Categorifying Induction: From Modules to Birepresentations

with an application to extended affine type A Soergel bimodules

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Guiding question

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We want to pass from

modules to birepresentations.

Roadmap

The talk has three parts.

1. From classical induction to categorical actions.
2. Algebra objects as a mechanism for constructing birepresentations.
3. The extended affine type A Soergel example, where homotopy categories naturally appear.

The main message is that parabolic induction has a categorical analogue, but in affine type this analogue naturally lives in a homotopical setting.

The story in one picture

classical induction	$\longrightarrow$	categorical induction
$R \otimes_S -$	$\longrightarrow$	induction via algebra objects
parabolic embeddings	$\longrightarrow$	K^b(Soergel categories)

The affine type A example forces the categorified construction into homotopy and triangulated categories.

Classical induction

Let $S \subset R$ be an inclusion of algebras.

If M is a left S-module, then its induced R-module is

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This is the kind of construction that we want to categorify.

The dictionary

Classical representation theory	Categorified representation theory
algebra R	monoidal category $\mathcal{A}$
vector space or module M	category $\mathcal{M}$
$R \longrightarrow \mathrm{End}_{\Bbbk}(M)$	$\mathcal{A} \longrightarrow \mathrm{END}_{\Bbbk}(\mathcal{M})$
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The goal is to make the last line precise.

Representations

A finite-dimensional representation consists of

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where

- R is a finite-dimensional $\Bbbk$-algebra;
- M is a finite-dimensional $\Bbbk$-vector space;
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Equivalently, M is an R-module.

Birepresentations

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A birepresentation is a monoidal $\Bbbk$-linear functor

$$\mathcal{A} \longrightarrow \mathrm{END}_{\Bbbk}(\mathcal{M}).$$

In the terminology used in our work, such an action is called a *birepresentation*.

This is close in spirit to the finitary 2-representation theory of Mazorchuk–Miemietz.

Finitary categories

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Thus finitary categories are categorical analogues of finite-dimensional algebras.

This is the setting underlying finitary 2-representation theory, for example in work of Mazorchuk–Miemietz.

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For the induction construction in this talk, one also has to pass to homotopy and triangulated categories.

Algebra objects as a mechanism for induction

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The useful idea is to use algebra objects to construct and, in good cases, encode birepresentations.

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For example, the unit object $\mathbb{1}$ is always an algebra object.

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So algebra objects generalize ordinary algebras.

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This is a first example where algebra objects live in a nontrivial monoidal category.

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This is a useful bridge toward Soergel bimodules (which are monoidal categories of bimodule-type objects).

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We denote by

$$\mathrm{mod}_{\mathcal{C}}(X)$$

the category of right X-module objects in $\mathcal{C}$.

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So algebra objects provide a convenient *formal* way to construct birepresentations, and to encode them under suitable hypotheses.

$$X \quad \rightsquigarrow \quad \text{mod}_{\mathcal{C}}(X).$$

This is the standard algebra-object/module-category philosophy from tensor category theory, as in work of Ostrik and of Etingof–Nikshych–Ostrik.

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- Algebra objects provide a mechanism to produce birepresentations:

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- At present, induction via algebra objects is best understood as a guiding mechanism rather than a fully developed general theory.

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Define, whenever the relevant module categories exist,

$$\mathrm{Ind}_{\mathcal{B}}^{\mathcal{A}}(\mathrm{mod}_{\mathcal{B}}(X)) = \mathrm{mod}_{\mathcal{A}}(\Psi(X))$$

Induction via algebra objects: what is actually known

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- The monoidal parabolic embedding is constructed in the maximal parabolic case.
- What is fully worked out at the level of induced birepresentations via algebra objects is the case $n = 2, \ k = 1$.
- In this example, induction of the trivial birepresentation already requires working in a completion, since the corresponding algebra object is an infinite coproduct.

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- In this example, induction of the trivial birepresentation already requires working in a completion, since the corresponding algebra object is an infinite coproduct.

- Other expected examples of induction, even in finite type A, remain to be written down in this language.
- Conceptually, these examples have a similar caveat: the need for completions makes it hard to identify explicitly the induced birepresentations associated to an algebra object.

