kac-Moody algebra and $\mathfrak{p} \subseteq \mathfrak{g}$ a parabolic subalgebra. These algebras can be thought as a mix between the KLR algebras $R(v)$ the algebras $A_n$.

Consider the collection of KLR diagrams where regions can be decorated with *floating dots*, drawn as hollow dots ○.

- Floating dots are labeled by simple roots in $I_r$ as a subscript, together with a non-negative integer as a superscript. By convention, we do not write the superscript of a floating dot whenever it is 0.
- Two floating dots are not allowed to be at the same height in a diagram.
- These diagrams are taken up to the isotopies allowed for KLR diagrams that preserve the relative height of floating dots.
- We assign a *parity* to these diagrams by declaring that floating dots are odd while crossings and dots are even.

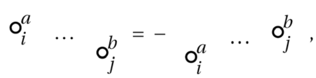
An example of such a diagram is given below, for $i, j, k \in I$ with $i, k \in I_r$,



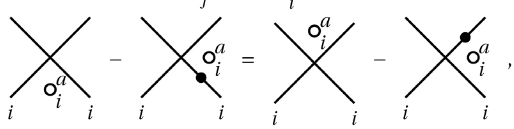

(14)

**Definition 3.28 :** Let $R_{\mathfrak{p}}(v)$ be the $\Bbbk$-super algebra generated by the KLR diagrams together with the floating dots, with multiplications as defined in the KLR algebra.

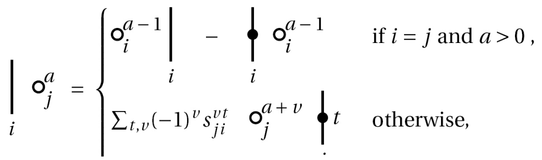
The diagrams are subjected to the *KLR relations* together with the local relations (15)-(19) involving floating dots, where we suppose all subscripts are in $I_r$:



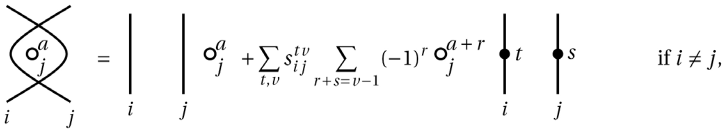
(15)




(16)




(17)




(18)

for all $a, b \in \mathbb{N}$ and $i, j, k \in I$. Moreover, we also demand a floating dot in the leftmost region to be zero:

(19)

We put

$$R_{\mathfrak{p}} = \bigoplus_{\nu \in \mathbb{N}[I]} R_{\mathfrak{p}}(\nu),$$

and call it the *extended KLR algebra associated to the pair* $(\mathfrak{p}, \mathfrak{q})$ ($\mathfrak{p}$-KLR algebra for short). Taking $\mathfrak{p} = \mathfrak{g}$ recovers the usual KLR algebra.

**Definition 3.29 :** $R_{\mathfrak{p}} = \bigoplus_{\nu \in \mathbb{N}[I]} R_{\mathfrak{p}}(\nu)$.

**Remark 3.30 :**

- Relation (15) means that, up to a sign, floating dots can move freely within regions.
- It also means that a diagram containing two floating dots with the same subscript and superscript in the same region is zero.
- Relation (19) implies the diagram in (14) is zero since the floating dot with subscript $i$ slides to the left over the strand with label $j$ by (17), and reaches the leftmost region.
- Whenever $i \cdot j = 0$ we have $t = v = 0$ and the sums in (17) give $t_{ij}$, so that the floating dot jumps over the strand at the cost of multiplying by an invertible scalar. Also in this case the sums over $t, v$ in (18) vanish, since we must have $v = 0$.

The algebra $R_{\mathfrak{p}}(v)$ is *multigraded* and the degree of a floating dot is not defined locally: it depends on the strands at its left. In order to still be able to write equations in a compact form, we introduce for each diagram a function that takes a region and spits a $|I|$-tuple $K = \sum k_i \cdot i$ where $k_i \in \mathbb{N}$ is the number of strands labeled $i$ at its left.

