A tour on quantum topology and categorification (a biased and incomplete story)

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Roadmap

Categorification
3

3- manifold
invariants
2

Khovanov homology
4

Jones
polynomial
of knots
1

Consequences
&
modern trends
5

The Jones revolution

V. Jones
Field Medal 1990
(together with
Drinfeld and
Witten)

"Jones discovered
an astonishing
relationship between
von Neuman algebras
and geometric topology"

The Jones polynomial $\mathcal{J} : \{\text{links}\}/\text{isotopy} \longrightarrow \mathbb{Z}[q^{\pm 1}]$

1985

1989

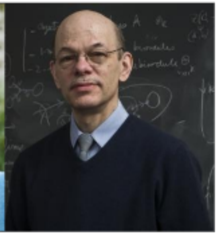
1991

1987

Physics
CS theory (TQFT)
G= SU(2)

Ed. Witten

Representation theory

N. Reshetikhin V. Turaev

Statistical mechanics

- proof of the Tait conjectures (~150 years)

L. Kauffman,
K. Murasugi,
M. Thistlethwaite

- Generalization of Jones (sl2) to all quantum groups

&

- Generalization to 3-manifolds

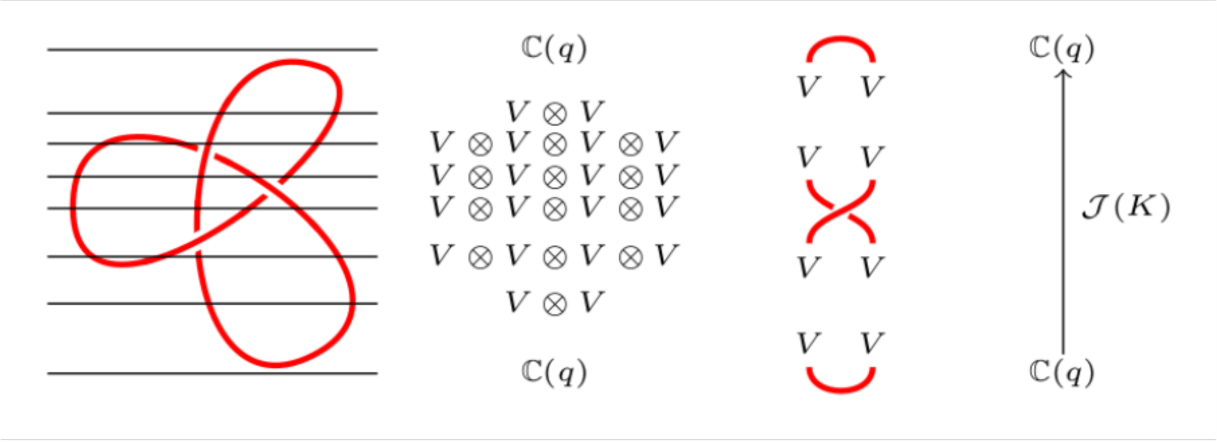
$(\zeta^n = 1)$

3-manifold
invariants

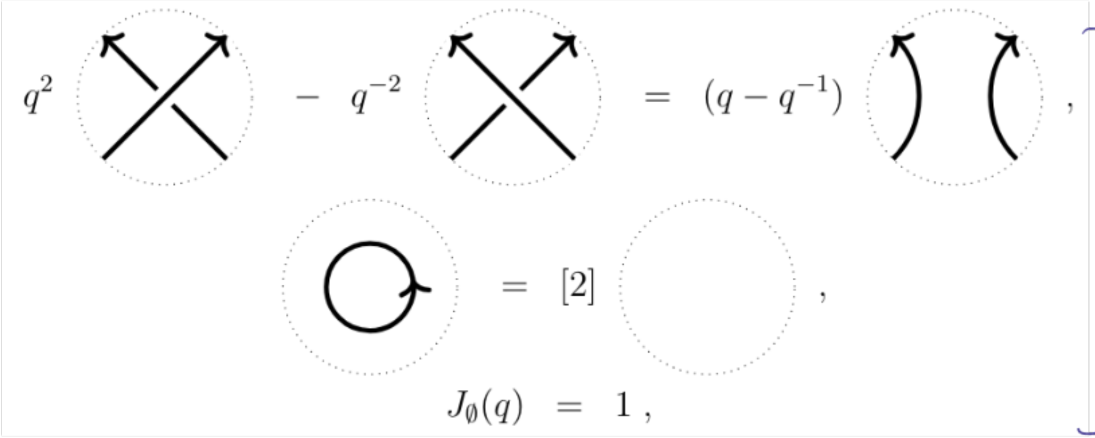
These are the same!

