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Odd Khovanov's arc algebra

by

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Abstract. We construct an odd version of Khovanov's arc algebra H^n. Extending the center to elements that anticommute, we get a subalgebra that is isomorphic to the oddification of the cohomology of the (n,n)-Springer variety. We also prove that the odd arc algebra can be twisted into an associative algebra.

1. Introduction. Arc algebras were first introduced by Khovanov [20] to extend his categorification of Jones' link invariant [19] to tangles. One of the main ingredients was a certain Frobenius algebra of rank 2, which coincides with the cohomology ring of complex projective space. In a follow-up paper [21], Khovanov showed that the arc algebra H^n from [20] is closely related to the geometry of Springer varieties. Indeed, he proved that the center of H^n is isomorphic to the cohomology ring of the (n,n)-Springer variety. Later on, Chen and Khovanov [11] defined subquotients of H^n with the aim of giving an explicit categorification of the action of tangles on tensor powers of the fundamental representation of quantum $\mathfrak{sl}(2)$. To do so, they categorified the n-fold tensor power of the fundamental representation of $U_q(\mathfrak{sl}(2))$ together with its weight space decomposition. Additionally, Khovanov's arc algebra was further studied in the sequence of papers [5–9], where most of its interesting representation-theoretic properties were revealed.

Khovanov's arc algebra was generalized in several directions by several authors, first to $\mathfrak{sl}_3$-web algebras [26] and then to $\mathfrak{sl}_n$-web algebras [25], and was further studied in [35, 36]. In [33], a version of the arc algebra associated with $\mathfrak{gl}(1|1)$ was constructed, motivated by a representation-theoretic categorification of the Alexander polynomial. In [14, 15], a Khovanov algebra of type D was introduced, in connection with orthosymplectic Lie algebras. More recently, a variant of Khovanov's arc algebra based on Blanchet's ver-

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sion of Khovanov homology [3] was constructed in [16]. This was extended in [17] to $\mathfrak{gl}_2$-arc and web algebras associated with the variants of Khovanov homology from [10] and [12]. One of the main properties of the arc/web algebras above is that, except maybe the ones from [14, 15], they all admit topological constructions using cobordisms or foams.

In [28], Ozsváth, Rasmussen and Szabó used an exterior version of Khovanov's original Frobenius algebra to give an odd version of Khovanov homology. Odd Khovanov homology agrees with the even (usual) Khovanov homology from [19] modulo 2, but they differ over fields of characteristic other than 2. Moreover, both categorify the Jones polynomial (see for example [34] for further properties). Odd Khovanov homology was given a (Bar–Natan style [2]) topological set-up by Putyra [30]. He introduced the so-called chronological cobordisms, which are cobordisms together with some extra structure related to a height function.

In this paper we use the set-up from [28, 30] and construct an odd version of Khovanov's arc algebra from [20].

1.1. Sketch of the construction and main results. The first step in the construction of an odd version of Khovanov's arc algebra is to replace the TQFT obtained by the Frobenius algebra from [19] by the chronological TQFT from [28, 30]. As explained in [30], in order to get a well-defined category of cobordisms one has to choose an orientation for each of the local Morse moves. It was proved in [30] (and in [28] in an algebraic set-up) that any consistent choice of orientations gives the same link homology. This is no longer the case if one tries to extend odd Khovanov homology to tangles. In particular, a priori there is no reason for two different choices of orientations to result in isomorphic odd arc algebras.

Our first result is that we get a family of odd arc algebras indexed by all possible choices. We denote by OH^n_C the odd arc algebra associated with the choice C of chronological cobordisms. As a second result, we find that for all C and all $n \geq 2$, the odd arc algebra OH^n_C is non-associative. This is done in Section 3.

In Section 4, we prove an odd version of Khovanv's results from [21]. Namely, we prove that the odd center of OH^n_C is isomorphic to the odd cohomology of the (n,n)-Springer variety as given by Lauda and Russell [24]. They constructed an oddification of the cohomology of the Springer variety associated to any partition, by replacing polynomial rings and symmetric functions by their odd counterparts.

As mentioned above, the algebra OH^n_C is not associative. But this is not too big a problem, since it is a quasialgebra in the sense of Albuquerque and Majid [1], i.e. a non-associative graded algebra with an associator given by a 3-cocycle coming from a higher structure, that is, a monoidal category.

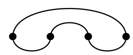
In Section 5, we introduce a grading on OH^n by a groupoid and prove the quasiassociativity of OH^n_C, the associator depending only on C. The idea of looking at an odd version of Khovanov's arc algebra as a quasialgebra goes back to the attempts of Putyra and Shumakovitch to extend the odd Khovanov homology to tangles. An extended discussion over generalized quasialgebras can be found in the unpublished work of Putyra [29]. We prove that the associator is a coboundary and thus admits a primitive τ. Twisting the multiplication of OH^n_C by this τ defines an associative algebra which keeps the odd flavor of OH^n_C. In addition, we prove that all choices of C and of the twist lead to isomorphic algebras.

2. Reminders. To begin, we recall the three main constructions we will use: the Khovanov arc algebra, the TQFT from odd Khovanov homology and the oddification of the cohomology of the Springer varieties.

2.1. Khovanov's arc algebra. As the construction in this paper follows Khovanov's original set-up from [20], we give below a sketch of the construction of the arc algebra H^n.

Crossingless matchings. Let B^n be the set of *crossingless matchings* of $2n$ points, that is, all ways one can pair $2n$ points on a horizontal line by non-crossing arcs placed below this line. For $b \in B^n$, we denote by $W(b)$ the reflection of b across the horizontal line and by $W(b)a$ the gluing of $W(b)$ on the top of $a \in B^n$. It is clear that $W(b)a$ is a disjoint union of circles. For example, we have in B^2:

$a =$  , $b =$  ,

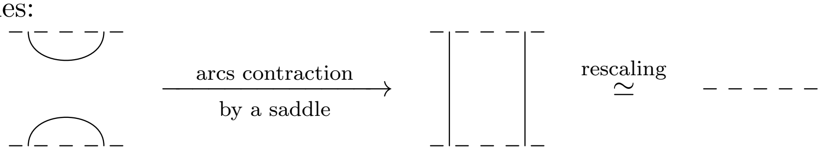
$W(b) =$  , $W(b)a =$  .

We also write $W(d)cW(b)a$ for the concatenation of $W(d)c$ on top of $W(b)a$, which is the disjoint union of $W(d)c$ and $W(b)a$ (see for example (2.2)).

Contraction cobordisms. Given a diagram $W(c)bW(b)a$, we construct a cobordism

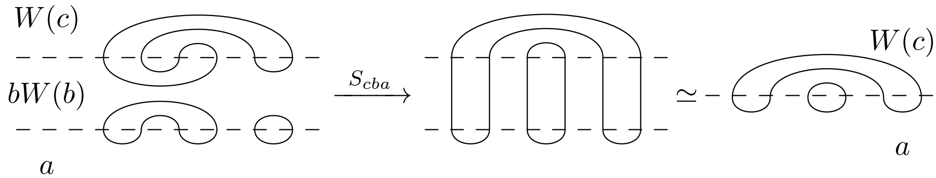
$$S_{cba} : W(c)bW(b)a \to W(c)a \tag{2.1}$$

by contracting the arcs of b with their symmetric counterparts in $W(b)$ by saddles:



This gives a surface with one saddle point for each arc in b. Therefore, S_{cba} has (minimal) Euler characteristic $-n$, is embedded in $\mathbb{R}^2 \times [0, 1]$ and is unique up to isotopy. Indeed, contracting the symmetric arcs in two different orders gives rise to homeomorphic surfaces, and thus the construction does not depend on any choice. Moreover, S_{cba} can be given a canonical orientation. The picture to keep in mind is

(2.2)



Frobenius algebra. Let $A := \mathbb{Z}[X]/(X^2)$ be the $\mathbb{Z}$-graded abelian group with grading given by $\deg 1 = -1$ and $\deg X = 1$. This group has the structure of a $\mathbb{Z}$-algebra when equipped with polynomial multiplication. However, this multiplication has degree 1 and does not give a graded algebra structure. We turn A into a Frobenius algebra by defining a trace,

$$\mathrm{tr} : A \to \mathbb{Z}, \quad \mathrm{tr}(1) = 0, \quad \mathrm{tr}(X) = 1.$$

As the trace is non-degenerate, this defines a TQFT,

$$F : 2Cob \to \mathbb{Z}\text{-grmod},$$

where $\mathbb{Z}$-grmod is the category of $\mathbb{Z}$-graded free $\mathbb{Z}$-modules with finite rank and $2Cob$ the category of oriented cobordisms between 1-manifolds [23]. From now on, unless stated otherwise, we will always assume that graded means $\mathbb{Z}$-graded.

Thus, we get $F(W(b)a) \simeq A^{\otimes |W(b)a|}$ for $|W(b)a|$ the number of circle components in $W(b)a$. Moreover, the comultiplication map is explicitly given by

$$\varDelta : A \to A \otimes A, \quad \varDelta(1) = X \otimes 1 + X \otimes 1, \quad \varDelta(X) = X \otimes X.$$

Applying this TQFT on the cobordism (2.1), we get a morphism

$$F(W(c)b) \otimes_{\mathbb{Z}} F(W(b)a) \simeq F(W(c)bW(b)a) \xrightarrow{F(S_{cba})} F(W(c)a). \tag{2.3}$$

This morphism has degree n since the multiplication and comultiplication maps in A have degree 1, and S_{cba} possesses n saddle points.

Arc algebra. Define the graded abelian groups

$$H^n := \bigoplus_{a,b \in B^n} b(H^n)a, \qquad b(H^n)a := F(W(b)a)\{n\},$$

where the notation $\{n\}$ means that we shift the degree up by n. Therefore, as the maximal number of components in $W(b)a$ is n, every element $x \in F(W(b)a)$ has degree $\deg x \geq -n$, and thus H^n is a $\mathbb{Z}_+$-graded group. To define a multiplication in H^n, we first let the product $d(H^n)c \otimes_{\mathbb{Z}} b(H^n)a \to H^n$

be zero whenever $c \neq b$. Then, for the other cases, we define the multiplication so that the diagram

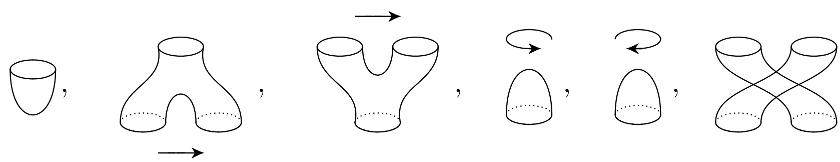
$$\begin{tikzcd} c(H^n)b \otimes_{\mathbb{Z}} b(H^n)a \arrow[rr] \arrow[d, "\simeq"'] & & c(H^n)a \\ F(W(c)b) \otimes_{\mathbb{Z}} F(W(b)a)\{2n\} \arrow[rr, "(2.3)"] & & F(W(c)a)\{n\} \arrow[u, "\simeq"'] \end{tikzcd}$$

commutes. The associativity of the multiplication follows from the fact that F is a TQFT. Moreover, the sum $\sum_{a \in B^n} 1_a$, with 1_a the unit in $a(H^n)a \simeq A^{\otimes n}\{n\}$, is a unit for H^n. All of this sums up to:

PROPOSITION 2.1 (Khovanov, [20, Proposition 1]). *The structures above make H^n into a $\mathbb{Z}_+$-graded associative unital $\mathbb{Z}$-algebra.*

2.2. Odd Khovanov homology. Ozsváth, Rasmussen and Szabó [28] constructed an odd version of Khovanov homology using some "projective TQFT" replacing F (projective meaning here that it is well-defined only up to sign). Putyra [30] extended the work of Bar–Natan [2] on Khovanov homology by giving a topological framework for the odd homology: the chronological cobordisms. In addition, Putyra's work allows the construction of odd Khovanov homology using a well-defined functor. In this subsection, we mainly follow the exposition in [30].

Chronological cobordisms. Recall that a *chronological 2-cobordism* is a 2-cobordism equipped with a *chronology*, that is, a Morse function with one critical point at each critical level. Moreover, at each critical point, we choose an orientation of the space of unstable directions in the gradient flow induced by the chronology. We indicate that choice by an arrow. These chronological 2-cobordisms, taken up to isotopy which preserves the orientations and the chronology, form a category with composition given by gluing. We denote it by $2ChCob$. Every chronological 2-cobordism can be built from the six elementary chronological 2-cobordisms:

(2.4) 

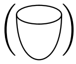
which are called respectively a *birth*, a *merge*, a *split*, a *positive death*, a *negative death* and a *twist*. As we are only interested in chronological 2-cobordisms, we will forget the prefix 2-.