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- The algebra objects may be infinite countable coproducts of indecomposable objects.
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Nevertheless, in extended affine type A, the construction can be made to work.

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- in which induction can be categorified,
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- and where homotopy/triangulated phenomena are unavoidable.

All the ideas of the talk make sense independently of this example.

The extended affine type A Soergel example

Soergel bimodules are monoidal categories introduced by Soergel to categorify Hecke algebras.

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It is a $\mathbb{C}$-linear monoidal category generated by objects

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Think of this as a well-behaved monoidal category categorifying the extended affine Hecke algebra of type A.

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This categorifies the parabolic embedding underlying classical induction.

The two embeddings

There are two monoidal functors

$$\Psi_L : \mathcal{S}_k^{\text{ext}} \longrightarrow K^b(\mathcal{S}_n^{\text{ext}})$$

and

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Rouquier complexes appear naturally in the images of the rotation objects.

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The compatibility is the categorical replacement for the commuting parabolic subalgebras in the classical situation.

Main theorem

The main construction in joint work with Mackaay and Miemietz is:

The symmetric pair Ψ_L, Ψ_R induces a monoidal functor

$$\Psi_{k,n-k} : \mathcal{S}_k^{\mathrm{ext}} \boxtimes \mathcal{S}_{n-k}^{\mathrm{ext}} \longrightarrow K^b(\mathcal{S}_n^{\mathrm{ext}}).$$

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On objects,

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On morphisms,

$$\Psi_{k,n-k}(f \boxtimes g) = \Psi_L(f)\Psi_R(g).$$

Main theorem

The main construction in joint work with Mackaay and Miemietz is:

The symmetric pair Ψ_L, Ψ_R induces a monoidal functor

$$\Psi_{k,n-k} : \mathcal{S}_k^{\mathrm{ext}} \boxtimes \mathcal{S}_{n-k}^{\mathrm{ext}} \longrightarrow K^b(\mathcal{S}_n^{\mathrm{ext}}).$$

On objects,

$$\Psi_{k,n-k}(X \boxtimes Y) = \Psi_L(X)\Psi_R(Y).$$

On morphisms,

$$\Psi_{k,n-k}(f \boxtimes g) = \Psi_L(f)\Psi_R(g).$$

This categorifies the parabolic embedding underlying classical parabolic induction.

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Let X be an algebra object encoding a birepresentation of

$$\mathcal{S}_k^{\text{ext}} \boxtimes \mathcal{S}_{n-k}^{\text{ext}}.$$

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This is the categorified analogue of inducing a module.

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Let V be the trivial birepresentation of

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The induced triangulated birepresentation is obtained from the triangulated closure of the additive category generated by objects of the form

$$FY, \qquad F \in K^b(\mathcal{S}_2^{\mathrm{ext}}).$$

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In extended affine type A, this naturally leads to homotopy and triangulated categories.

Summary

classical induction $\rightsquigarrow$ categorical induction $\rightsquigarrow$ triangulated birepresentations

- Algebra objects construct, and under suitable hypotheses encode, birepresentations.
- Monoidal functors send algebra objects to algebra objects.
- This gives a formal mechanism for categorical induction.
- The monoidal functor $\Psi_{k,n-k}$ categorifies the parabolic embedding.
- In extended affine type A, the induced objects naturally live in homotopy and triangulated categories.

Open questions

- Can one develop a general theory of irreducible triangulated birepresentations?
- Is there a triangulated analogue of the Zelevinsky classification?
- Can this construction be extended beyond affine type A?
- What are the connections with affine Springer theory, character sheaves, and knot homology?

Thank you

Thank you!

References and landmarks

- **Finitary 2-representation theory:** Mazorchuk–Miemietz.
- **Locally finitary and locally wide finitary extensions:** Macpherson.
- **Almost finitary birepresentations and affine Soergel bimodules:** Mackaay–Miemietz–Vaz.
- **Soergel bimodules and Hecke categorification:** Soergel, Elias–Khovanov, Elias–Williamson.
- **Rouquier complexes:** Rouquier.