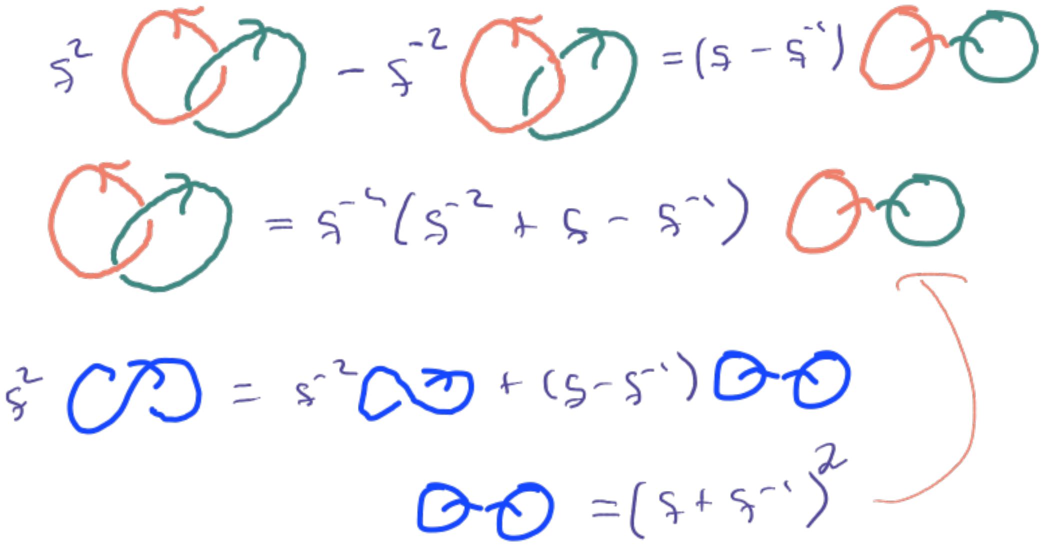
Computing the Jones polynomial

BT:



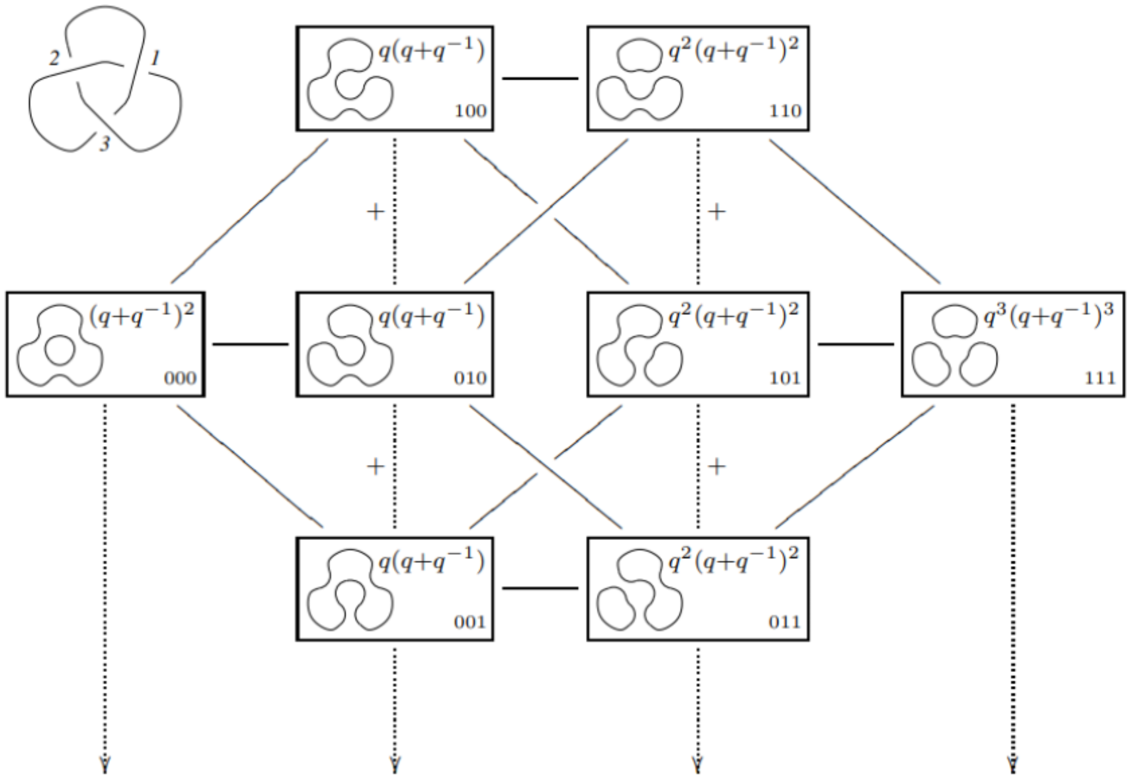
Skein:





Computing the Jones polynomial: the Kauffmann bracket

$$\langle \emptyset \rangle = 1; \quad \langle \bigcirc L \rangle = (q + q^{-1}) \langle L \rangle; \quad \langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle.$$



$$(q+q^{-1})^2 \quad - \quad 3q(q+q^{-1}) \quad + \quad 3q^2(q+q^{-1})^2 \quad - \quad q^3(q+q^{-1})^3$$

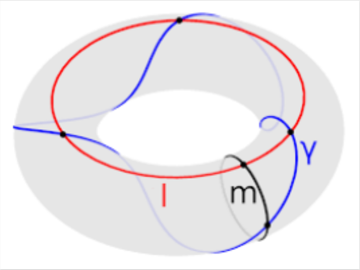
$$= q^{-2} + 1 + q^2 - q^6 \xrightarrow[\text{(with } (n_+, n_-) = (3,0))]{\cdot(-1)^{n_-} q^{n_+ - 2n_-}} q + q^3 + q^5 - q^9 \xrightarrow{\cdot(q+q^{-1})^{-1}} J(\circledast) = q^2 + q^6 - q^8.$$

Open question:
Does the Jones polynomial detect the unknot?

From knots to 3-manifolds

Drilling and filling : Dehn surgery (1910')

Take $K \subset S^3$ framed



$T_K \subset S^3 \rightsquigarrow S^3 \backslash T_K \rightsquigarrow S^3 \backslash T_K \sqcup T/(m \sim \gamma) := \mathcal{M}_K$

a torus

Lickorish-Wallace theorem (early 60's): $\mathcal{M}$ a closed, connected, orientable 3-manifold.

- $\exists\ K$ framed link s.t. $\mathcal{M} \cong \mathcal{M}_K$.