The odd functor. We describe the functor $OF : 2ChCob \to \mathbb{Z}$-grmod from [28]. Morally, objects of $2ChCob$ are disjoint unions of circles. For S such a union we denote by $V(S)$ the free abelian group generated by the components of S with a grading such that each generator has degree 2. We define

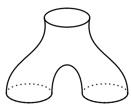
$$OF(S) := \bigwedge^* V(S)\{-|S|\},$$

with $\bigwedge^* V(S)$ being the exterior algebra generated by the elements of $V(S)$ and $|S|$ the number of components.

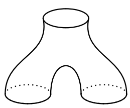
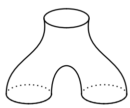
We now define the functor on each of the elementary cobordisms (2.4). Let S_1 and S_2 be objects of $2\mathit{ChCob}$ with S_2 containing one circle more than S_1. For a birth of a circle from S_1 to S_2, there is a canonical inclusion $V(S_1) \subset V(S_2)$ (the new generator being the circle cupped by the birth cobordism). This induces a morphism

$OF\Big($  $\Big) : \bigwedge^* V(S_1) \xhookrightarrow{\subset} \bigwedge^* V(S_2), \quad v \mapsto 1 \wedge v.$

Consider a merge of two circles a_1, a_2 in S_2 to a single one in S_1 with an arrow $a_1 \to a_2$. The arrow represents one of the two possible choices of orientation of the merge, the other being denoted $a_2 \to a_1$. There is an isomorphism of groups $V(S_1) \simeq V(S_2)/\{a_1 - a_2\}$ and thus the canonical projection $V(S_2) \to V(S_2)/\{a_1 - a_2\}$ induces a morphism

$OF\Big($  $a_1 \longrightarrow a_2$ $\Big) : \bigwedge^* V(S_2) \to \bigwedge^* \left(V(S_2)/\{a_1 - a_2\}\right) \simeq \bigwedge^* V(S_1).$

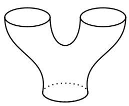
It is not hard to see that the choice of orientation does not change the result in this case and we get

$OF\Big($  $a_1 \longrightarrow a_2$ $\Big) = OF\Big($  $a_1 \longleftarrow a_2$ $\Big).$

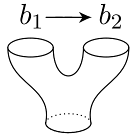
Now say we have a split sending $a \in S_1$ to $b_1 \to b_2$ in S_2. Again, there is a natural identification $V(S_2) \simeq V(S_1)/\{b_1 - b_2\}$, but now we also use the isomorphism

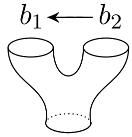
$$\bigwedge^* \left(V(S_1)/\{b_1 - b_2\}\right) \simeq (b_1 - b_2) \wedge \bigwedge^* V(S_1)$$

to get a morphism

$OF\Big($ $b_1 \longrightarrow b_2$  $\Big) : \bigwedge^* V(S_2) \simeq (b_1 - b_2) \wedge \bigwedge^* V(S_1) \xhookrightarrow{\subset} \bigwedge^* V(S_1).$

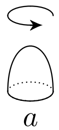
As a matter of fact, this morphism is easily computable by replacing the occurences of a by b_1 (or b_2) and multiplying by $b_1 - b_2$. For example, 1 is sent to $b_1 - b_2$ and a is sent to $(b_1 - b_2) \wedge b_1 = b_1 \wedge b_2$. Note also that reversing the orientation changes the sign of the morphism:

$OF\Bigg($ 

$\Bigg) = -OF\Bigg($ 

$\Bigg).$

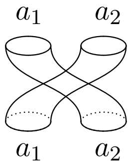
Suppose we have a positive (in other words, anticlockwise oriented) death of $a \in S_2$. We associate to it the morphism given by contraction with the dual of a:

$OF\Bigg($ 

$\Bigg) : \bigwedge^* V(S_2) \to \bigwedge^* V(S_1), \qquad v \mapsto a^*(v).$

The negative one is given by the opposite.

Finally, the twist is given by a the permutation of the corresponding terms:

$OF\Bigg($ 

$\Bigg) : \bigwedge^* V(S_2) \to \bigwedge^* (V(S_2)), \qquad \begin{cases} a_1 \mapsto a_2, \\ a_2 \mapsto a_1, \\ a_1 \wedge a_2 \mapsto a_2 \wedge a_1. \end{cases}$

Remark 2.2. Since changing the orientations of the merges does not change the result of the functor, we will ignore them in our discussion.

2.3. Odd cohomology of the Springer varieties. First, let us recall the definition of a Springer variety.

Definition 2.3. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of m, E_m be a complex vector space of dimension m and $z_\lambda : E_m \to E_m$ be a nilpotent linear endomorphism with $|\lambda|$ nilpotent Jordan blocks of size $\lambda_1, \ldots, \lambda_m$. The *Springer variety* for the partition λ is

$$\mathfrak{B}_\lambda := \{\text{complete flags in } E_m \text{ stabilized by } z_\lambda\}.$$

The cohomology ring of $\mathfrak{B}_\lambda$ can be computed by quotienting the polynomial ring in m variables by the ideal of partially symmetric functions (see [13] for more details). Write (n, n) for the partition $\lambda = (n, n)$ of $2n$.

Theorem 2.4 (Khovanov, [21, Theorem 1.1]). *There is an isomorphism of graded algebras*

$$Z(H^n) \simeq H(\mathfrak{B}_{n,n}, \mathbb{Z}).$$

Lauda and Russell [24] constructed an oddification of the cohomology of the Springer varieties, denoted $OH(\mathfrak{B}_\lambda, \mathbb{Z})$. Like the usual cohomology is obtained as a quotient of the polynomials by the partially symmetric functions, they constructed $OH(\mathfrak{B}_\lambda, \mathbb{Z})$ as a quotient of the ring $OPol_m$ of odd polynomials:

$$OPol_m := \frac{\mathbb{Z}\langle x_1, \dots, x_m\rangle}{\langle x_i x_j + x_j x_i = 0 \text{ for all } i \neq j\rangle}, \qquad \deg(x_i) = 2,$$

by some ideal. Since we only need the case $m = 2n$ and $\lambda = (n, n)$ for our discussion, we restrict to this case from now on.

Definition 2.5 (Lauda & Russell, [24]). The *odd cohomology* of the (n, n)-Springer variety is the quotient

$$OH(\mathfrak{B}_{n,n}, \mathbb{Z}) := OPol_{2n}/OI_n$$

where OI_n is the left ideal generated by the set of odd partially symmetric functions:

$$OC_n := \left\{ \epsilon_r^S := \sum_{1\leq i_1<\cdots<i_r\leq 2n} x_{i_1}^S \dots x_{i_r}^S \;\middle|\; \begin{array}{c} k \in \{1, \dots, n\}, |S| = n + k, \\ r \in \{n - k + 1, n - k, \dots, n + k\} \end{array} \right\},$$

for all ordered subsets S of $\{1, \dots, 2n\}$ of cardinality $n + k$ and

$$x_{i_j}^S := \begin{cases} 0 & \text{if } i_j \notin S, \\ (-1)^{S(i_j)-1} x_{i_j} & \text{otherwise,} \end{cases}$$

with $S(i_j)$ the position of i_j in S.

In general, the odd cohomology of a Springer variety is only a module over the odd polynomials. However, in case $\lambda = (n, n)$, it is a graded algebra. This is due to the fact that, thanks to [24, Lemma 3.6], $x_i^2 \in OI_n$ for all i. Thus, OI_n is a 2-sided ideal. As a matter of fact, $OH(\mathfrak{B}_{n,n}, \mathbb{Z})$ also has the structure of a superalgebra with superdegree given by dividing the degree by 2. Finally, by construction, the algebra $OH(\mathfrak{B}_{n,n}, \mathbb{Z})$ is isomorphic to $H(\mathfrak{B}_{n,n}, \mathbb{Z})$ modulo 2.

Example 2.6. $OH(\mathfrak{B}_{2,2}, \mathbb{Z})$ is given by the odd polynomials in four variables x_1, x_2, x_3, x_4 quotiented by the (not minimal) relations

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 &= 0, \\ -x_i x_j + x_i x_k - x_j x_k &= 0, \quad \forall i < j < k \in [1, 4], \\ -x_1x_2 + x_1x_3 - x_1x_4 - x_2x_3 + x_2x_4 - x_3x_4 &= 0, \\ x_i x_j x_k &= 0, \quad \forall i < j < k \in [1, 4], \\ -x_1x_2x_3 + x_1x_2x_4 - x_1x_3x_4 + x_2x_3x_4 &= 0, \\ x_1x_2x_3x_4 &= 0. \end{aligned}$$

3. Odd arc algebra. In this section, we construct an odd version of the Khovanov arc algebra H^n. Therefore, we will closely follow the construction above, replacing the TQFT F by the odd functor OF from Section 2.2. First, for all $n \geq 0$ we define the following graded abelian groups:

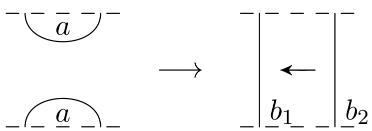
$$OH^n := \bigoplus_{a,b \in B^n} b(OH^n)a, \qquad b(OH^n)a := OF(W(b)a)\{n\},$$

so that OH^n is $\mathbb{Z}_+$-graded.

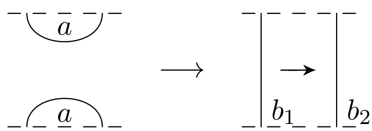
The first difficulty we encounter when we try to define multiplication as in Section 2.1 is that we have to choose a chronology and signs for the splits. In the odd Khovanov homology from [28], the signs are forced by the requirement that the cube of resolutions anticommutes (all possible choices leading to isomorphic cubes). However, in our case, there is no condition other than that the cobordisms must be embedded in $\mathbb{R}^2 \times [0,1]$. This means we have to consider all possible choices.

Contraction cobordisms. For each $a, b, c \in B^n$, there is a canonical cobordism with minimal number of critical points (up to homeomorphism and embedded in $\mathbb{R}^2 \times [0,1]$) from the diagram $W(c)bW(b)a$ to $W(c)a$. This corbordism is given by contracting the arcs of b with their symmetric counterparts in $W(b)$, as in the definition of H^n in Section 2.1. To be able to apply OF, we need to define a chronological cobordism, and there are several ways to do so:

- We have to choose a chronology, in other words, we have to choose an order in which we contract the symmetric arcs of $bW(b)$, taking care never to contract two arcs before the one surrounding them. This is required to get an embedded surface.
- We have to give an orientation for the critical points, especially for the splits (we do not need to orient the merges by Remark 2.2). We express each of the two possibilities by an arrow:



or



  meaning that we split the component a into two components b_1, b_2 with the orientation $b_2 \to b_1$ in the first case and $b_1 \to b_2$ in the second one.

Remark 3.1. There is always at least one possible choice: it suffices to go through the end points of b from left to right and contract whenever we encounter an arc which was not already contracted, then orient the splits from left to right (i.e. put an arrow from the component passing through the left point to the one passing through the right point).

We now assume that for each triplet $a, b, c \in B^n$ we have chosen a chronological cobordism. We write it C_{cba}, and denote the collection of all of them by $C := \{C_{cba} \mid a,b,c \in B^n\}$. In addition, we write $\mathcal{C}^n$ for the set of all possible choices of such a set C.

Multiplication. As in the even case, we let the multiplication

$$d(OH^n)c \otimes_{\mathbb{Z}} b(OH^n)a \to \{0\} \subset OH^n$$

be zero for $c \neq b$. We define the multiplication $c(OH^n)b \otimes_{\mathbb{Z}} b(OH^n)a \to c(OH^n)a$ using the morphism $OF(C_{cba})$. More precisely, there is a morphism

$$OF(W(c)b) \otimes_{\mathbb{Z}} OF(W(b)a) \to OF(W(c)bW(b)a) : (x,y) \mapsto x \wedge y \tag{3.1}$$

induced by the inclusions $W(c)b \subset W(c)bW(b)a$ and $W(b)a \subset W(c)bW(b)a$. We compose it with $OF(C_{cba})$ to obtain the multiplication by making the following diagram commute:

$$\begin{tikzcd}
c(OH^n)b \otimes b(OH^n)a \arrow[rr] \arrow[d, "\simeq"'] & & c(OH^n)a \\
OF(W(c)b) \otimes OF(W(b)a) \arrow[r, "(3.1)"'] & OF(W(c)bW(b)a) \arrow[r, "OF(C_{cba})"'] & OF(W(c)a) \arrow[u, "\simeq"']
\end{tikzcd}$$

This map is degree preserving thanks to the minimality hypothesis on the Euler characteristic of C_{cba} which guarantees that the degree of $OF(C_{cba})$ is n.

