

# The limits of knot detection: alternating links, mutation, and (categorified) quantum invariants

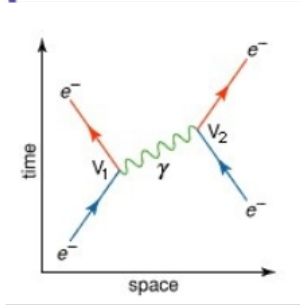
Pedro Vaz, Université catholique de Louvain

Lille, February 20, 2026



*Slides*

# Why quantum (polynomial categorified) invariants matter ?

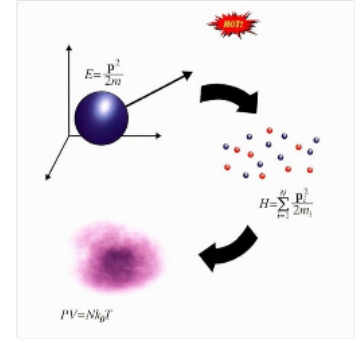


quantum field theory

$$M^p(\lambda) = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p})} V$$

representation theory

statistical mechanics

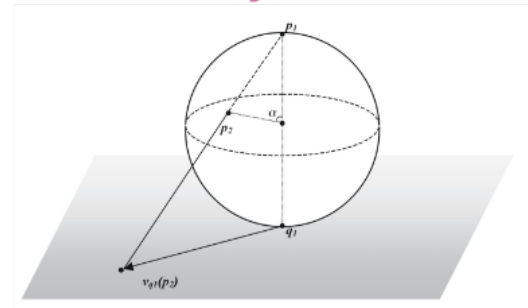
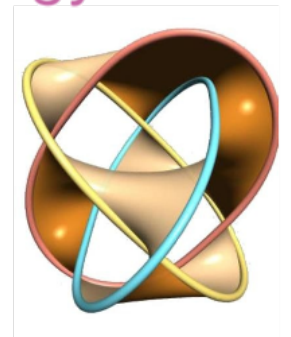


quantum computation

Quantum invariants

Low-Dim. Topology

Geometry



Topological Quantum Computation

Braid Gate

Computation

Physics

readout

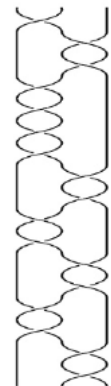
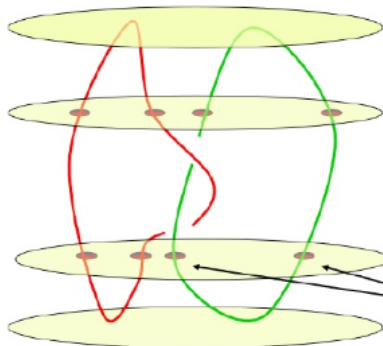
fusion

applying circuits

braiding anyons

initialize

create anyon pairs



Prove Conjectures!



# How effective are these invariants as classifiers?

n	Jones polynomial
3	100
5	100
6	100
7	100
8	100
9	100
10	96.38
11	90.13
12	83.00
13	73.31
14	64.49
15	55.74
16	49.42
17	44.84
18	41.62
19	40.17
20	?

Jones polynomial  
on prime knots

For each n :  
 $\#(\text{polynomials}) / \#(\text{knots})$

# How effective are these invariants as classifiers?

## Question

Put all knots in a bag, shake, and draw one.  
How likely is a link invariant to single it out?

Khovanov homology detects the unknot (Kronheimer-Mrowka),  
the trefoil & the cinquefoil (Baldwin, Sivek), ...



## A refinement

- P1: What is the probability  $P(n)$  that an invariant detects a knot among all knots with crossing number up to  $n$ ?
- P2: What is the behavior of  $P(n)$ , as a function of  $n$ ?

# Theorem (T. Kelomäki, A. Lacabanne, D. Tubbenhauer, P.V., V.L. Zhang)



The following invariants detect alternating links with **probability zero**:

① **Polynomials**: Jones, Alexander,  $SL(N)$ , HOMFLYPT,  $SO(2N+1)$ ,  $SO(2N)$ ,  $SP(2N)$ , Kauffman,  $G_2$ , colored Jones, colored  $SL(N)$  (ext, sym and  $(N-1,1)$ ), colored HOMFLYPT (ext, sym).