Concretely, when we write a local relation with a $K$ placed somewhere in a region, it means it is embedded in a diagram where there are $k_i$ strands labeled $i$ at the left of this region.

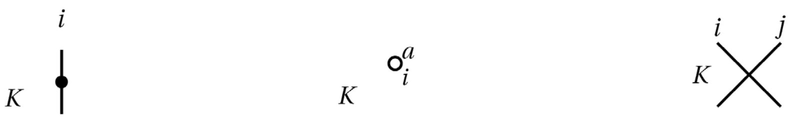
For a fixed $K$ it will be sometimes useful to also consider $K_X = \sum_{i\in I} k_i \cdot i_X \in X$. This allows compact notations such as $\langle i, K_X\rangle = -\sum_{j\in I} k_j d_{ij}$. In particular $|K|$ counts the total number of strands at the left of the considered region. We will also abuse notation and write $K - i$ instead of $K - 1 \cdot i$.

For example, we can now write relation (19) as a local relation

$$K \quad \circ^{a}_{i} \quad = \quad 0 \qquad\qquad \text{whenever } |K| = 0.$$

We introduce a multigrading on $R_{\mathfrak{p}}(v)$ consisting of a quantum grading $q$ and $|I_r|$ homological gradings $\lambda = \{\lambda_i\}_{i\in I_r}$. We write the degree of an element as a pair $(r, L)$ with $r \in \mathbb{Z}$ being the $q$-degree, and $L = \sum_{i\in I_r} \ell_i \cdot i \in \mathbb{Z}^{|I_r|}$ being the homological multigrading.

We fix the degree of the generators by




$(i \cdot i, 0)$ $\qquad$ $((1 + a - \langle i, K_X\rangle + k_i)\, i \cdot i, 2 \cdot i)$ $\qquad$ $(-i \cdot j, 0)$

Relations (10-19) above are clearly homogeneous for this multigrading, and $R_{\mathfrak{p}}(v)$ becomes a multigraded superalgebra (recall dots and crossings are even and floating dots are odd). To keep the notation simple we will write grading shifts by monomials in variables $q$ and $\lambda_i$s, and the parity shift by $\Pi$.

- $R_{\mathfrak{p}}(v)$ contains the KLR algebra $R(v)$ as a graded subalgebra if we extend its $q$-grading to a multigrading trivially.
- If $I = I_r = \{i\}$ with $i \cdot i = 2$ we recover the algebra $A_n$. In this case, a floating dot with nonzero subscript is a linear combination of floating dots with zero subscript with coefficients being (partially symmetric) polynomials on dots.
- In general for $\mathfrak{p} \subset \mathfrak{p}'$, with $I_f \subset I_{f'}$, we can obtain $R_{\mathfrak{p}'}(v)$ as a quotient (or resp. a sub-algebra) of $R_{\mathfrak{p}}(v)$ by killing floating dots with subscript in $I_{f'} \backslash I_f$ (resp. by restricting to floating dots with subscript in $I_{f'} \subset I_f$).

- As in the case of the KLR algebras and $A_n$ the algebras $R_{\mathfrak{p}}(v)$ act faithfully on a supercommutative ring.

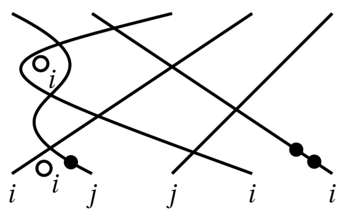
3.3.4 Tight floating dots and a basis for $R_{\mathfrak{p}}$

Contrary to the algebra $A_n$, in general we cannot slide all floating dots to the bottom (or to the top) of a diagram from $R_{\mathfrak{p}}$. This implies that the basis of $A_n$ in Proposition 3.8 do not extend to basis of $R_{\mathfrak{p}}(v)$ unless $v = v_i \cdot i$. This is different from situation in KLR algebras where we have a basis given by diagrams split in two regions, one containing all crossings (labelled by elements of $S_{|v|}$) and one containing dots:

- *considering diagrams of $R_{\mathfrak{p}}(v)$ that are split in three regions (containing crossings, dots and floating dots respectively) does not give a basis of $R_{\mathfrak{p}}(v)$.*

**Definition 3.31 :** We say that a floating dot is tight if it has superscript 0 and it is placed directly at the right of the leftmost strand.