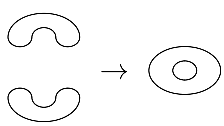
Unit. We write 1_a for the unit in the exterior algebra $\bigwedge^* V(W(a)a)$ and we easily check that the sum $\sum_{a \in B^n} 1_a$ is a unit for the multiplication defined in the paragraph above. We also write ${}_b1_a$ for the unit in the exterior algebra $\bigwedge^* V(W(b)a)$ (notice that ${}_b1_a$ is not an idempotent).

Proposition 3.2. *For $n \geq 2$ and any choice $C \in \mathcal{C}^n$, the multiplication defined above is not associative.*

Proof. First, suppose that $n = 2$ and let $C \in \mathcal{C}^2$ be an arbitrary choice of cobordisms. Consider $a, b \in B^2$ such that

$a =$ , $\quad b =$ .

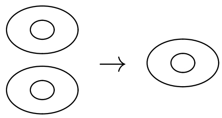
Take $x = b_1 \in b(OH^n)b$, $y = {}_b1_a \in b(OH^n)a$ and $z = {}_a1_b \in a(OH^n)b$, where b_1 is the element in the exterior algebra coming from the outer circle in the diagram $W(b)b$. We write b_2 for the element generated by the inner circle. Then we compute $x(yz)$ as follows. The cobordism

C_{bab} : 

is given by a merge followed by a split such that the element yz is

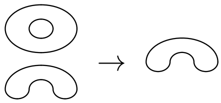
$$yz = \alpha(b_1 - b_2),$$

where $\alpha \in \{\pm 1\}$ depends on the orientation chosen for the unique split in C_{bab}. Then $x(yz)$ is given by



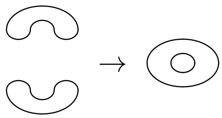
$$C_{bbb} : \quad \rightarrow \quad , \quad x(yz) = -\alpha b_1 \wedge b_2,$$

since C_{bbb} is composed of two merges. Now, we compute $(xy)z$ by



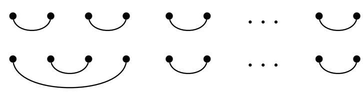
$$C_{bba} : \quad \rightarrow \quad , \quad xy = c_1,$$

with c_1 the element coming from the unique circle in $W(b)a$, and then



$$C_{bab} : \quad \rightarrow \quad , \quad (xy)z = \alpha b_1 \wedge b_2.$$

This means that for every $C \in \mathcal{C}^2$, $(xy)z = -x(yz)$. To conclude the proof, we observe that this example can be extended to all $n \geq 2$ by adding the same arcs to the right of a and b,



$$a_n := \quad \cdots \quad ,$$
$$b_n := \quad \cdots \quad ,$$

so that exactly the same computation can be done for all $n \geq 2$. ∎

Definition 3.3. We denote by OH^n_C the $\mathbb{Z}_+$-graded, non-associative, unital $\mathbb{Z}$-algebra given by OH^n with the multiplication obtained from a $C \in \mathcal{C}^n$.

Remark 3.4. We sometimes write $b(OH^n_C)a$. By this, we mean that we take the elements of the group $b(OH^n)a$, but viewed as elements in OH^n_C.

As for each C we get a family OH^n_C of algebras, it is legitimate to ask if one can classify them. We give some partial answer to this question in Section 5.

Remark 3.5. From now on, unless otherwise specified, all assertions are valid for all $n \in \mathbb{N}$ and $C \in \mathcal{C}^n$ and we do not specify n, C.

To ensure that OH^n_C is an odd version of H^n, we have to check that the two algebras agree modulo 2.

Proposition 3.6. *There is an isomorphism of graded algebras*

$$H^n \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \simeq OH^n \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}.$$

Proof. The result follows directly from the construction since the functor F used to define H^n agrees up to sign with OF. ∎

It is interesting (and will be useful) to note that OH^n_C contains a collection of exterior algebras as subalgebras.

Proposition 3.7. *There is an inclusion of graded algebras*

$$\bigoplus_{a\in B^n} \textstyle\bigwedge^* \mathbb{Z}^n \simeq \displaystyle\bigoplus_{a\in B^n} a(OH^n_C)a \subset OH^n_C.$$

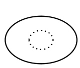
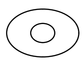
Proof. It is enough to see that $OF(W(a)a) \simeq \bigwedge^* \mathbb{Z}^n$ as $W(a)a$ is a collection of n circles, and to remark that the multiplication

$$a(OH^n_C)a \otimes_{\mathbb{Z}} a(OH^n_C)a \to a(OH^n_C)a$$

is the usual product in the exterior algebra. Indeed, the cobordism C_{aaa} consists of n merges and no splits and thus gives the exterior product. ■

Diagrammatic notation. To simplify the notation, we will write the generators of OH^n using diagrams. There is an order on the components of $W(b)a$, for $a, b \in B^n$, given by reading the diagram from left to right. More precisely, for $a_1, a_2 \in W(b)a$, we write $a_1 < a_2$ whenever a_1 passes through an end point of a (or equivalently b) which is at the left of all end points contained in a_2.

An element $x_1 \wedge \cdots \wedge x_k$ in $b(OH^n)a$ is written as the diagram $W(b)a$ where we draw the components with a solid line for each x_i and a dashed one for the others. Moreover, we require that $x_1 \wedge \cdots \wedge x_k$ is in the order induced by reading the diagram from left to right, otherwise we add a sign to recover this order. Thus, for example in OH^2 we get:

 $= b_1 \wedge 1,$ $-$  $= b_2 \wedge b_1,$

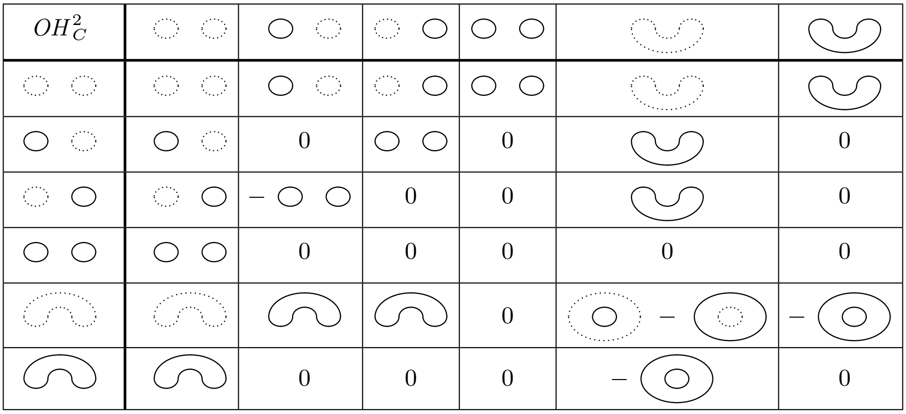
 $= {}_a1_b,$  $= c_1,$

with a_i, b_i and c_i following the conventions from the proof of Proposition 3.2.

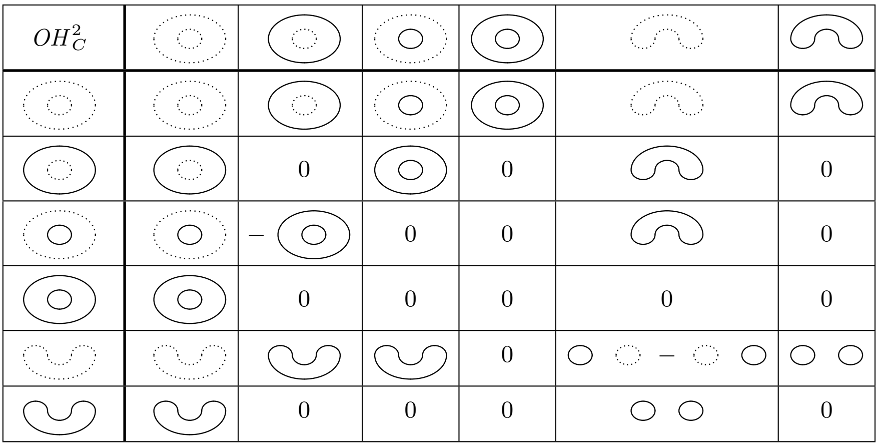
3.1. An example: OH^2_C. We construct explicit multiplication tables for OH^2_C, with C the choice from Remark 3.1. Since the multiplication maps for $*_2(OH^2)a \otimes b(OH^2)*_1 \to 0$ and $*_2(OH^2)b \otimes a(OH^2)*_1 \to 0$ are zero for all $*_1, *_2 \in \{a, b\}$, we give the tables only for $*_2(OH^2)a \otimes a(OH^2)*_1 \to *_2(OH^2)*_1$ and $*_2(OH^2)b \otimes b(OH^2)*_1 \to *_2(OH^2)*_1$. Moreover, these tables are written with the convention

	y
x	xy

By direct computation we get



and



3.2. The odd center of OH^n_C. When talking about exterior algebras (or in general superalgebras), it is common to consider the supercenter, which is an extension of the center to the elements that anticommute. In the same spirit, we define the odd center for OH^n_C.

DEFINITION 3.8. The *parity* of a homogeneous element $z \in a(OH^n)b$ is defined by

$$p(z) := \frac{\deg(z) - \deg({}_a1_b)}{2} = \frac{\deg(z) - n + |W(b)a|}{2} \bmod 2,$$

with $|W(b)a|$ the number of circle components in $W(b)a$.

One can easily see that this number counts the factors of $z = a_1 \wedge \cdots \wedge a_m$, that is,

$$p(a_1 \wedge \cdots \wedge a_m) = m \bmod 2.$$

Definition 3.9. The *odd center* of OH^n_C is the subset

$$OZ(OH^n_C) := \{z \in OH^n_C \mid zx = (-1)^{p(x)p(z)}xz, \forall x \in OH^n_C\}.$$

Remark 3.10. The parity does not give a grading on OH^n_C, since there are elements x, y in OH^n_C such that

$$p(xy) \neq p(x) + p(y) \bmod 2.$$

This means that OH^n_C is not a superalgebra with respect to p. As a matter of fact, the parity descends to a degree on the antisymmetric subalgebra $\bigoplus_{a\in B^n} a(OH^n_C)a \subset OH^n_C$, in which the odd center lives.

Proposition 3.11. *There are inclusions*

$$Z(OH^n_C) \subset OZ(OH^n_C) \subset \bigoplus_{a\in B^n} a(OH^n_C)a.$$

Proof. The second inclusion is immediate since, for every $z \in OZ(OH^n_C)$, one can decompose $z = \sum_{a,b\in B^n} {}_bz_a$ with ${}_bz_a \in b(OH^n_C)a$ and get ${}_bz_a = 1_bz1_a = z(1_b1_a) = 0$, unless $b = a$. The first inclusion is obtained by first noting that $Z(OH^n_C) \subset \bigoplus_{a\in B^n} a(OH^n_C)a$ by the same argument as before, and then observing that every central element has even parity. ∎

Moreover, one can check that the odd center is an *associative* superalgebra with superdegree given by the parity, and is characterized by the following property.

Proposition 3.12. *An element $z = \sum_{a\in B^n} z_a$ is in $OZ(OH^n_C)$ if and only if $z_b \cdot {}_b1_a = {}_b1_a \cdot z_a$ for all $a, b \in B^n$.*

Proof. An element $z = \sum_{a\in B^n} z_a$ commutes with $x = \sum_{a,b\in B^n} {}_bx_a$ if and only if z commutes with every ${}_bx_a$. Moreover, $z \cdot {}_bx_a = (z_b \cdot {}_b1_a) \wedge {}_bx_a = (-1)^{p(z)p(x)}{}_bx_a \wedge (z_b \cdot {}_b1_a)$, and we have ${}_bx_a \wedge (z_b \cdot {}_b1_a) = {}_bx_a \wedge ({}_b1_a \cdot z_a) = {}_bx_a \cdot z$ if and only if $z_b \cdot {}_b1_a = {}_b1_a \cdot z_a$. ∎

The following result allows us to write $OZ(OH^n)$ with no ambiguity.