**Homological**: Khovanov homology, odd Khovanov homology

**Other**: signature, determinant, (HF) double-branched cover, algebraic concordance, finite type of degree  $\leq 10$ .

② **Most likely** (experimental data): Khovanov-Rozansky over  $\mathbb{Z}$ , HOMFLYPT homology over  $\mathbb{Z}$ , several HFK over  $\mathbb{F}_2$ .

# Contents



Main Theorem



Experimental  
data



Theorem



# Counting alternating diagrams and mutation classes

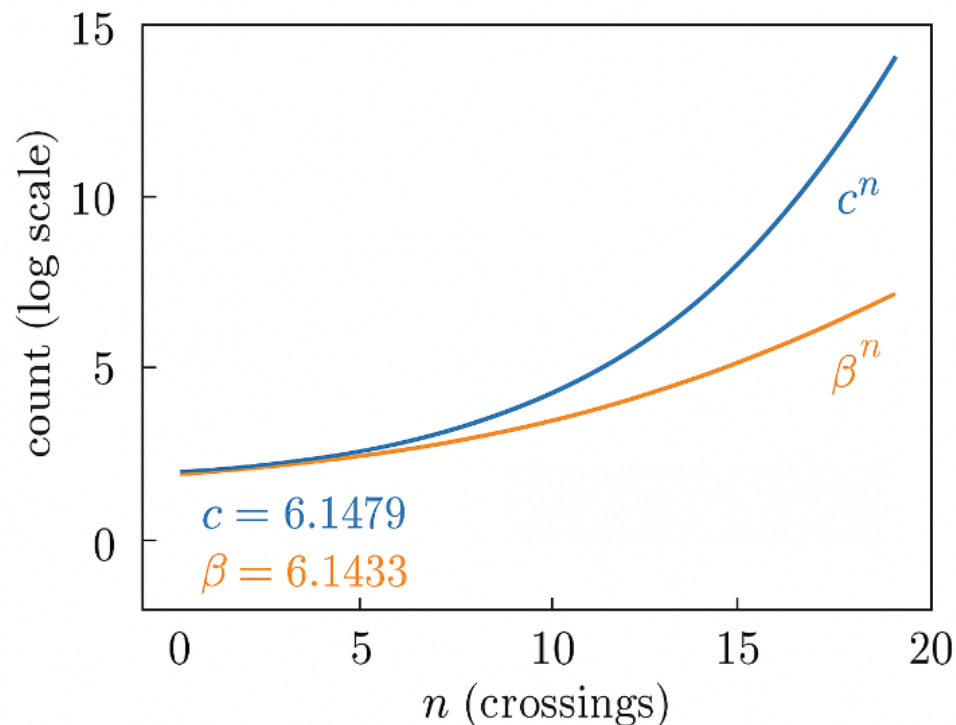
- **Alternating links** can be counted combinatorially (Sundberg-Thistlethwaite '98)  
Generation-function analysis gives exponential growth:

$$L_n \sim C c^n, \quad c = 6.147902686 \dots$$

- The number of mutation-invariant classes grows slightly slower:

$$M_n \sim C' \beta^n, \quad \beta = 6.143312242 \dots$$

- N.B.  $\beta < c$ .



# Why mutation-invariant invariants fail to detect links

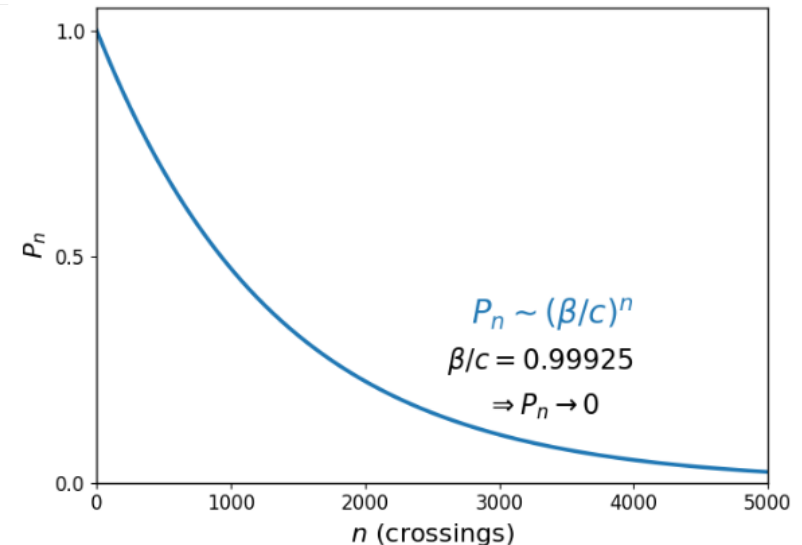
- If an invariant  $I$  is invariant under mutation, then all links in the same mutation class share the same  $I$ -value.
- Therefore, the number of distinct invariant values  $\leq$  number of mutation classes:

$$\#\text{Im}(I_n) \leq M_n.$$

- Detection probability:  $P_n = \frac{\#\text{Im}(I_n)}{L_n} \leq \frac{M_n}{L_n} \sim \left(\frac{\beta}{c}\right)^n = (0.99925\dots)^n \rightarrow 0.$

- So every mutation-invariant polynomial or categorified invariant detects alternating links with probability zero.

- The decay is exponential.



# Mutation-invariant invariants

## ■ Which invariants ignore mutation:

- Any polynomial quantum invariant that satisfies a skein relation or is multiplicity free is invariant under mutation.
- $SL(N)$  for  $(N-1,1)$  : Morton-Cromwell '96 (HOMFLYPT: stabilization).
- Khovanov homology over  $F_2$  or odd Khovanov homology: Wehrli '10, Bloom '10.
- Remaining ones: classical result, or Green '13 (double-branched cover related).

## Theorem (T. Kelomäki, A. Lacabanne, D. Tubbenhauer, P.V., V.L. Zhang)

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# Why detection probabilities vanish?

## ■ Takeaway

Mutation is a **symmetry** invisible to quite a few quantum invariants.

It's exponential abundance among alternating links forces their **detection probabilities to decay to zero.**

# Big data and the limits of detection



Computational and experimental results

# Setup

- Large-scale computations on knots up to 19 crossings
- Invariants tested:
  - Khovanov homology: **KnotTheory Mathematica package** - the knot atlas
  - Odd Khovanov homology: **KnotJob** program - D. Shüetz
  - Jones polynomial : homemade program
  - HFK homology: **Knot Floer homology calculator** - P. Ozsváth, Z. Szabó
  - Alexander polynomial: **KnotTheory Mathematica package** - the knot atlas
  - $SL(3)$  Khovanov-Rozansky homology: **KnotJob** program - D. Shüetz
  - $SL(3)$  polynomial: **KnotTheory Mathematica package** - the knot atlas
  - HOMFLYPT homology: program by K. Nakagane et S. Taketo
  - HOMFLYPT polynomial: **KnotTheory Mathematica package** - the knot atlas
- Detection probability = proportion of distinct values among prime knots

## Resources:

- Katana computational cluster, supporter by UNSW Sydney
- Laboratoire Mathématique Blaise Pascal