By (19), for a tight floating dot not to be zero, it must have the same subscript as the strand at its left and so we will always assume it is the case. For example,




Thanks to relations (17), (18) and (19) these can be brought to the region immediately at the right of the first strand, and this shows that any diagram of $R_{\mathfrak{p}}(v)$ can be written as a $\Bbbk$-linear combination of diagrams involving only KLR generators and tight floating dots. Moreover, we can also construct a basis in terms of these type of diagrams.

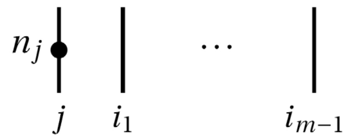
**Proposition 3.32 :** *There is a basis of $R_{\mathfrak{p}}(v)$ given in terms of KLR generators and involving only tight floating dots.*

This basis generalizes the basis of $A_n$ in Proposition 3.11, it is defined combinatorially and its particular form is not important for this lectures (see 18, §3.3).

3.3.5. Semi-cyclotomic $\mathfrak{p}$-KLR algebras

Fix $\beta$ and $N = \{n_j\}_{j \in I_f}$ as before.

Define the *semi-cyclotomic ideal* $I^N(\nu) \subset R_{\mathfrak{p}}(\nu)$ as the 2-sided ideal generated by the diagrams




with $ji_1 \dots i_{m-1} \in Seq(\nu)$ and $j \in I_f$.

**Definition 3.33 :** The semi-cyclotomic $\mathfrak{p}$-KLR algebra $R^N_{\mathfrak{p}}(\nu)$ is the quotient

$$R^N_{\mathfrak{p}}(\nu) = R_{\mathfrak{p}}(\nu) / I^N(\nu).$$

We put

$$R^N_{\mathfrak{p}} = \bigoplus_{\nu \in \mathbb{N}[I]} R^N_{\mathfrak{p}}(\nu).$$

By taking $\mathfrak{p} = \mathfrak{g}$ we recover the usual cyclotomic KLR algebras.

3.3.6. Categorical $\mathfrak{g}$-action

Adding a vertical strand labeled $i$ at the right of a diagram from $R_{\mathfrak{p}}(v)$ defines a homomorphism $R_{\mathfrak{p}}(\nu) \to R_{\mathfrak{p}}(\nu + i)$. Define the functors

$$Q_i = \coprod_{a \geq 0} q_i^{2a+1} \Pi : R^N_{\mathfrak{p}}(\nu) - s\,\mathrm{mod} \to R^N_{\mathfrak{p}}(\nu) - s\,\mathrm{mod},$$

and

$$F_i = Ind_{\nu}^{\nu+i} : R^N_{\mathfrak{p}}(\nu) - s\,\mathrm{mod} \to R^N_{\mathfrak{p}}(\nu + i) - s\,\mathrm{mod},$$

$$E_i = \begin{cases} q_i^{1+\langle i, \nu_X \rangle} \lambda_i^{-1} \mathrm{Res}_{\nu}^{\nu+i} : R^N_{\mathfrak{p}}(\nu + i) - s\,\mathrm{mod} \to R^N_{\mathfrak{p}}(\nu) - s\,\mathrm{mod} & \text{if } i \in I_r, \\ q_i^{1-n_i+\langle i, \nu_X \rangle} \mathrm{Res}_{\nu}^{\nu+i} : R^N_{\mathfrak{p}}(\nu + i) - s\,\mathrm{mod} \to R^N_{\mathfrak{p}}(\nu) - s\,\mathrm{mod} & \text{otherwise.} \end{cases}$$

We write $1_v$ for the identity functor of $R_{\mathfrak{p}}(v) - smod$.