Proposition 3.13. *For all $C, C' \in \mathcal{C}^n$, there is an isomorphism of graded (super)algebras*

$$OZ(OH^n_C) \simeq OZ(OH^n_{C'}).$$

Proof. The condition $z_b \cdot {}_b1_a = {}_b1_a \cdot z_a$ from Proposition 3.12 depends only on the multiplication maps

$$b(OH^n)b \otimes_{\mathbb{Z}} b(OH^n)a \to b(OH^n)a, \qquad b(OH^n)a \otimes_{\mathbb{Z}} a(OH^n)a \to b(OH^n)a.$$

It is not hard to see that those are defined using only cobordisms without split so that they do not depend on C. Moreover, by Proposition 3.7, $\bigoplus_{a\in B^n} a(OH^n_C)a$ is isomorphic to a direct sum of exterior algebras, and thus does not depend on C. ∎

4. Odd center and (n, n)-Springer variety. We are now ready to prove one of the main results of this paper, which constructs an explicit isomorphism between the odd cohomology of the (n, n)-Springer variety and the odd center of the odd Khovanov arc algebra.

THEOREM 4.1. *There is an isomorphism of graded (super)algebras between $OZ(OH^n)$ and $OH(\mathfrak{B}_{n,n}, \mathbb{Z})$. Moreover, this isomorphism is given by*

$$h : OH(\mathfrak{B}_{n,n}, \mathbb{Z}) \to OZ(OH^n), \qquad x_i \mapsto \sum_{a \in B^n} a_i,$$

where a_i is generated by the circle component of $W(a)a$ passing through the ith end point of a, counting from the left.

The proof of this theorem will occupy the rest of this section and is split into four steps. Firstly, we define a morphism $h_0 : OPol_{2n} \to OZ(OH^n_C)$ and prove that OI_n from Definition 2.5 lies in the kernel of this map, inducing the map h on $OH(\mathfrak{B}_{n,n}, \mathbb{Z})$. Secondly, we show that h is injective using the equivalence up to sign between the odd and the even case together with Theorem 2.4. Thirdly, we show that the ranks of the two algebras are equal using the cohomology of a geometric construction based on hypertori. Finally, we prove the theorem using all those ingredients.

Existence of h. To construct h, we first define the algebra homomorphism

$$h_0 : OPol_{2n} \to OH^n_C, \qquad x_i \mapsto \sum_{a \in B^n} a_i,$$

where a_i is generated by the circle component of $W(a)a$ passing through the ith end point of a, counting from the left. It is well-defined since

$$h_0(x_i x_j) = \sum_{a \in B^n} a_i \wedge a_j = -\sum_{a \in B^n} a_j \wedge a_i = -h_0(x_j x_i).$$

LEMMA 4.2. *The image of h_0 lies in the odd center of OH^n_C:*

$$h_0(OPol_{2n}) \subset OZ(OH^n).$$

Proof. The proof is straightforward from Proposition 3.12 and the fact that for all $a, b \in B^n$, we have

$$(h_0(x_i))_b 1_a = b_i \cdot {}_b 1_a = {}_b 1_a \cdot a_i = {}_b 1_a (h_0(x_i)),$$

since a_i and b_i are both sent to the component of $W(b)a$ passing through the ith point. ■

Now, we want to show that ϵ^S_r is in the kernel of h_0 for all S and r as in Definition 2.5. This is equivalent to showing that ϵ^S_r lies in the kernel of the homomorphism

$$h_a : OPol_{2n} \to a(OH^n_C)a, \qquad x_i \mapsto a_i,$$

for all $a \in B^n$, since $h = \sum_{a \in B^n} h_a$.

For the sake of simplicity, we fix an element $a \in B^n$. We also denote by E_{2n} the ordered set $\{1, \dots, 2n\}$ and we see it as the set of end points of a, from left to right. For $S \subset E_{2n}$, we call a pair of distinct points in S which are linked by an arc in a an *arc* of S. The other points of S are called *free points*. For $R = \{i_1, \dots, i_r\} \subset S$, we also write

$$\epsilon_R^S := x_{i_1}^S \dots x_{i_r}^S \quad \text{so that} \quad \epsilon_r^S = \sum_{R \subset S,\, |R|=r} \epsilon_R^S.$$

LEMMA 4.3. *Let $S \subset E_{2n}$ be a subset with $|S| = n + k$. Then S contains at least k arcs and at most $n - k$ free points.*

Proof. There are at most n free points and we pick $n + k$ points, thus we have to pick at least k arcs. Then there remain $n + k - 2k$ points which can be free. ∎

LEMMA 4.4. *If $R \subset S \subset E_{2n}$ contains an arc of S, then $h_a(\epsilon_R^S) = 0$.*

Proof. This assertion follows from the fact that if i is connected to i' in a, then $a_i = a_{i'}$, and thus $h_a(x_i x_{i'}) = a_i \wedge a_{i'} = 0$. ∎

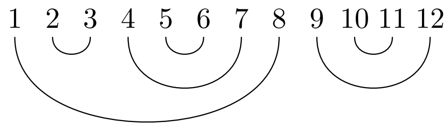
LEMMA 4.5. *For all $R \subset S \subset E_{2n}$ with $|R| > n$, one has $h_a(\epsilon_R^S) = 0$.*

Proof. There are at most n free points in S, but R contains at least $n+1$ points. So, R contains an arc and the result follows from the preceding lemma. ∎

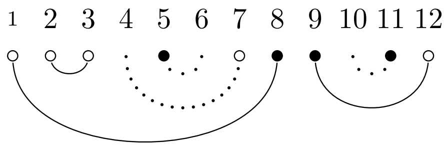
LEMMA 4.6. *For all $R \subset S \subset E_{2n}$ with $|S| = n + k$ and $|R| \geq n - k + 1$, there exists an arc (j, j') in S with j or j' in R.*

Proof. We have to choose $n - k + 1$ points in S, but there are at most $n - k$ free points by Lemma 4.3. ∎

EXAMPLE 4.7. It is probably time to stop here for a moment and look at an example that we will generalize below. So suppose

$a =$ 

with $n = 6$, $r = 4$, $k = 3$, $S = \{1, 2, 3, 5, 7, 8, 9, 11, 12\}$ and $R = \{5, 8, 9, 11\}$ $\subset S$. Putting a circle at the end points of a which are in S, a bullet at the end points of R, and drawing the arcs that are not in S with dotted lines we get



The free points of S are thus $\{5,7,11\}$. We have $\epsilon_R^S = (-x_5)(-x_8)x_9(-x_{11})$ and in $a(OH_C^n)a$ we obtain

$$h_a(\epsilon_R^S) = (-A_4) \wedge (-A_1) \wedge A_5 \wedge (-A_6),$$

where A_i corresponds to the ith circle in $W(a)a$ counting from the left (see the paragraph about diagrammatic notation in Section 3). Now we look how $h_a(\epsilon_R^S)$ behaves when we modify R.

Take $R' = \{5,8,11,12\}$, obtained from R by exchanging the end points of the 5th arc. Then we get $\epsilon_{R'}^S = (-x_5)(-x_8)(-x_{11})x_{12}$ and

$$h_a(\epsilon_{R'}^S) = (-A_4) \wedge (-A_1) \wedge (-A_6) \wedge A_5 = -h_a(\epsilon_R^S).$$

For $R'' = \{1,5,9,11\}$, obtained by replacing 8 with 1, we compute $\epsilon_{R''}^S = x_1(-x_5)x_9(-x_{11})$ and

$$h_a(\epsilon_{R''}^S) = A_1 \wedge (-A_4) \wedge A_5 \wedge (-A_6) = h_a(\epsilon_R^S).$$

Fortunately, by replacing 9 with 12 in R'' we get an element that will cancel with $h_a(\epsilon_{R''}^S)$. If we remove 11 or 9 from R and take 7 instead, then exchanging 8 with 1 will also create two elements that cancel with each other. If we take 2 or 3 instead of 7, then exchanging those points will also give 0. In general, the number of free points in S which are not in R plus the number of points in R that are not free, both between the end points of an arc, will say if exchanging those end points changes the sign in the image of h_a. We can observe that for each choice of R there exists a possibility to exchange two points in such a way that they cancel with each other in the image of h_a. At the end, we get $h_a(\epsilon_r^S) = 0$.

For $x \in S \subset E_{2n}$, we write

$$F_S(x) := \begin{cases} 1 & \text{if } x \text{ is free in } S, \\ 0 & \text{otherwise,} \end{cases}$$

and, for $R \subset S$ and (j,j') an arc of S such that $j \in R$ or $j' \in R$, we define

$$\begin{aligned} p_{R,S}((j,j')) &:= \sum_{\substack{x \in S\setminus R \\ j<x<j'}} F_S(x) + \sum_{\substack{y\in R \\ j<y<j'}} (1 - F_S(y)) \bmod 2 \\ &\equiv \sum_{\substack{x \in S \\ j<x<j'}} F_S(x) + \sum_{\substack{y\in R \\ j<y<j'}} 1 \bmod 2. \end{aligned}$$

We say that a point x (resp. an arc (k,k')) belongs to an arc (j,j') if $j < x < j'$ (resp. $j < k < k' < j'$). Therefore, $p_{R,S}((j,j'))$ counts the number of free points of S belonging to the arc (j,j') and which are not in R plus the number of points of R also belonging to (j,j') and which are not free. Denote by $]j,j'[_R$ the set of all subarcs of (j,j') with an extremity in R. Also, write $]j,j[_R^{\max}$ for the set of all maximal subarcs of (j,j') with an extremity in R, that is, the subarcs not belonging to any other arcs from $]j,j'[_R$.

Example 4.8. Suppose we take S and R as in Example 4.7. Then we obtain $p_{R,S}((1,8)) \equiv 3 \bmod 2$ and $p_{R,S}((9,12)) \equiv 2 \bmod 2$.

Lemma 4.9. *If no arc of $S \subset E_{2n}$ belongs to any $R \subset S$, then for every arc (j,j') in S with j or j' in R we have*

$$p_{R,S}((j,j')) = \sum_{(k,k')\in]j,j'[_R^{max}} (p_{R,S}((k,k'))+1) + \sum_{\substack{x\in S\setminus R \\ x\notin (k,k'),\, \forall (k,k')\in]j,j'[_R}} F_S(x) \bmod 2,$$

with the last sum over all $x \in S \setminus R$ which are not in any subarcs of $]j,j'[_R$.

Proof. All points belonging to a maximal subarc (k,k') from the left sum belong to (j,j'). Moreover, k or k' is a free point in R, and thus contributes $+1$. ∎

Lemma 4.10. *Let $R \subset S \subset E_{2n}$ be subsets with $|S| = n+k$ and $n-k+1 \le |R| \le n$. If no arc of S belongs to R, then there exists an arc (j,j') of S with j or j' in R and such that $p_{R,S}((j,j')) = 0 \bmod 2$.*

Proof. First, by Lemma 4.6 there is at least one arc of S with an extremity in R. We write $L \neq \emptyset$ for the set of all such arcs. Now, we suppose for contradiction that $p_{R,S} = 1 \bmod 2$ on all those arcs. By Lemma 4.9, for all $(j,j') \in L$ we get

$$p_{R,S}((j,j')) = \sum_{\substack{x\in S\setminus R \\ x\notin (k,k'),\, \forall (k,k')\in]j,j'[_R}} F_S(x) \bmod 2,$$

with x not belonging to any subarc (k,k') of (j,j') such that k or k' is in R. Since by the contradiction hypothesis this sum must be $1 \bmod 2$, there is at least one such x which is free, and thus at least $|L|$ free points in $S \setminus R$. However, we know that there are at least $n-k-|L|+1$ free points in R, and by Lemma 4.3 there are at most $n-k$ free points in S, which is a contradiction. ∎

Lemma 4.11. *Let $R \subset S \subset E_{2n}$ be subsets and (j,j') be an arc of S with $j \in R$ (resp. $j' \in R$). Then*

$$h_a(\epsilon_R^S) = (-1)^{(p_{R,S}((j,j'))+1)} h_a(\epsilon_{R'}^S),$$

with R' obtained by taking R where we replace j by j' (resp. j' by j), respecting the order of S.