## Knot tables:



Knotinfo  
3-16 Xings



Regina  
17-19 Xings



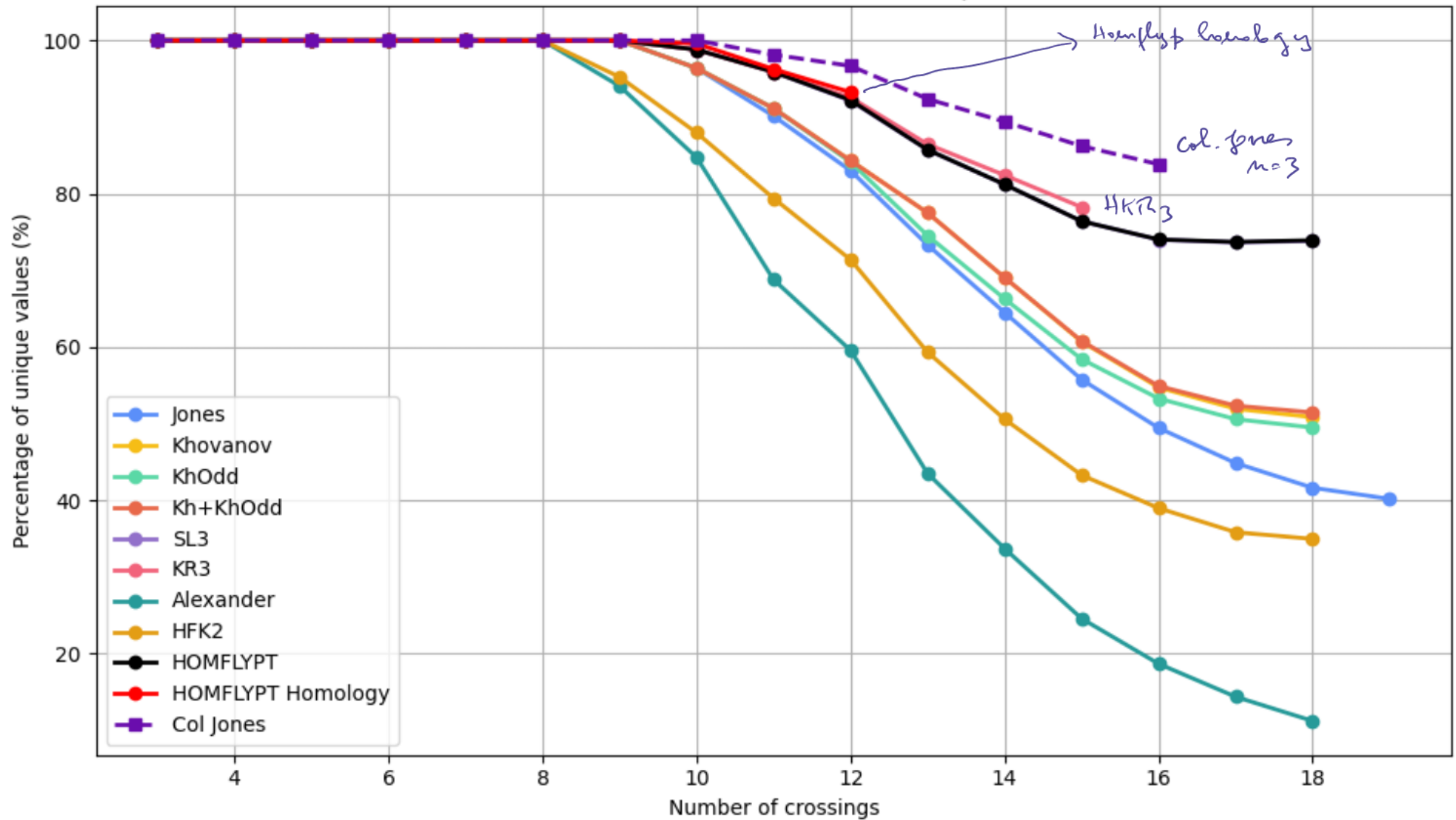
Stoimenow  
Braid presentations

# Results



interactive playground

Effectiveness of Knot and Link Invariants (Updated Data)



HOMFLYPT polynomial  
SL(3) polynomial

HFK + OKR Odd  
HFK  
HFK Odd

Jones  
HFK

Alexander

# Results

n	Jones	Khovanov	HKhOdd	Kh+KhO	Col Jones n = 3	SL3	KR3	Alexander	HFK2	HOMFLYPT	HOMF Hgy
3	100	100	100	100	100	100	100	100	100	100	100
4	100	100	100	100	100	100	100	100	100	100	100
5	100	100	100	100	100	100	100	100	100	100	100
6	100	100	100	100	100	100	100	100	100	100	100
7	100	100	100	100	100	100	100	100	100	100	100
8	100	100	100	100	100	100	100	100	100	100	100
9	100	100	100	100	100	100	100	94.04	95.23	100	100
10	96.38	96.38	96.38	96.38	100	98.79	98.79	84.73	87.95	98.79	99.59
11	90.13	91.13	91.13	91.13	98.13	95.88	95.88	68.78	79.40	95.88	96.25
12	83.00	84.31	84.11	84.31	96.67	92.13	92.54	59.55	71.38	92.20	93.24
13	73.31	77.50	74.53	77.50	92.36	85.71	86.45	43.49	59.31	85.73	-
14	64.49	69.04	66.28	69.05	89.39	81.2	82.41	33.69	50.57	81.21	-
15	55.74	60.69	58.40	60.77	86.21	76.4	78.24	24.55	43.27	76.40	-
16	49.42	54.71	53.27	54.90	83.83	74.00	-	18.65	38.96	74.02	-
17	44.84	51.93	50.57	52.33	-	73.65	-	14.35	35.82	73.70	-
18	41.62	50.83	49.48	51.45	-	73.84	-	11.19	34.95	73.91	-
19	40.17	-	-	-	-	-	-	-	-	-	-