**Theorem 3.34 :** *Functors $Q_i$, $F_i$, $E_i$ are exact and send projectives to projectives. Moreover, there are natural, non-split, short exact sequences*

$$0 \to F_i E_i 1_\nu \to E_i F_i 1_\nu \to q_i^{-\langle i, \nu_X \rangle} \lambda_i Q_i 1_\nu \oplus q_i^{\langle i, \nu_X \rangle} \lambda_i^{-1} \Pi Q_i \to 0,$$

*for all $i \in I_r$, and natural isomorphisms*

$$E_j F_j 1_\nu \cong F_j E_j 1_\nu \oplus_{[n_j - \langle j, \nu_X \rangle]_{q_j}} 1_\nu \quad \text{if} \quad n_j - \langle j, \nu_X \rangle \geq 0,$$

$$F_j E_j 1_\nu \cong E_j F_j 1_\nu \oplus_{[\langle j, \nu_X \rangle - n_j]_{q_j}} 1_\nu \quad \text{if} \quad n_j - \langle j, \nu_X \rangle \leq 0,$$

*for all* $j \in I_f$. *There are also natural isomorphisms*

$$F_i E_j 1_\nu \cong E_j F_i 1_\nu,$$

*and*

$$\bigoplus_{a=0}^{\lfloor (d_{ij}+1)/2 \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a \end{bmatrix}_{q_i} F_i^{2a} F_j F_i^{d_{ij}+1-2a} 1_\nu \cong \bigoplus_{a=0}^{\lfloor d_{ij}/2 \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a+1 \end{bmatrix}_{q_i} F_i^{2a+1} F_j F_i^{d_{ij}-2a} 1_\nu,$$

$$\bigoplus_{a=0}^{\lfloor (d_{ij}+1)/2 \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a \end{bmatrix}_{q_i} E_i^{2a} E_j E_i^{d_{ij}+1-2a} 1_\nu \cong \bigoplus_{a=0}^{\lfloor d_{ij}/2 \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a+1 \end{bmatrix}_{q_i} E_i^{2a+1} E_j E_i^{d_{ij}-2a} 1_\nu,$$

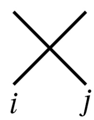
*for all* $i, j \in I$ *with* $i \neq j$.

Choosing $\mathfrak{p} = \mathfrak{g}$ (and thus $I_r = \varnothing$) in the theorem above yields the direct sums decompositions used to prove the Khovanov-Lauda cyclotomic categorification conjecture, which was proven by Kang-Kashiwara [5] and Webster [21].

3.3.7 $R_\mathfrak{b}$ and $R_\mathfrak{p}^N$ as a DGAs

(1) *The case* $\mathfrak{p} = \mathfrak{b}$: Choose a subset $I_f \subset I$ and consider the corresponding parabolic subalgebra $\mathfrak{p}$. Fix also some $N = \{n_j\}_{j \in I_f}$. We will see below that $R_\mathfrak{p}^N$ can be obtained as the homology of $R_\mathfrak{b}$ with respect to a differential $d_N$.

We equip $R_\mathfrak{b}(\nu)$ with the differential $d_N$ below. Firstly, it acts trivial on KLR generators,

$d_N\Big($  $\Big) = 0,$ $\qquad$ $d_N\Big($  $\Big) = 0,$

all $i, j \in I$.

To define $d_N$ on floating dots we decompose it into diagrams involving only tight floating dots and put

$$d_N(\omega_j) = (-1)^{n_j} x_1^{n_j}, \tag{20}$$

for $j \in I_f$ and $\omega_j$ supposed to be tight, that is in a region with $K = 1 \cdot j$.