Proof. We use induction on the size of $S\cap]j,j'[$ and R. If $S\cap]j,j'[= \emptyset$, then clearly $p_R((j,j')) = 0 \bmod 2$ and we get the result since $S(j') = S(j)+1$ and thus $x_j^S = -x_{j'}^S$. To get the result in the general case, we check how the sign of the exterior product changes when we add free points and arcs to S and points to R. We leave the details to the reader. ∎

Proposition 4.12. *For all $a \in B^n$, $k \in \{1,\dots,n\}$, $r \geq n-k+1$ and $S \subset E_{2n}$ such that $|S| = n+k$, we have*

$$h_a(\epsilon_r^S) = 0.$$

Proof. This directly follows from Lemmas 4.5, 4.4, 4.10, and 4.11. ∎

Corollary 4.13. *The map h_0 induces a homomorphism of graded (super)algebras given by*

$$h : OH(\mathfrak{B}_{n,n},\mathbb{Z}) \to OZ(OH^n), \qquad x_i \mapsto \sum_{a\in B^n} a_i.$$

Injectivity of h. We will need the following result.

Lemma 4.14. *The algebra homomorphism*

$$\overline{h} : OH(\mathfrak{B}_{n,n},\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to OZ(OH^n) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

induced by h is an isomorphism.

Proof. The assertion results from the commutativity of the diagram

$$\begin{array}{ccc} OH(\mathfrak{B}_{n,n},\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\overline{h}} & OZ(OH^n) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \\ \simeq \big\downarrow & & \big\downarrow \simeq \\ H(\mathfrak{B}_{n,n},\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\simeq} & Z(H^n) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \end{array}$$

which comes from the equivalence modulo 2 between the odd and the even cases, upon observing that $H(\mathfrak{B}_{n,n},\mathbb{Z})$ is isomorphic to $Z(H^n)$ by a morphism similar to h; see [21, Section 5.3] for more details. ∎

Proposition 4.15. *The homomorphism h is injective.*

Proof. From [24, Theorem 3.8] we know that $OH(\mathfrak{B}_{n,n},\mathbb{Z})$ is a free $\mathbb{Z}$-module. Hence the injectivity of h follows from the lemma above and the fact that if a $\mathbb{Z}$-linear map between two free $\mathbb{Z}$-modules induces an isomorphism over $\mathbb{Z}/2\mathbb{Z}$ then it is injective. ∎

Computing the rank of $OZ(OH^n)$. To show the existence of an isomorphism between $Z(H^n)$ and $H(\mathfrak{B}_{n,n})$, Khovanov [21] constructed a manifold $\widetilde{S}$ using products of 2-spheres. We make a similar construction, but using circles. The inspiration from Khovanov's work should be clear. All cohomology groups and rings in this section are supposed to be taken over $\mathbb{Z}$.

Definition 4.16. For an $a \in B^n$, let $T_a \subset T^{2n} := S^1 \times \overset{2n}{\cdots} \times S^1$ be the set of all points $(x_1, \dots, x_{2n}) \in T^{2n}$ such that if i is linked to j by an arc of a, then $x_i = x_j$. We also define

$$\widetilde{T} := \bigcup_{a \in B^n} T_a \subset T^{2n}.$$

One can notice that $T_a \simeq T^n$ as we equalize n pairs of coordinates. In the same spirit, we have $T_b \cap T_a \simeq T^{|W(b)a|}$, with $x_k = x_l$ whenever the kth and the lth end points are in the same component of $W(b)a$. As a result, $\widetilde{T}$ is a collection of hypertori identified with each other on certain subtori.

It is well-known that the cohomology ring of an n-torus is the exterior algebra generated by n elements of degree 1. If we forget the grading, then we get an isomorphism $a(OH^n_C)a \simeq H(T_a)$ of superrings and an isomorphism ${}_b(OH^n_C)_a \simeq_{ab} H(T_b \cap T_a)$ of abelian groups for all $a, b \in B^n$. Lifting the (super)ring structure from $H(T_b \cap T_a)$, we get a (super)ring structure on ${}_b(OH^n_C)_a$ and (super)ring morphisms

$$\gamma_{a;b,a} : a(OH^n_C)a \to b(OH^n_C)a, \quad x \mapsto {}_b1_a x,$$
$$\gamma_{b;b,a} : b(OH^n_C)b \to b(OH^n_C)a, \quad x \mapsto x_b 1_a.$$

The inclusions $T_b \cap T_a \subset T_a$ and $T_b \cap T_a \subset T_b$ induce ring morphisms on cohomology:

$$\psi_{a;b,a} : H(T_a) \to H(T_b \cap T_a), \qquad \psi_{b;b,a} : H(T_b) \to H(T_b \cap T_a).$$

Lemma 4.17. *The morphisms defined above are such that the following diagram of ring morphisms commutes:*

$$\begin{tikzcd}
H(T_b) \arrow[r, "\psi_{b;b,a}"] \arrow[d, "\simeq"] & H(T_b \cap T_a) \arrow[d, "\simeq"] & H(T_a) \arrow[l, "\psi_{a;b,a}"'] \arrow[d, "\simeq"] \\
b(OH^n_C)b \arrow[r, "\gamma_{b;b,a}"] & b(OH^n_C)a & a(OH^n_C)a \arrow[l, "\gamma_{a;b,a}"']
\end{tikzcd}$$

Proof. Say $H(T_b) \simeq \bigwedge^*\{t_1, \dots, t_n\}$. Then the map $\psi_{b;a,b}$ identifies t_k with t_l whenever the kth and the lth end points of b are in the same component of $W(b)a$. The map $\gamma_{b;b,a}$ does exactly the same on the generators of $b(OH^n_C)b$ as C_{bba} merges the corresponding components. ∎

Definition 4.18. Let I and J be finite sets and A_i, B_j be rings for all $i \in I$ and $j \in J$. Moreover, let $\beta_{i,j} : A_i \to B_j$ be ring morphisms for some pairs $(i, j) \in I \times J$ with

$$\beta := \sum \beta_{i,j} : \prod_{i \in I} A_i \to \prod_{j \in J} B_j.$$

We define the *equalizer* $\mathrm{Eq}(\beta)$ of β as the subring of $\prod A_i$ such that for

$(a_i)_{i\in I} \in \mathrm{Eq}(\beta)$ we have

$$\beta_{i,j}(a_i) = \beta_{k,j}(a_k)$$

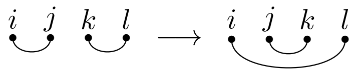
whenever $\beta_{i,j}$ and $\beta_{k,j}$ are defined.

By Proposition 3.12, we get $OZ(OH^n) = \mathrm{Eq}(\gamma)$ for $\gamma := \sum_{a\neq b} \gamma_{a;b,a} + \gamma_{b;b,a}$. Thus, if we define $\psi := \sum_{a\neq b} \psi_{a;b,a} + \psi_{b;b,a}$, then by Lemma 4.17, we get a commutative diagram

$$\begin{tikzcd} H(\widetilde{T}) \arrow[r, "\kappa"] & \mathrm{Eq}(\psi) \arrow[r] \arrow[d, "\simeq"] & \bigoplus_{a\in B^n} H(T_a) \arrow[r, "\psi"] \arrow[d, "\simeq"] & \bigoplus_{a\neq b\in B^n} H(T_b\cap T_a) \arrow[d, "\simeq"] \\ OZ(OH^n) \arrow[r, "\simeq"] & \mathrm{Eq}(\gamma) \arrow[r] & \bigoplus_{a\in B^n} a(OH^n_C)a \arrow[r, "\gamma"] & \bigoplus_{a\neq b\in B^n} b(OH^n_C)a \end{tikzcd} \tag{4.1}$$

with κ coming from the factorization by $\mathrm{Eq}(\psi)$ of the map $\phi : H(\widetilde{T}) \to \bigoplus_{a\in B^n} H(T_a)$ induced by the inclusions $T_a \hookrightarrow \widetilde{T}$. This factorization exists since $\operatorname{im}\phi \subset \mathrm{Eq}(\psi)$. Our goal now is to prove that κ is an epimorphism such that $\mathrm{rk}(OZ(OH^n)) \leq \mathrm{rk}(H(\widetilde{T}))$.

Definition 4.19. We say that *there is an arrow $a \to b$ for $a, b \in B^n$* if there exists a quadruplet $1 \leq i < j < k < l \leq 2n$ such that $(i,j),(k,l) \in a$ and $(i,l),(j,k) \in b$. Visually, we have



This leads to a partial order: $a \prec b$ if there exists a chain $a \to a_1 \to \cdots \to a_k \to b$. We extend (arbitrarily) this partial order to a total order $<$ on B^n.

Lemma 4.20. *For all $a \in B^n$, we have*

$$T_{<a} \cap T_a = \bigcup_{b\to a} (T_b \cap T_a)$$

with $T_{<a} := \bigcup_{b<a} T_b$.

Proof. We use similar arguments to those in [21, Lemma 3.4], replacing S by T. ∎

Lemma 4.21. *There exists a cellular decomposition of T_a which restricts to a decomposition of $T_{<a}\cap T_a$, which itself restricts to the decomposition of $T_b \cap T_a$ for all $b \to a$. This decomposition is such that there are $\binom{n}{k}$ cells of dimension k in T_a.*

Proof. We construct a similar decomposition to that in [21, Lemma 3.5]. We stress that now the cells are not in even degree only. ∎

COROLLARY 4.22. *The morphism*

$$H(T_{<a}\cap T_a)\to \bigoplus_{b<a} H(T_b\cap T_a)$$

induced by the inclusions $T_b\cap T_a\subset T_{<a}\cap T_a$ *is injective.*

We remark that $T_{\leq a}=T_{<a}\cup T_a$, so there is a Mayer–Vietoris sequence

$$\begin{tikzcd}[ampersand replacement=\&] \& \& \cdots \arrow[r] \& H^{m-1}(T_a\cap T_{<a}) \arrow[dll, "\delta"'] \\ \& H^m(T_a\cup T_{<a}) \arrow[r] \& H^m(T_a)\oplus H^m(T_{<a}) \arrow[r] \& H^m(T_a\cap T_{<a}) \arrow[dll, "\delta"'] \\ \& H^{m+1}(T_a\cup T_{<a}) \arrow[r] \& \cdots. \& \end{tikzcd} \tag{4.2}$$

PROPOSITION 4.23. *The following sequence is exact:*

$$H(T_{\leq a})\xrightarrow{\phi}\bigoplus_{b\leq a} H(T_b)\xrightarrow{\psi^-}\bigoplus_{b<c\leq a} H(T_b\cap T_c),$$

where ϕ *is induced by the morphisms* $T_b\hookrightarrow T_{\leq a}$, *and where we define*

$$\psi^- := \sum_{b<c\leq a}(\psi_{b,c}-\psi_{c,b})$$

with $\psi_{b,c}=\psi_{b;b,c}: H(T_b)\to H(T_b\cap T_c)$ *induced by the inclusion* $T_b\cap T_c\hookrightarrow T_b$.