Computing cost  
17/18 Kings

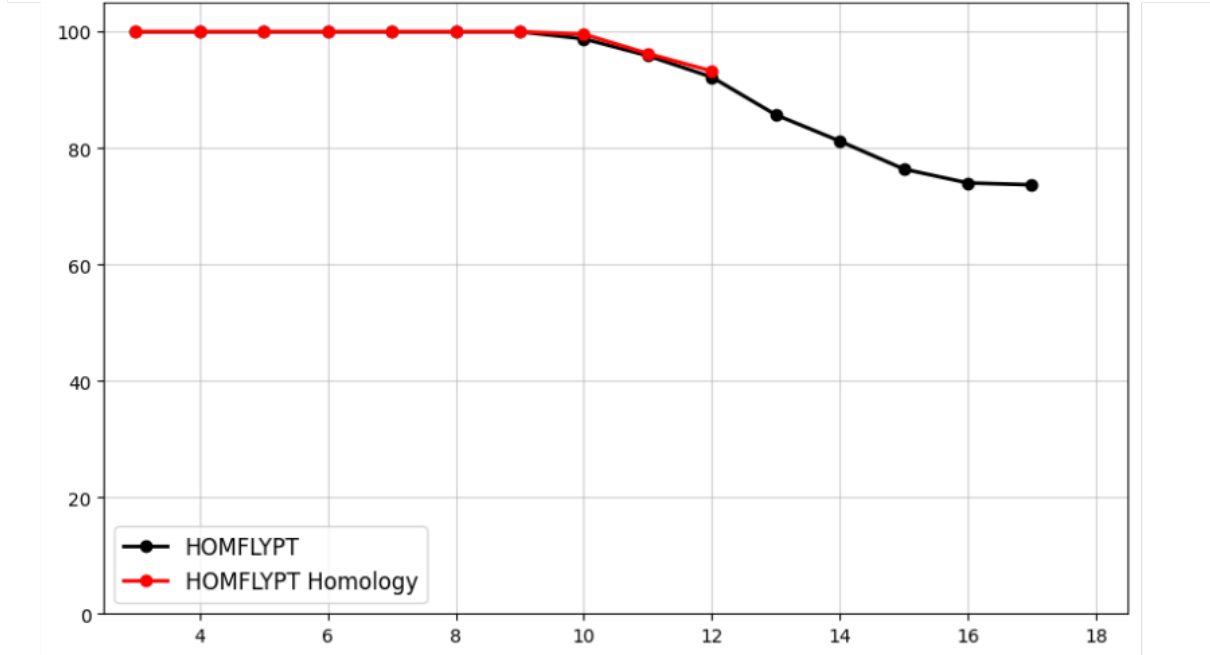
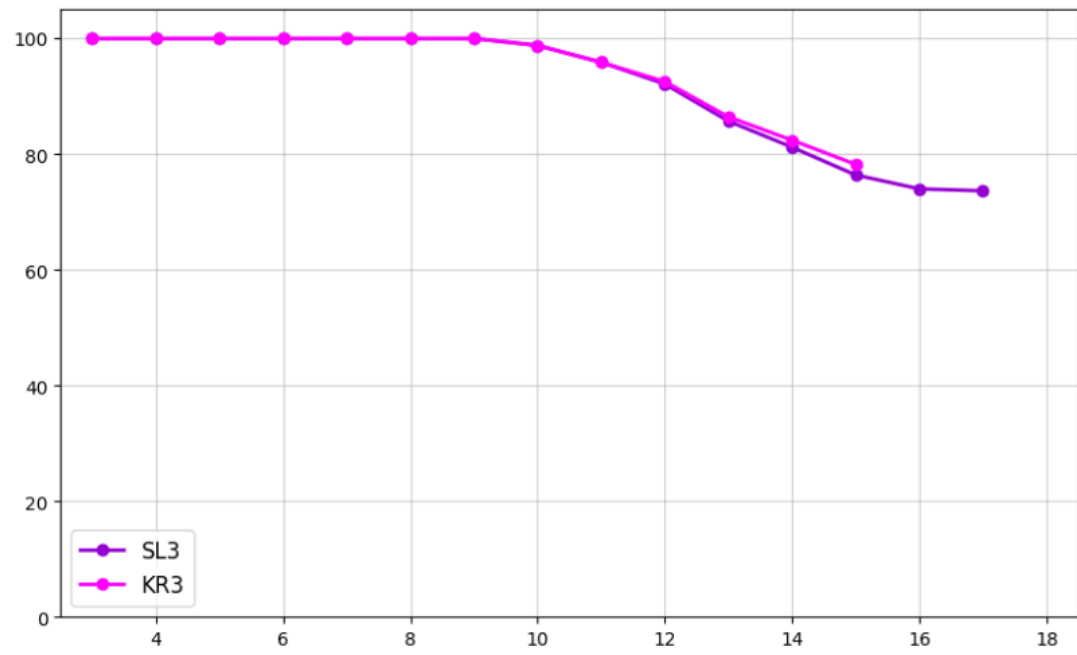