**Remark 3.35 :** There is an explicit formula for the action of $d_N$ on an arbitrary floating dot [18]. In this case $d_N$ returns a linear combination of partially symmetric polynomials.

**Remark 3.36 :** The case $\mathfrak{p} = \mathfrak{b}$ gives the cyclotomic KLR algebra $R^N$.

**Proposition 3.37 :** *The algebra $R_{\mathfrak{b}}(v)$ equipped with $d^N$ forms a formal DG algebra $(R_{\mathfrak{p}}(v), d_N)$ whose homology is isomorphic to the cyclotomic quotient $R^N_{\mathfrak{p}}(v)$. Moreover, if $\langle j, \nu_X \rangle > n_j + \nu_j$ for $j \in I_{f'}$ then $(R_{\mathfrak{p}}(v), d_N)$ is acyclic.*

This results fits the idea that a parabolic Verma module is a mix between a finite-dimensional representation and a Verma module.

(2) *The case of general* $\mathfrak{p}$: Suppose that $I_f$ and $\mathfrak{p}$ are fixed.

- Choose a subset $I_f \subsetneq I_{f'} \subset I$ such that $\mathfrak{p} \subsetneq \mathfrak{p}'$.
- For each $j \in I_{f'} \setminus I_f$, choose a non-negative integer $n_j \in \mathbb{N}$.
- Write $N' = \{n_j \mid j \in I_{f'} \setminus I_f\}$.

The same formulas as before endow $R^N_{\mathfrak{p}}(v)$ with a differential $d_{N'}$.

**Proposition 3.38 :** *$(R^N_{\mathfrak{p}}, d_{N'})$ is a formal DG algebra and*

$$H^*(R^N_{\mathfrak{p}}, d_{N'}) \cong R^{N \cup N'}_{\mathfrak{p}'}(\nu).$$

The following diagram summarizes the several extended KLR DG algebras and differentials.

$$R_{\mathfrak{b}}(\nu) \overset{d_N}{\rightsquigarrow} R^N_{\mathfrak{p}}(\nu) \overset{d_{N'}}{\rightsquigarrow} R^{N \cup N'}_{\mathfrak{p}'}(\nu) \overset{d_{N''}}{\rightsquigarrow} R^{N \cup N' \cup N''}(\nu).$$

We have various (commutative) ways of going from $R_{\mathfrak{b}}(v)$ to $R^{N''}(v)$.

3.3.8. New bases for cyclotomic KLR algebras

Under Proposition 3.38, the basis in Proposition 3.32 give new basis of cyclotomic KLR algebras (for all types). It would be interesting to know if in type $A$ these are related to the Hu--Matthas graded cellular bases [4].

3.4. *Categorification of Verma modules*

Let

- $R^N_{\mathfrak{p}}(\nu) - \text{psmod}_{lfg}$ be the category of cone bounded, locally finitely generated projective, left $R^N_{\mathfrak{p}}(v)$-supermodules.

These are the projective modules generated by a collection of elements such that the (infinite) sum of the monomials corresponding to their degrees gives an element of $\mathbb{Z}((q, \beta))$. This category is cone complete (i.e. it contains all cone bounded, locally finite coproducts), and possesses the local Krull-Schmidt property. Indeed, the indecomposable projectives have all locally finite dimensions contained in cones compatible with $\prec$, and their part in minimal degree is isomorphic to $\Bbbk$. The topological split Grothendieck group $\mathbf{K}_0(R^N_{\mathfrak{p}}(v))$ is a free $\mathbb{Z}_\pi((q, \beta))$-module, with $\mathbb{Z}_\pi = \mathbb{Z}[\pi]/(\pi^2 - 1)$, generated by the classes of indecomposable projective modules, up to shift.

We consider also

- $R^N_{\mathfrak{p},\mu}(\nu) - \text{smod}_{lf}$, the category of cone bounded, locally finite dimensional $R^N_{\mathfrak{p}}(v)$-modules.