Proof. We proceed by induction on a using Corollary 4.22 as in [21, Proposition 3.8] with the only difference that we lose the left part $0\to$ in our sequence. ∎

PROPOSITION 4.24. *There is an epimorphism of superrings*

$$k: H(\widetilde{T})\to OZ(OH^n).$$

Proof. We take a maximal in Proposition 4.23 giving an exact sequence

$$H(\widetilde{T})\xrightarrow{\phi}\bigoplus_{b} H(T_b)\xrightarrow{\psi^-}\bigoplus_{b<c} H(T_b\cap T_c),$$

and we observe that by definition $\mathrm{Eq}(\psi)=\ker\psi^-=\operatorname{im}\phi$, so that $\kappa: H(\widetilde{T})\to\mathrm{Eq}(\psi)$ from diagram (4.1) is surjective. ∎

Now we show that $H(\widetilde{T})$ is a free abelian group with the same rank as $OH(\mathfrak{B}_{n,n},\mathbb{Z})$. Thanks to [24, Corollary 3.9], we already know that

$$\mathrm{rk}(OH(\mathfrak{B}_{n,n},\mathbb{Z}))=\binom{2n}{n}.$$

LEMMA 4.25. *For all $k \geq 0$, the cohomology groups*

$$H^k(T_a \cup T_{<a}), \quad H^k(T_a), \quad H^k(T_{<a}), \quad H^k(T_a \cap T_{<a})$$

are free and their ranks satisfy

$$\mathrm{rk}(H^k(T_a \cup T_{<a})) = \mathrm{rk}(H^k(T_a)) + \mathrm{rk}(H^k(T_{<a})) - \mathrm{rk}(H^k(T_a \cap T_{<a})).$$

Proof. We use induction on B^n. We consider the Mayer–Vietoris sequence (4.2) and we claim that the morphisms $H^k(T_a) \to H^k(T_a \cap T_{<a})$ are surjective, and thus the boundary operators δ are zero. Indeed, the decomposition from Lemma 4.21 has the same number of k-cells as the rank of $H^k(T_a)$, so characteristic functions on them are independent generators for the cohomology. Since the cell decomposition restricts to $T_a \cap T_{<a}$, the cohomology groups $H^k(T_a \cap T_{<a})$ are free with ranks given by the cell decomposition, and the morphisms are surjective. This gives an isomorphism

$$H^k(T_a \cap T_{<a}) \simeq \frac{H^k(T_a) \oplus H^k(T_{<a})}{H^k(T_a \cup T_{<a})}.$$

If a is minimal, then the lemma is trivial. For the general case, we get the result by induction since $H^k(T_a)$, $H^k(T_{<a})$ and $H^k(T_a \cap T_{<a})$ are free, and so is $H^k(T_a \cup T_{<a})$. ∎

PROPOSITION 4.26. *$H(\widetilde{T})$ is a free abelian group of rank*

$$\mathrm{rk}(H(\widetilde{T})) = \binom{2n}{n}.$$

Proof. We obtain a cellular partition of $\widetilde{T}$ by first taking the cellular decomposition of T_{a_0} from Lemma 4.21, with $a_0 \in B^n$ the minimal element, and then by adding the cells $T_{a_m} \setminus T_{<a_m}$ for all $a_m \in B^n$ following the total order. We claim that the rank of $H(\widetilde{T})$ is the number of cells of the partition. Indeed, all cohomology groups are free and the relation from Lemma 4.25 gives the claim since $\mathrm{rk}(H(T_{a_m})) - \mathrm{rk}(H(T_{a_m} \cap T_{<a_m}))$ counts exactly the number of cells of $T_{a_m} \setminus T_{<a_m}$. Finally, as in [21] (and as proved in [27, Lemma 3.64]), the number of cells is $\binom{2n}{n}$. ∎

COROLLARY 4.27. *$OZ(OH^n)$ is a free abelian group and*

$$\mathrm{rk}(OZ(OH^n)) = \mathrm{rk}(OH(\mathfrak{B}_{n,n}, \mathbb{Z})).$$

Proof. By Proposition 4.15, we have

$$\mathrm{rk}(OZ(OH^n)) \geq \mathrm{rk}(OH(\mathfrak{B}_{n,n}, \mathbb{Z}))$$

and by Propositions 4.24 and 4.26, and [24, Corollary 3.9], we get

$$\mathrm{rk}(OZ(OH^n)) \leq \mathrm{rk}(H(\widetilde{T})) = \binom{2n}{n} = \mathrm{rk}(OH(\mathfrak{B}_{n,n}, \mathbb{Z})). \ ∎$$

Proof of Theorem 4.1. In order to prove Theorem 4.1, we construct a surjective homomorphism $\pi : OH(\mathfrak{B}_{n,n}, \mathbb{Z}) \to H(\widetilde{T})$ such that the following diagram commutes:

$$\begin{tikzcd} OH(\mathfrak{B}_{n,n}, \mathbb{Z}) \arrow[rr, "h"] \arrow[dr, "\pi"'] & & OZ(OH^n) \\ & H(\widetilde{T}) \arrow[ur, "k"'] & \end{tikzcd}$$

Following [21, Section 4], let $\imath^* : H(T^{2n}) \to H(\widetilde{T})$ be the homomorphism induced by the inclusion $\widetilde{T} \subset T^{2n}$.

Lemma 4.28. *The map* $\imath^* : H(T^{2n}) \twoheadrightarrow H(\widetilde{T})$ *is an epimorphism.*

Proof. The map induced on homology $H_1(\widetilde{T}) \hookrightarrow H_1(T^{2n})$ is split injective, and both groups are free. Indeed the cellular decomposition of $\widetilde{T}$ has all 1-dimensional cells given by linearly independent diagonals in T^{2n}, which can be completed to a basis for $H_1(T^{2n})$. Hence, by the universal coefficient theorem the map induced on cohomology $H^1(T^{2n}) \twoheadrightarrow H^1(\widetilde{T})$ is surjective. The same applies for all $H^k(T^{2n}) \to H^k(\widetilde{T})$, with linearly independent hyperplanes of dimension k. Indeed, cells of $T_a \setminus T_{<a}$ are given by hyperplanes with at least one generating vector linearly independent of the cells of $T_{<a}$. ∎

We write $X_i \in H(\widetilde{T})$ for the image by $\imath^*$ of the generating element in the cohomology of T^{2n} given by the characteristic function on the ith component. In other words, if p_i is the projection $T^{2n} \to S^1$ onto the ith component, then $X_i = i^* \circ p_i^*(X)$, where $H(S^1) = \bigwedge^*\{X\}$.

Then we define a ring homomorphism $\pi_0 : OPol_{2n} \to H(\widetilde{T})$ by $\pi_0(x_i) = X_i$. Clearly we have an isomorphism $H(T^{2n}) \cong \bigwedge^*\{p_1^*(X), \dots, p_{2n}^*(X)\}$, and so π_0 is surjective by the lemma above.

Lemma 4.29. *The map* $k : H(\widetilde{T}) \to OZ(OH^n)$ *is an isomorphism.*

Proof. It is an epimorphism between free abelian groups of the same rank. ∎

From this, we deduce the homomorphism $j^* : H(\widetilde{T}) \to \bigoplus_{a \in B^n} H(T_a) \cong \bigoplus_{a \in B^n} a(OH_C^n)a$, induced by the inclusions $T_a \subset \widetilde{T}$, is an injection. Moreover, by construction of π_0, the following diagram commutes:

$$\begin{tikzcd} OPol_{2n} \arrow[rr, "h_0"] \arrow[dr, "\pi_0"'] & & \bigoplus_{a \in B^n} a(OH_C^n)a \\ & H(\widetilde{T}) \arrow[ur, "j^*"'] & \end{tikzcd}$$

Lemma 4.30. *We have* $\pi_0(OC_n) = 0$.

Proof. By the commutativity of the diagram above, we get $j^* \circ \pi_0(OC_n) = h_0(OC_n)$, which is zero by Proposition 4.12. This concludes the proof since j^* is injective. ∎

Therefore, there is an induced map $\pi : OH(\mathfrak{B}_{n,n}, \mathbb{Z}) \to H(\widetilde{T})$ given by $\pi(x_i) = X_i$. Since π_0 is surjective, so is π.

Proof of Theorem 4.1. By construction, $h = k \circ \pi$. Since k and π are both surjective, h is surjective as well. By Proposition 4.15, it is also injective and thus an isomorphism. ∎

5. Turning OH_C^n into an associative algebra. In this section, we show that we can twist the multiplication of OH_C^n, turning it into an associative $\mathbb{Z}[i]$-algebra. To do so, we begin by proving that OH_C^n is a quasialgebra in the sense of Albuquerque–Majid [1], graded by a groupoid as in [29]. Finally, we give some classification results for such algebras.

5.1. The Putyra–Shumakovitch associator. The material in this subsection is due to Putyra and Shumakovitch [31] ([1]).

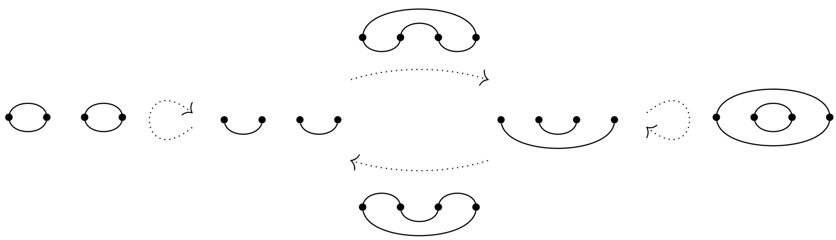
Grading by a groupoid. A *groupoid* is a small category with every morphism admitting an inverse. We say that a ring R is *graded* by a groupoid $\mathcal{G}$ if

$$R = \bigoplus_{g \in \mathrm{Hom}(\mathcal{G})} R_g, \quad \text{and} \quad R_{g_1} R_{g_2} \subset R_{g_1 \circ g_2}$$

whenever g_1 and g_2 are composable, and $R_{g_1} R_{g_2} = 0$ otherwise.

Arc grading. Let $\mathcal{G}^n$ be the groupoid with objects given by the elements of B^n and with a unique morphism $a \to b$ for all $a, b \in B^n$. By uniqueness of the morphisms, the composition is such that $a \to b \to c$ is equal to $a \to c$. We can view the morphism $a \to b$ as the diagram $W(b)a$ with the composition defined for all $a, b, c \in B^n$ by $W(c)b \circ W(b)a = W(c)a$. It is clear that $\mathcal{G}^n$ is a groupoid as every morphism $a \to b$ has an inverse $b \to a$ and $a \to a$ is the identity.

Example 5.1. We can put $\mathcal{G}^n$ in the form of a diagram. For example, $\mathcal{G}^2$ can be pictured as



([1]) And we would like to thank Krzysztof Putyra for explaining it to us.

The decomposition

$$OH^n = \bigoplus_{a,b\in B^n} b(OH^n)a = \bigoplus_{W(b)a\in\mathrm{Hom}(\mathcal{G}^n)} b(OH^n)a$$

gives a grading of OH^n_C by $\mathcal{G}^n$. The integer degree, coming from the grading of $\bigwedge^* V(S)$, will be called the *quantum degree* and written $|\cdot|_q$. The degree coming from the grading of $\mathcal{G}^n$ will be called the *arc degree* and denoted $|\cdot|_B$. We get a bigrading, written $|\cdot|$, by the groupoid $\mathcal{G}^n \times \mathcal{Z}$, with $\mathcal{Z}$ being the abelian group $\mathbb{Z}$ viewed as a category with one abstract object $\star$, morphisms given by the integers and composition obtained by taking the sum, i.e.

$$\star \xrightarrow{z_1} \star \xrightarrow{z_2} \star = \star \xrightarrow{z_1+z_2} \star.$$

Notice that $\mathrm{Hom}(\mathcal{G}^n \times \mathcal{Z}) \simeq \mathrm{Hom}(\mathcal{G}^n) \times \mathbb{Z}$.

As a matter of fact, OH^n_C is graded by a subgroupoid with arrows given by diagrams $W(b)a$ and a quantum degree in $2\mathbb{Z}$ or $2\mathbb{Z}+1$ depending on whether $|W(b)a| \equiv n \bmod 2$ or not.

Definition 5.2. We denote by $\mathcal{G}^n \times \mathcal{Z}_2$ the groupoid given by the same objects as $\mathcal{G}^n \times \mathcal{Z}$, but with hom-spaces defined as

$$\hom((a,\star),(b,\star)) = \left\{ (W(b)a, k) \;\middle|\; \begin{matrix} k \in 2\mathbb{Z} & \text{if } |W(b)a| \equiv n \bmod 2, \\ k \in 2\mathbb{Z}+1 & \text{otherwise.} \end{matrix} \right\}.$$

Quasialgebras. Recall that we proved in Proposition 3.2 that OH^n_C is not an associative algebra. We claim that it is almost one: it is a quasialgebra in the sense of [1] but graded by a groupoid as in [29]. Before defining quasialgebras, recall that the *nerve* of a category $\mathcal{C}$ is the simplicial set generated by its morphisms. We denote it by $N(\mathcal{C})$, and $N_k(\mathcal{C})$ is the set of compositions of k morphisms in $\mathcal{C}$.

Definition 5.3. A *quasialgebra* A is a (non-associative) R-algebra graded by a groupoid $\mathcal{G}$ with a 3-cocycle

$$\phi : N_3(\mathcal{G}) \to R^*,$$

where $R^* \subset R$ are the invertible elements, such that

$$(xy)z = \phi(|x|,|y|,|z|)x(yz) \tag{5.1}$$

for all $x, y, z \in A$ (homogeneous with compatible degrees). We call ϕ the *associator* of A.