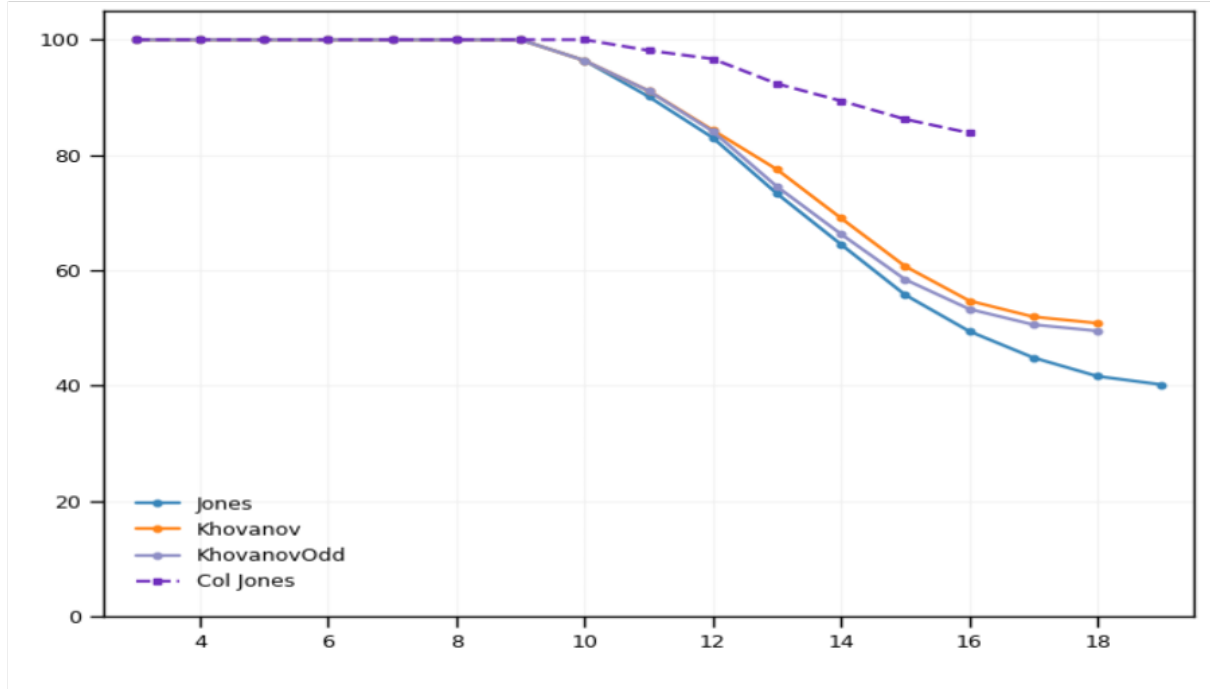
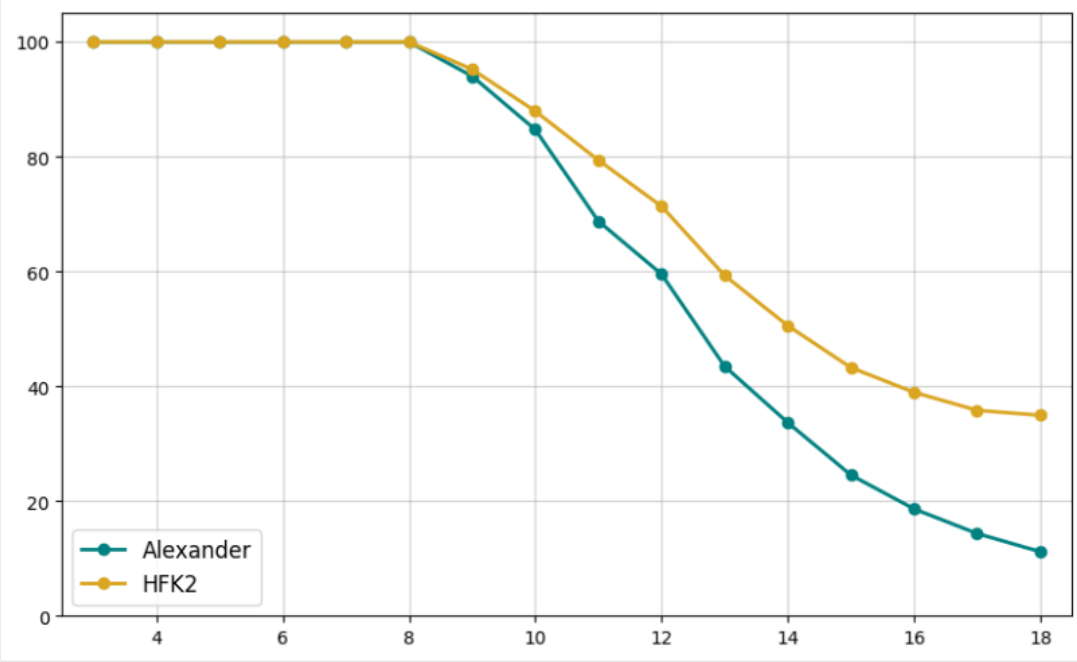
In Q3 2025, Restech provided over