Here, the graded dimension of the modules seen as $\Bbbk$-vector spaces are in $\mathbb{Z}((q, \beta))$. It is also cone complete and possesses the local Jordan-Hölder property. Therefore its topological Grothendieck group $\mathbf{G}_0(R^N_{\mathfrak{p}}(v))$ is also a $\mathbb{Z}_\pi((q, \beta))$-module, freely generated by the classes of simple modules. When specializing the parameter $\pi = -1$ and extending the scalars to $\mathbb{Q}$, we write

$$\widetilde{\mathbf{G}_0}(R^N_{\mathfrak{p}}(\nu)) = \mathbf{G}_0(R^N_{\mathfrak{p}}(\nu)) \otimes_{\mathbb{Z}_\pi} \mathbb{Q}[\pi]/(\pi+1),$$

and the same for $\widetilde{\mathbf{K}}_0(R^N_{\mathfrak{p}}(\nu))$.

Taking projective resolutions of the simple objects yields a change of basis, and $\mathbf{K}_0(R^N_{\mathfrak{p}}(\nu))$ is also freely generated by the classes of projective modules. This justifies the choice $q \prec \lambda_i$ in the order chosen to define $\mathbb{Z}((q, \beta))$.

The functor $Q_i$ descends onto the Grothendieck groups as

$$[Q_i M] = -q_i(1 + q_i^2 + q_i^4 + \dots)[M] = \frac{1}{q_i - q_i^{-1}}[M],$$

explaining the choice $0 \prec q$.

**Example 3.39 :** Take a simple object $S = \mathbb{Q}$ and projective $R_{\mathfrak{p}}^{N}(i) \cong \mathbb{Q}[\xi] \otimes \bigwedge^{\bullet}\langle\omega\rangle$ with $\deg(\xi) = (2i \cdot i, 0)$ and $\deg(\omega) = (-2i \cdot i, 2 \cdot i)$, viewed as modules over $R_{\mathfrak{p}}^{N}(i)$. Then $S$ admits a projective resolution yielding

$$[S] = (1-q_i^2)(1-\pi q_i^{-2}\lambda_i + q_i^{-4}\lambda_i^4 - \ldots)[R_{\mathfrak{p}}^{N}(i)].$$

We want to take $(1+q_i^{-2}\lambda_i + q_i^{-4}\lambda_i^4 + \ldots)$ as the inverse of $(1-q_i^{-2}\lambda_i^2)$.

**Theorem 3.40 :** *The functors* $\{E_i, F_i\}_{i\in I}$ *induce an action of* $\mathfrak{g}$ *on* $\widetilde{\mathbf{G}_0}(R_{\mathfrak{p}}^{N})$. *In particular there is an isomorphism of* $\mathfrak{g}$*-modules*

$$\widetilde{\mathrm{G}_0}(R_{\mathfrak{p}}^{N}) \cong M^{\mathfrak{p}}(\lambda q^{-1}, N),$$

*sending classes of indecomposable projective modules to divided canonical basis elements and classes of simple modules to dual canonical basis elements.*

Here $(\lambda q^{-1})_i$ is $\lambda_i q_i^{-1}$ if $i \in I_f$ or $n_i$ if $i \in I \backslash I_f$.

Similarly, $\{E_i, F_i\}_{i\in I}$ induce an action of $\mathfrak{g}$ on $\widetilde{\mathbf{K}}_0(R_{\mathfrak{p},}^{N})$ and we have an isomorphism of $\mathfrak{g}$-modules $\widetilde{\mathbf{K}}_0(R_{\mathfrak{p}}^{N}) \cong M^{\mathfrak{p}}(\lambda q^{-1}, N)$. However in this case dual canonical basis elements are only given by formal power series of the classes of projectives.

The Shapovalov form admits a nice interpretation in term of graded (super)dimensions of some vector spaces.