Remark 5.4. The condition of ϕ being a 3-cocycle means

$$\phi(h,k,l)\phi(g,hk,l)\phi(g,h,k) = \phi(gh,k,l)\phi(g,h,kl)$$

for all sequences $e \xleftarrow{g} d \xleftarrow{h} c \xleftarrow{k} b \xleftarrow{l} a \in \mathcal{G}$. We also require $\phi(h, \mathrm{id}_b, l) = 1$ whenever id_b is an identity morphism. Hence the following diagram commutes:

(5.2)

$$\begin{tikzcd}
((A\otimes A)\otimes A)\otimes A \arrow[rr, "\phi\otimes\mathrm{id}"] \arrow[d, "\phi"'] & & (A\otimes(A\otimes A))\otimes A \arrow[d, "\phi"] \\
(A\otimes A)\otimes(A\otimes A) \arrow[dr, "\phi"'] & & A\otimes((A\otimes A)\otimes A) \arrow[dl, "\mathrm{id}\otimes\phi"] \\
& A\otimes(A\otimes(A\otimes A)) &
\end{tikzcd}$$

Notice that if we have a non-associative R-algebra A graded by a groupoid $\mathcal{G}$ and if a $\phi : N_3(\mathcal{G}) \to R^*$ respecting (5.1) exists, then its definition on $\deg A := \{\deg(x) \in \mathrm{Hom}(\mathcal{G}) \mid x \in A\}$ is forced by (5.1). Moreover, it will be a 3-cocycle on this subgroupoid since (5.2) must commute for the multiplication in A to be well-defined.

We can view $\mathbb{Z}^* = \{\pm 1\}$ as the group $\mathbb{Z}/2\mathbb{Z}$ by the isomorphism $x \in \mathbb{Z}/2\mathbb{Z} \mapsto (-1)^x$. Because of that, we will write the associator of OH^n_C as a map with codomain $\mathbb{Z}/2\mathbb{Z}$.

Lemma 5.5. *There exists a unique map*

$$\varphi^{\mathrm{ch}}_C : N_3(\mathcal{G}^n) \to \mathbb{Z}/2\mathbb{Z}$$

such that for all $a, b, c, d \in B^n$ we have

$$OF(C_{dba} \circ C_{dcb}\,\mathrm{Id}_{W(b)a}) = (-1)^{\varphi^{\mathrm{ch}}_C(W(d)c, W(c)b, W(b)a)}\, OF(C_{dca} \circ \mathrm{Id}_{W(d)c}\, C_{cba}).$$

Proof. The two cobordisms $C_{dba} \circ C_{dcb}\,\mathrm{Id}_{W(b)a}$ and $C_{dca} \circ \mathrm{Id}_{W(d)c}\, C_{cba}$ have the same Euler characteristic and the same source and target, and thus are homeomorphic. This means they are related by changes of chronology and orientations, which induce only potential changes of sign. ∎

Lemma 5.6. *For all choices $C \in \mathcal{C}^n$, the cobordism C_{cba} is composed of the same number of splits.*

Proof. This is immediate as all C_{cba} have the same Euler characteristic. ∎

We define

$$\varphi^{\mathrm{com}} : N_3(\mathcal{G}^n \times \mathcal{Z}) \subset (\mathrm{Hom}(\mathcal{G}^n) \times \mathbb{Z})^3 \to \mathbb{Z}/4\mathbb{Z}$$

by

$$((W(d)c, k), (W(c)b, l), (W(b)a, m)) \\ \mapsto (k - n + |W(d)c|)s(W(c)b, W(b)a) \bmod 4,$$

where $s(W(c)b, W(b)a)$ is the number of splits coming from the cobordism C_{cba}, which does not depend on C by Lemma 5.6. If we take $x, y, z \in OH^n_C$, then

$$\varphi^{\mathrm{com}}(|x|,|y|,|z|) = 2p(x)s(|y|_B,|z|_B). \tag{5.3}$$

In this spirit, we extend the definition of parity to any $(W(d)c,k) \in \mathrm{Hom}(\mathcal{G}^n \times \mathcal{Z})$ by setting $p((W(d)c,k)) := (k-n+|W(d)c|)/2 \in \{0,1/2,1,3/2\}$. Note that the parity gives an integer for every element in the subgroupoid $\mathcal{G}^n \times \mathcal{Z}_2$ from Definition 5.2.

Lemma 5.7. *The map*

$$\varphi_C := \varphi_C^{\mathrm{ch}} + \varphi^{\mathrm{com}}/2 : N_3(\mathcal{G}^n \times \mathcal{Z}_2) \to \mathbb{Z}/2\mathbb{Z}$$

is such that

$$(xy)z = (-1)^{\varphi_C(|x|,|y|,|z|)} x(yz)$$

for all $x, y, z \in OH^n_C$.

Proof. Suppose $x \in d(OH^n_C)c$, $y \in c(OH^n_C)b$ and $z \in b(OH^n_C)a$. We compute

$$\begin{aligned} xy &= S(d,c,b) \wedge x \wedge y, \\ (xy)z &= S(d,b,a) \wedge S(d,c,b) \wedge x \wedge y \wedge z, \\ yz &= S(c,b,a) \wedge y \wedge z, \\ x(yz) &= S(d,c,a) \wedge x \wedge S(c,b,a) \wedge y \wedge z, \end{aligned}$$

where $S(d,c,b)$ are the terms coming from the splits of the cobordism C_{dcb}. Note that we abuse notation by identifying x, y with their images in $d(OH^n_C)b$ in the first line, and so on.

This computation means that the non-associativity comes from two phenomena:

- The commutation between the elements coming from the splits of the product yz and the left term x, that is,

  $$\begin{aligned} &S(d,c,a) \wedge x \wedge S(c,b,a) \wedge y \wedge z \\ &\qquad = (-1)^{p(x)p(S(c,b,a))} S(d,c,a) \wedge S(c,b,a) \wedge x \wedge y \wedge z. \end{aligned}$$

  By (5.3), we have

  $$p(x)p(S(c,b,a)) = p(x)s(W(c)b, W(b)a) = \varphi^{\mathrm{com}}(|x|,|y|,|z|)/2.$$

- The change of chronology and orientations between the cobordisms $C_{dba} \circ C_{dcb}\,\mathrm{Id}_{W(b)a}$ and $C_{dca} \circ \mathrm{Id}_{W(d)c}\,C_{cba}$, meaning that

  $$S(d,b,a) \wedge S(d,c,b) = (-1)^{\varphi_C^{\mathrm{ch}}(|x|_B,|y|_B,|z|_B)} S(d,c,a) \wedge S(c,b,a)$$

  by Lemma 5.5.

To conclude, we have

$$(xy)z = (-1)^{\varphi_C^{\mathrm{ch}}(|x|_B,|y|_B,|z|_B) + \varphi^{\mathrm{com}}(|x|,|y|,|z|)/2} x(yz). \quad \blacksquare$$

LEMMA 5.8. *The map φ_C is a 3-cocycle. More generally, the map*

$$\psi_C := 2\varphi_C^{\mathrm{ch}} + \varphi^{\mathrm{com}} : N_3(\mathcal{G}^n \times \mathcal{Z}) \to \mathbb{Z}/4\mathbb{Z}$$

is a 3-cocycle.

Proof. We mainly use Remark 5.4. Take $a, b, c, d, e \in B^n$. Substituting ${}_e1_d$, ${}_d1_c$, ${}_c1_b$ and ${}_b1_a$ in (5.2), we get

$$\mathrm{ch}_{edcb} + \mathrm{ch}_{edba} + \mathrm{ch}_{dcba} = s_{edc}s_{cba} + \mathrm{ch}_{ecba} + \mathrm{ch}_{edca} \tag{5.4}$$

where $s_{cba} = s(W(c)b, W(b)a)$, $\mathrm{ch}_{dcba} = \phi_C^{\mathrm{ch}}(W(d)c, W(c)b, W(b)a)$, and so on. Now suppose we have a sequence $(e,\star) \xleftarrow{g} (d,\star) \xleftarrow{h} (c,\star) \xleftarrow{k} (b,\star) \xleftarrow{l} (a,\star) \in \mathcal{G}^n \times \mathcal{Z}$. We compute

$$\begin{aligned}\psi_C(h,k,l) &+ \psi_C(g,hk,l) + \psi_C(g,h,k)\\ &= 2(p(h)s_{cba} + \mathrm{ch}_{dcba} + p(g)s_{dba} + ch_{edba} + p(g)s_{dcb} + \mathrm{ch}_{edcb}),\end{aligned}$$

and

$$\psi_C(gh,k,l) + \psi_C(g,h,kl) = 2\big(p(gh)s_{cba} + \mathrm{ch}_{ecba} + p(g)s_{dca} + \mathrm{ch}_{edca}\big).$$

It is easy to see that $p(gh) = p(g) + p(h) + s_{edc}$ and

$$s_{dba} + s_{dcb} = s_{cba} + s_{dca} = \#\text{splits } W(d)cW(c)bW(b)a \to W(d)a,$$

so that by (5.4) we get

$$\psi_C(h,k,l) + \psi_C(g,hk,l) + \psi_C(g,h,k) = \psi_C(gh,k,l) + \psi_C(g,h,kl),$$

which concludes the proof for ψ_C. We get the claim for φ_C by seeing that $\varphi_C = \psi_C|_{\mathcal{G}\times\mathcal{Z}_2}/2$. ∎

THEOREM 5.9. *The non-associative ring OH_C^n is a quasialgebra with associator φ_C.*

Proof. This is an immediate consequence of Lemmas 5.7 and 5.8. ∎

We call φ_C the *Putyra–Shumakovitch associator.*

5.2. Twisting OH_C^n

Twisted multiplication. The idea of twisting a $\mathcal{G}$-graded R-algebra A by a map $\tau : N_2(\mathcal{G}) \to R^*$ is to define a new algebra A_τ to have the same elements as A, but with a multiplication given by

$$A_\tau \otimes_R A_\tau \to A_\tau, \qquad (x,y) \mapsto x *_\tau y := \tau(|x|,|y|)xy,$$

for all $x, y \in A_\tau$, where xy is the product in A.

PROPOSITION 5.10. *Let A be a quasialgebra graded by $\mathcal{G}$ with associator ϕ. If ϕ is a coboundary, then there exists a twist τ such that A_τ is associative.*

Proof. By definition of coboundary, there exists a map $\tau : N_2(\mathcal{G}) \to R^*$ such that

$$\phi(g,h,k) = \tau(g,h)\tau(g,hk)^{-1}\tau(gh,k)\tau(h,k)^{-1} \tag{5.5}$$

for all sequences $d \xleftarrow{g} c \xleftarrow{h} b \xleftarrow{k} a \in \mathcal{G}$. Let A_τ be the twisting of A by this τ. Then

$$\begin{aligned}(x *_\tau y) *_\tau z &= \tau(|x|,|y|)\tau(|xy|,|z|)(xy)z,\\ x *_\tau (y *_\tau z) &= \tau(|x|,|yz|)\tau(|y|,|z|)x(yz),\end{aligned}$$

for all $x,y,z \in A_\tau$, and thus, by (5.1) and (5.5), we conclude that A_τ is associative. ■

The geometric realization of the nerve of a category $\mathcal{C}$, denoted $|N(\mathcal{C})|$, is a topological space constructed by gluing simplices respecting the simplicial structure of the nerve.

LEMMA 5.11. *The geometric realization of $N(\mathcal{G}^n)$ is a simplex of dimension $C_n - 1$, with C_n the nth Catalan number:*

$$|N(\mathcal{G}^n)| \simeq \Delta^{(C_n-1)}.$$

Proof. The proof is immediate from the fact that B^n has cardinality C_n, and that $\mathcal{G}^n$ has a unique morphism between each pair of objects. ■

LEMMA 5.12. *The cohomology groups of $\mathcal{G}^n \times \mathcal{Z}$ are*

$$\begin{aligned}H^0(N(\mathcal{G}^n \times \mathcal{Z}), \mathbb{Z}/4\mathbb{Z}) &\simeq \mathbb{Z}/4\mathbb{Z},\\ H^1(N(\mathcal{G}^n \times \mathcal{Z}), \mathbb{Z}/4\mathbb{Z}) &\simeq \mathbb{Z}/4\mathbb{Z},\\ H^{\geq 2}(N(\mathcal{G}^n \times \mathcal{Z}), \mathbb{Z}/4\mathbb{Z}) &\simeq 0.\end{aligned}$$

Proof. First, by Lemma 5.11, we find that $|N(\mathcal{G}^n \times \mathcal{Z})| \simeq \Delta^m \times S^1$ for $m = C_n - 1$. By the Künneth formula, we get

$$H^k(N(\mathcal{G}^n \times \mathcal{Z}), \mathbb{Z}/4\mathbb{Z}) \simeq \bigoplus_{i+j=k} H^i(\Delta^m, \mathbb{Z}/4\mathbb{Z}) \otimes_{\mathbb{Z}/4\mathbb{Z}} H^j(S^1, \mathbb{Z}/4\mathbb{Z}),$$

which proves the claim. ■

Some technicalities still remain to be solved, before we can apply Proposition 5.10 to OH^n_C: we do not know the cohomology of $\mathcal{G}^n \times \mathcal{Z}_2$, and thus we are not able to show that the Putyra–Shumakovitch associator is a coboundary. Therefore, we work with ψ_C, which has an image in $\mathbb{Z}/4\mathbb{Z}$ and gives square roots of -1. Hence, we must consider the extended version $OH^n_C \otimes_{\mathbb{Z}} \mathbb{Z}[i]$ to the Gaussian integers, with $\mathbb{Z}[i]^* = \{1, i, -1, -i\} \simeq \mathbb{Z}/4\mathbb{Z}$. By Lemmas 5.7 and 5.8, $OH^n_C \otimes_{\mathbb{Z}} \mathbb{Z}[i]$ is a quasialgebra graded by $\mathcal{G}^n \times \mathcal{Z}$ with ψ_C as associator.