**18 million core-hours**

of compute time through our **Katana Compute Cluster**,

and your account was among the most active, with a total of **1,030,265**

**core-hours** consumed. The estimated/reference value is approximately **\$41,211**.



# Observations

- **Exponential decay** in detection probability in crossing number
- **Categorified invariants**  $\approx$  **polynomial ones** in performance
- Odd Khovanov and HOMFLYPT homologies surprisingly weak

# What this results tell us

## ■ Why this is surprising

- Categorification was expected to strengthen detection: more structure, richer gradings.
- Yet, data and asymptotics show no improvement over polynomial invariants.

## ■ What it reveals

- Mutation invariance acts as a bottleneck: most known invariants collapse on the same class.
- The set of distinct invariant values grows exponentially slower than the set of links.

## ■ The message

- Most known knot invariants fail as detectors.
- Categorification refines meaning, not distinguishing power.

We learned how quantum invariants fail beautifully

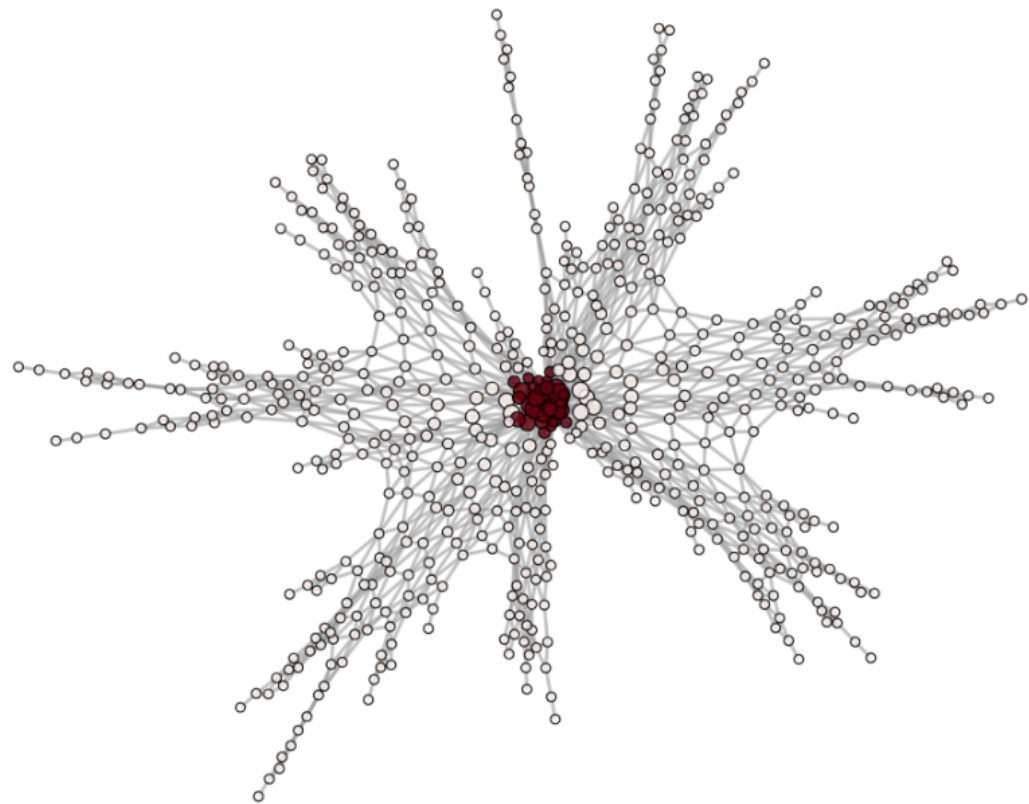
# What now?