**Proposition 3.41 :** *For each* $M, N \in R_{\mathfrak{p}}^{N} - smod_{lf}$ *we have*

$$([M],[N]) = \operatorname{sdim} M^{\psi} \otimes_{R_{\mathfrak{p}}^{N}} N,$$

*where* $(-,-)$ *is the universal Shapovalov form, and* $M^{\psi}$ *is the right* $R_{\mathfrak{p}}^{N}$*-module given by the anti-involution on* $R_{\mathfrak{p}}^{N}$ *that reverses diagrams along the horizontal axis.*

The induction and restriction functors $E_i$, $F_i$ have their derived counterparts given by replacing objects with their bar resolution, so that $\mathcal{D}^{lc}(R_{\mathfrak{p}}^{N}, d_{N'})$ and $\mathcal{D}^{lf}(R_{\mathfrak{p}}^{N}, d_{N'})$ are equipped with a categorical action of $\mathfrak{g}$. This categorical action induces in turn an action of $\mathfrak{g}$ on both topological Grothendieck groups.

**Theorem 3.42 :** *There are equivalences of triangulated categories*

$$\mathcal{D}^{lc}(R_{\mathfrak{p}}^{N}, d_{N'}) \cong \mathcal{D}^{lc}(R_{\mathfrak{p}'}^{(N\cup N')}, 0), \quad \mathcal{D}^{lf}(R_{\mathfrak{p}}^{N}, d_{N'}) \cong \mathcal{D}^{lf}(R_{\mathfrak{p}'}^{(N\cup N')}, 0).$$

**Corollary 3.43 :** *There are isomorphisms of* $\mathfrak{g}$*-modules*

$$\mathbf{K}_0(\mathcal{D}^{lc}(R_{\mathfrak{p}}^{N}, d_{N'})) \otimes \mathbb{Q} \cong \mathbf{K}_0(\mathcal{D}^{lc}(R_{\mathfrak{p}'}^{(N\cup N')}, 0)) \otimes \mathbb{Q} \cong \widetilde{\mathbf{K}}_0(R_{\mathfrak{p}'}^{(N\cup N')}) \cong M^{\mathfrak{p}'}(\Lambda', N\cup N'),$$

*and*

$$\mathbf{K}_0(\mathcal{D}^{lf}(R_{\mathfrak{p}}^{N}, d_{N'}))\otimes\mathbb{Q} \cong \mathbf{K}_0(\mathcal{D}^{lf}(R_{\mathfrak{p}'}^{(N\cup N')}, 0))\otimes\mathbb{Q} \cong \widetilde{\mathbf{G}_0}(R_{\mathfrak{p}'}^{(N\cup N')}) \cong M^{\mathfrak{p}'}(\beta', N\cup N'),$$

*where* $\beta' = \{\beta_i \mid i \in I_{r'}\}$.

We view this result as a categorification of the order on the parabolic Verma modules.

**Remark 3.44 :** Note that equipping $R_{\mathfrak{p}}^{N}$ with a trivial differential and passing to the derived category yields a natural way to specialize $\pi = -1$ in the Grothendieck group.

## References


[1] J. Chuang and R. Rouquier, Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification, *Ann. of Math.* (2) 167 (2008), no. 1, 245-298. arXiv: math/0407205 [math.RT].

[2] A. Ellis and A. Lauda, An odd categorification of quantum *sl*(2), *Quantum Topol.* 7 (2016), no. 2, 329-433. arXiv: 1307.7816 [math.QA].

[3] I. Frenkel, M. Khovanov and C. Stroppel, A categorification of finite-dimensional irreducible representations of quantum $\mathfrak{sl}_2$ and their tensor products, *Selecta Math.* (N.S.) 12 (2006), no. 3-4, 379-431. arXiv: math/0511467 [math.QA].