THEOREM 5.13. *For all $C \in \mathcal{C}^n$ there exists a map $\tau_C : N_2(\mathcal{G}^2 \times \mathcal{Z}) \to \mathbb{Z}/4\mathbb{Z}$ such that the twisted algebra $(OH^n_C \otimes_{\mathbb{Z}} \mathbb{Z}[i])_{\tau_C}$ is associative.*

Proof. By Lemma 5.12, we have $H^3(N(\mathcal{G}^n \times \mathcal{Z}), \mathbb{Z}/4\mathbb{Z}) \simeq 0$, and thus every 3-cocycle is a coboundary. In particular, the associator ψ_C is a coboundary and we can apply Proposition 5.10. ∎

Remark 5.14. The twist is not necessarily unique and thus we potentially get a family of associative algebras for each $C \in \mathcal{C}^n$.

Example 5.15. We construct an explicit $\left(OH^2_C\right)_{\tau_C}$ based on the choice C from Remark 3.1. We twist this algebra by setting

$$\tau_C((\quad, k), (\quad, \ell)) = i^k, \quad \tau_C((\quad, k), (\quad, \ell)) = i^{k-1},$$

for $k, \ell \in \mathbb{Z}$, and $\tau_C = 1$ everywhere else. In short, we get the multiplication table

$(OH^2_C)_{\tau_C}$						
		0		0	$-$	0
		$-$	0	0	$-$	0
		0	0	0	0	0
				0	$-$	$-$
		0	0	0		0

for a, and the one for b stays the same. An exhaustive computation (which can easily be done by computer) confirms that $d\tau_C$ gives the associator in this case.

Remark 5.16. For this example, the twisting in OH^n_C results in integer coefficients, which is not surprising since the geometric realization of $\mathcal{G}^2 \times \mathcal{Z}_2$ has dimension 2, and thus there exists a twist for the associator φ_C in this case. In general, this is not true and we lose the property that the algebra agrees modulo 2 with $H^n \otimes_{\mathbb{Z}} \mathbb{Z}[i]$.

5.3. Classification results. For now, we have a family $\{OH^n_C\}$ of quasialgebras indexed by $\mathcal{C}^n$ and a family of associative algebras indexed by $\mathcal{C}^n$ and the twists. In this section, we partially classify these families.

Proposition 5.17. *Let C, C' be two choices in $\mathcal{C}^n$ and $\varphi_C, \varphi_{C'}$ be the respective associators of OH^n_C and $OH^n_{C'}$. If $\varphi_C = \varphi_{C'}$, then the two quasialgebras are isomorphic, $OH^n_C \simeq OH^n_{C'}$.*

Proof. Since C_{cba} and C'_{cba} are related by a change of chronology and orientations, there is a map $\eta_{C,C'} : N_2(\mathcal{G}^n) \to \mathbb{Z}/2\mathbb{Z}$ such that

$$OF(C_{cba}) = (-1)^{\eta_{C,C'}(W(c)b,W(b)a)} OF(C'_{cba})$$

for all $a, b, c \in B^n$. Writing $*_C$ for the product in OH^n_C and $*_{C'}$ for the one in $OH^n_{C'}$, this means that $x *_C y = (-1)^{\eta_{C,C'}(|x|_B,|y|_B)} x *_{C'} y$ for all $x, y \in OH^n$. We compute

$$\begin{aligned}
&OF(C_{dba} \circ C_{dcb} \operatorname{Id}_{W(b)a}) \\
&\quad = (-1)^{\eta_{C,C'}(W(d)b,W(b)a)+\eta_{C,C'}(W(d)c,W(c)b)} OF(C'_{dba} \circ C'_{dcb} \operatorname{Id}_{W(b)a}), \\
&OF(C_{dca} \circ \operatorname{Id}_{W(d)c} C_{cba}) \\
&\quad = (-1)^{\eta_{C,C'}(W(d)c,W(c)a)+\eta_{C,C'}(W(c)b,W(b)a)} OF(C'_{dca} \circ \operatorname{Id}_{W(d)c} C'_{cba}),
\end{aligned}$$

so that by definition of the associators, we get

$$d\eta_{C,C'} = \varphi^{\mathrm{ch}}_{C'} - \varphi^{\mathrm{ch}}_{C}, \tag{5.6}$$

and thus, as $\varphi^{\mathrm{ch}}_C = \varphi^{\mathrm{ch}}_{C'}$ by hypothesis, $\eta_{C,C'}$ is a 2-cocycle. By Lemma 5.11, we know that $H^2(N(\mathcal{G}^n), \mathbb{Z}/2\mathbb{Z}) \simeq 0$, and so $\eta_{C,C'}$ is a coboundary. Hence, $\eta_{C,C'} = d\lambda_{C,C'}$ for some $\lambda_{C,C'} : N_1(\mathcal{G}^n) \to \mathbb{Z}/2\mathbb{Z}$. This means that the morphism

$$x \mapsto \lambda_{C,C'}(|x|_B)x : OH^n_C \to OH^n_{C'}$$

is an isomorphism of quasialgebras. ∎

For the associative twisted algebra, the case is much simpler and all algebras are isomorphic. This means that the choice of C and of the twist τ_C does not matter in the end.

Proposition 5.18. *For all choices $C, C' \in \mathcal{C}^n$ and all choices of twists $\tau_C, \tau_{C'}$, there is an isomorphism $(OH^n_C \otimes_{\mathbb{Z}} \mathbb{Z}[i])_{\tau_C} \simeq (OH^n_{C'} \otimes_{\mathbb{Z}} \mathbb{Z}[i])_{\tau_{C'}}$.*

Proof. For all $x, y \in OH^n$, we have

$$x *_{\tau_C} y = i^{\tau_C(|x|,|y|)+2\eta_{C,C'}(|x|_B,|y|_B)-\tau_{C'}(|x|,|y|)} x *_{\tau_{C'}} y,$$

where $*_{\tau_C}$ is the product in $(OH^n_C \otimes_{\mathbb{Z}} \mathbb{Z}[i])_{\tau_C}$, and we write

$$\theta_{C,C'} := \tau_C + 2\eta_{C,C'} - \tau_{C'} : N_2(\mathcal{G}^n \times \mathcal{Z}) \to \mathbb{Z}/4\mathbb{Z}.$$

We compute $d\theta_{C,C'} = \psi_C + 2d\eta_{C,C'} - \psi_{C'} \overset{(5.6)}{=} 0$, and by using similar arguments to those in the proof of Proposition 5.17, we conclude that the two algebras are isomorphic. ∎

Corollary 5.19. *The associative twisted algebra is uniquely determined up to isomorphism. We denote it OH^n_τ.*

Remark 5.20. Finding a twist is not an easy task, which can entail some serious difficulties for the construction of OH^n_τ.

PROPOSITION 5.21. *The following three arc algebras are not isomorphic:*

$$H^n \otimes \mathbb{Z}[i] \not\simeq OH^n_\tau \not\simeq OH^n_C \otimes \mathbb{Z}[i].$$

Proof. The first two are associative algebras as opposed to the last one. Let us begin with the case $n = 2$. We know that the center of H^2 has graded rank $1 + 3q^2 + 2q^4$. However, by a similar argument to Proposition 3.11, we have

$$Z(OH^2_\tau) \subset a(OH^2)a \oplus b(OH^2)b.$$

But OH^2_τ behaves like an exterior algebra on this subset, implying that elements anticommute and thus the graded rank of the center is 0 in degree 2. Therefore, H^2 and OH^2_τ have non-isomorphic centers and thus cannot be isomorphic as algebras.

This can be extended to all $n \geq 2$. Suppose $x, y \in a(OH^n_\tau)a$ with $|x|_q = |y|_q = 1$. Their products are given by

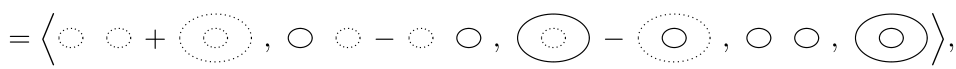
$$(x, y) \mapsto \tau(|x|, |y|)x \wedge y, \qquad (y, x) \mapsto \tau(|y|, |x|)y \wedge x,$$

since $OF(C_{aaa})$ is the product in the exterior algebra $\bigwedge^* V(W(a)a)$. However, $|x| = |y|$, implying $\tau(|x|, |y|) = \tau(|y|, |x|)$ and thus $xy = -yx$. ∎

PROPOSITION 5.22. *The odd center of OH^2_τ is not isomorphic to the algebra $OH(\mathfrak{B}_{2,2}, \mathbb{Z}[i])$.*

Proof. It is not hard to compute that the odd center of OH^2_τ satisfies

$OZ(OH^2_\tau)$



and thus has graded rank $1 + 2q^2 + 2q^4$. However, $OH(\mathfrak{B}_{2,2}, \mathbb{Z}[i])$ has graded rank $1 + 3q^2 + 2q^4$. ∎

Despite this result, it is easy to show that OH^n_τ contains a subalgebra isomorphic to $OH(\mathfrak{B}_{n,n}, \mathbb{Z}[i])$, by constructing an injective map, say $\tilde{h}$, similar to h_0 from Section 4. As $\bigoplus_a a(OH^n_\tau)a$ is isomorphic to an exterior algebra, $\tilde{h}$ will be well-defined. Moreover, the different arc algebras being isomorphic modulo 2, it is injective for the same reason as in Proposition 4.15.

6. Perspectives. One natural application of the work in this paper could be the construction of odd Khovanov homology for tangles (Putyra–Shumakovitch's work in progress using the structure of quasialgebras [31]). The fact that the twist τ is not explicit may cause several technical difficulties in defining (and working with) (OH^n_τ, OH^n_τ)-bimodules.

Another possibility consists in working with *quasibimodules*, that is, bimodules with the associativity axiom given by an associator, as in [29]. With such a theory at hand, it seems plausible that the braid group action on the category of complexes of (OH^n_C, OH^n_C)-quasibimodules up to homotopy descends to an action of the (-1)-Hecke algebra from [24, Section 4] on its (odd) center, paralleling the even case (see [20, Section 5.3]).

The twisted odd arc algebra OH^n_τ was defined over the Gaussian integers for technical reasons. One question that we leave open is whether it is possible to twist OH^n_C over the integers.

The construction in this paper shares several features with Ehrig–Stroppel's [14] Khovanov arc algebra of type D. It would be interesting to find a connection between these two arc algebras.

In [7], an action of the 2-Kac–Moody algebra of Rouquier [32] (and therefore of Khovanov–Lauda's [22]) on Khovanov's arc algebras was constructed. The results of Rouquier [32] on strong categorical actions, together with the fact that our associative arc algebra is not isomorphic to Khovanov's, imply that the 2-Kac–Moody algebra does not act on it. It seems plausible that the odd arc algebra admits an action of an algebra akin to Brundan and Kleshchev's [4] Hecke–Clifford superalgebra, which could be seen as a super counterpart of the cyclotomic KLR algebra.

Another challenging problem we mention is to find the representation-theoretic context (category $\mathcal{O}$) for the odd arc algebras. The analogy with [14] and [15], taken together with [18] and the results in [17, Section 5.1], might suggest a connection with category $\mathcal{O}$ for Lie superalgebras.

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