- Colored invariants with  $\mathbb{Z}/2\mathbb{Z}$  not multiplicity free (e.g. HOMFLYPT polynomial colored with  $\mathbb{Z}/2\mathbb{Z}$  detects mutation - Nawata-Ramadevi-Singh '17).
- Consider knots/links up to mutation.
- Tools from TDA (ballmaper) to analyze structure of data.

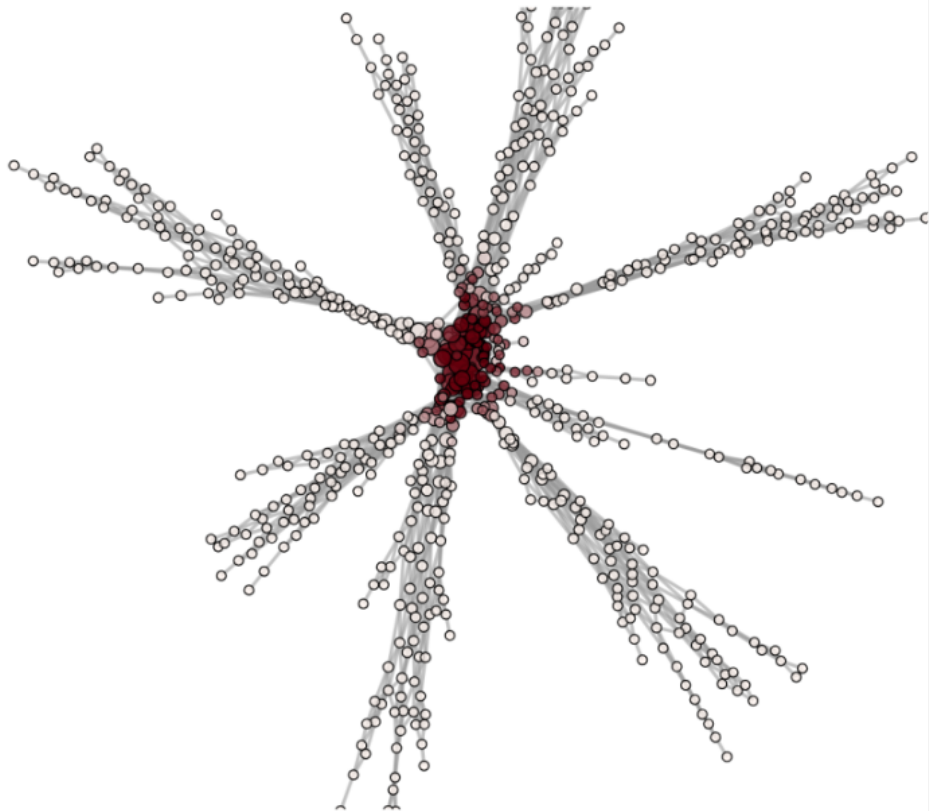
Invariant failure is not weakness - it reveals hidden symmetries in knot space

This is not the end of the story...

# Further analysis : TDA - the ballmaper



Jones



Khovanov

# Further analysis

- statistical analysis (avg computation time, ...)
- invariants up to mutation, compare categorification / non categorification
- root distribution
- other invariants (mod  $p$  reduction, Theta (BN-VDV) hyperbolic volume, ...)

Have look at our interactive playground :



Thank you !



Slides