When do birepresentations come from algebra objects?

There is no general correspondence between birepresentations and algebra objects.

A birepresentation of a monoidal category $\mathcal{C}$ corresponds to an algebra object only under strong hypotheses:

- $\mathcal{C}$ is additive, $\Bbbk$-linear, and idempotent complete;
- the birepresentation is generated by a single object;
- the internal endomorphism object exists (possibly after a completion).

In such cases, this is a Morita-type reconstruction: if m is a generator and the internal endomorphism object exists, then

$$X = \underline{\mathrm{End}}(m), \qquad \mathcal{M} \simeq \mathrm{mod}_{\mathcal{C}}(X).$$

In affine and triangulated settings (e.g. Soergel categories), these hypotheses typically fail: algebra objects are often infinite coproducts, forcing completions and making explicit identification of induced birepresentations difficult.

The rigid case $\mathrm{Rep}(G)$

Let G be a finite group, and work over a field $\Bbbk$ such that $\mathrm{Rep}(G)$ is semisimple.

The monoidal category $\mathrm{Rep}(G)$ is then semisimple, rigid, and finite.

In this setting:

- under standard finiteness assumptions, module categories over $\mathrm{Rep}(G)$ are equivalent to $\mathrm{mod}_{\mathrm{Rep}(G)}(A)$ for finite-dimensional algebra objects $A \in \mathrm{Rep}(G)$;
- no completions are needed, and the correspondence is well behaved.

This rigidity is precisely what fails in Soergel-type and affine settings, where algebra objects are often infinite coproducts and completions are unavoidable.

The extended affine Hecke algebra

The extended affine Hecke algebra H_n^{ext} of type A_{n-1} is generated by

$$T_0, T_1, \ldots, T_{n-1}, \qquad \rho, \rho^{-1},$$

with indices modulo n.

The T_i satisfy the affine type A Hecke relations:

$$(T_i - q)(T_i + q^{-1}) = 0,$$

together with the braid relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad T_i T_j = T_j T_i \quad \text{if } i \not\equiv j \pm 1.$$

The rotation satisfies

$$\rho T_i \rho^{-1} = T_{i+1}.$$

Schematically, under the chosen normalization,

$$T_i + q \quad \rightsquigarrow \quad B_i, \qquad \rho^{\pm 1} \quad \rightsquigarrow \quad B_{\rho^{\pm 1}}.$$

What are Soergel bimodules?

Fix a Coxeter system (W, S) and a reflection representation V. Let

$$R = \mathrm{Sym}(V^*)$$

be the graded polynomial ring.

For each simple reflection $s \in S$, one has a basic bimodule

$$B_s = R \otimes_{R^s} R$$

up to grading shift.

The Soergel category is obtained from these B_s's by taking

tensor products over R, direct sums, direct summands, grading shifts.

They form a monoidal category whose split Grothendieck group is the Hecke algebra.

Categorified parabolic induction (maximal parabolic)

- Categorical embeddings

$$\Psi_L \colon \widehat{\mathcal{S}}^{\mathrm{ext}}_k \to K^b(\widehat{\mathcal{S}}^{\mathrm{ext}}_n) \quad \text{and} \quad \Psi_R \colon \widehat{\mathcal{S}}^{\mathrm{ext}}_{n-k} \to K^b(\widehat{\mathcal{S}}^{\mathrm{ext}}_n)$$

$$\Psi_L(\mathrm{B}_i) = \mathrm{B}_i \qquad \Psi_R(\mathrm{B}_j) = \mathrm{B}_{j+k}$$

$$\Psi_L(\mathrm{B}_\rho) = \mathrm{B}_\rho \mathrm{T}_{n-1}\mathrm{T}_{n-2}\cdots\mathrm{T}_k \qquad \Psi_R(\mathrm{B}_\rho) = \mathrm{T}_k^{-1}\cdots\mathrm{T}_2^{-1}\mathrm{T}_1^{-1}\mathrm{B}_\rho$$

- Symmetric braiding $\Psi_L\Psi_R \cong \Psi_R\Psi_L$