[4] J. Hu and A. Mathas, Graded cellular bases for the cyclotomic Khovanov-Lauda-Rouquier algebras of type $\mathfrak{sl}_2$, *Adv. Math.* 225 (2010), no. 2, 598-642. arXiv: 0907.2985 [math.RT].

[5] S.-J. Kang and M. Kashiwara, Categorification of highest weight modules via Khovanov-Lauda-Rouquier algebras, *Invent. Math.*, (2012) 190:699-742. arXiv: 1102.4677 [math.QA].

[6] S. Kang, M. Kashiwara and S. Oh, Supercategorification of quantum Kac-Moody algebras. *Adv. Math.* 242 (2013), 116-162. arXiv: 1206.5933 [math.RT].

[7] S. Kang, M. Kashiwara and S. Tsuchioka, Quiver Hecke superalgebras, *J. Reine Angew. Math.* 711 (2016), 1-54. arXiv:1107.1039v2 [math.QA]

[8] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups I, *Repres. Theo.* 13 (2009), 309-347. arXiv: 0803.4121 [math.QA].

[9] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups II, *Trans. Amer. Math.* Soc. 363 (2011), no. 5, 2685-2700. arXiv: 0804.2080 [math.QA].

[10] M. Khovanov and A. Lauda, A categorification of quantum *sl*(*n*), *Quantum Topol.* 1 (2010), no. 1, 1-92. arXiv: 0807.3250 [math.QA]

[11] A. Lauda, A categorification of quantum $\mathfrak{sl}_2$, *Adv. Math.* 225 (2010), no. 6, 3327-3424. arXiv: 0803.3652 [math.QA].

[12] A. Lauda, An introduction to diagrammatic algebra and categorified quantum $\mathfrak{sl}_2$, *Bull. Inst. Math. Acad. Sin. (N.S.)* 7 (2012), no. 2, 165-270. arXiv: 1106.2128.

[13] A. Lauda, D. Rose and H. Queffelec, Khovanov homology is a skew Howe 2-representation of categorified quantum *sl*(*m*) *Algebraic and Geometric Topology* 15 (2015), 2517-2608. arXiv: 1212.6076 [math.QA].

[14] V. Mazorchuk, *Lectures on* $\mathfrak{sl}(2, \mathbb{C})$*-modules*, Imperial College Press, London, 2010. x+263 pp.

[15] G. Naisse and P. Vaz, An approach to categorification of Verma modules, *Proc. Lond. Math. Soc.* (3) 117 (2018), no. 6, 1181-1241. arXiv: 1603.04555v2 [math.RT] (2016).

[16] G. Naisse and P. Vaz, On 2-Verma modules for quantum $\mathfrak{sl}_2$, *Sel. Mat. New Ser.* 24 (2018), no. 4, 3763-3821. arXiv: math.RT/1704.08205.

[17] G. Naisse and P. Vaz, 2-Verma modules and the Khovanov-Rozansky link homologies, arXiv: math.QA/1704.08485 (2017).

[18] G. Naisse and P. Vaz, *2-Verma modules*, arXiv: 1710.06293 [math.RT] (2017).

[19] H. Queffelec and D. Rose, The $\mathfrak{sl}_n$ foam 2-category: A combinatorial formulation of Khovanov-Rozansky homology via categorical skew Howe duality, *Adv. Math.* 302 (2016), 1251-1339. arXiv: 1405.5920 [math.QA]

[20] R. Rouquier, *2-Kac-Moody algebras*, arXiv: 0812.5023.

[21] B. Webster, Knot invariants and higher representation theory, *Mem. Amer. Math. Soc.*, 250, 141. doi:10.1090/memo/1191 (2017). arXiv: math.GT/1309.3796.

[22] B. Webster, Tensor product algebras, Grassmannians and Khovanov homology. In *Physics and mathematics of link homology* (Vol. 680, pp. 23-58). Amer. Math. Soc., Providence, RI (2016). arXiv: 1312.7357 [math.GT].

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