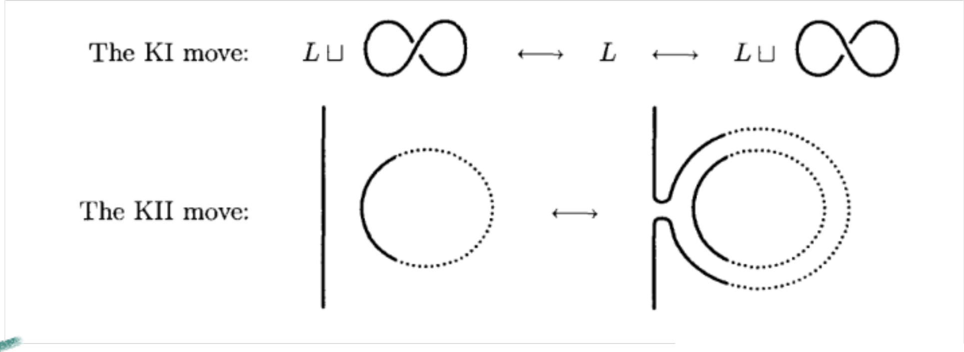
Examples: $\mathcal{M}_{\substack{\text{unknot} \\ 0}} \simeq S^2 \times S^1$ $\qquad \mathcal{M}_{\substack{\text{unknot} \\ b/a}} \simeq L(b,a)$

$[\gamma] = [a\,l + b\,m]$

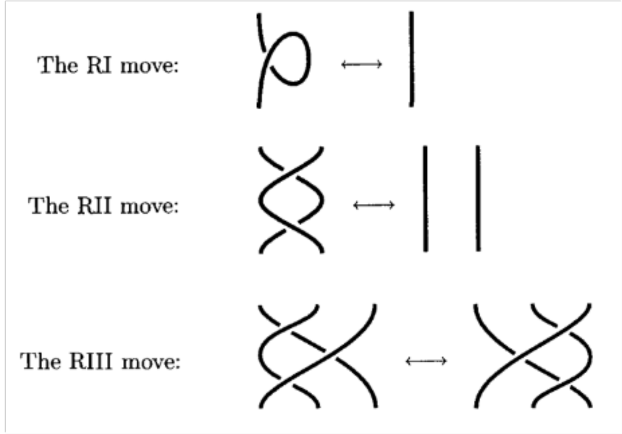
$\in \pi_1(T_K)$

From knot invariants to 3-manifold invariants

Q: invariants of knots $\rightsquigarrow$ invariants of 3-manifolds ?

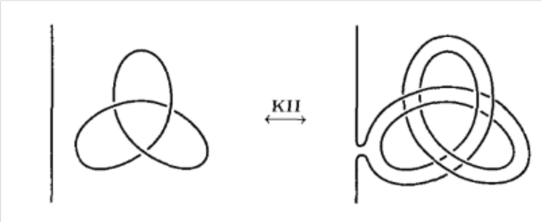


Kirby



Reidemeister

Example:

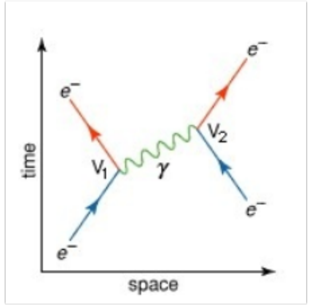


$\{\text{closed conn. ori. 3-manifols}\}/\text{homeomorphism} = \{\text{unor. framed links in } S^3\}/\text{isotopy} + \text{KI} + \text{KII}$

Combinatorial (can also use triangulations). Other: geometric/gauge theory (Witten)

Why study quantum invariants?

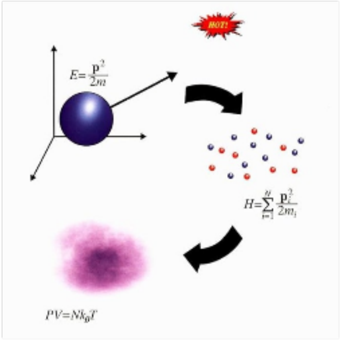
$$M^{\mathfrak{p}}(\lambda) = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p})} V$$



quantum field theory

representation theory

statistical mechanics



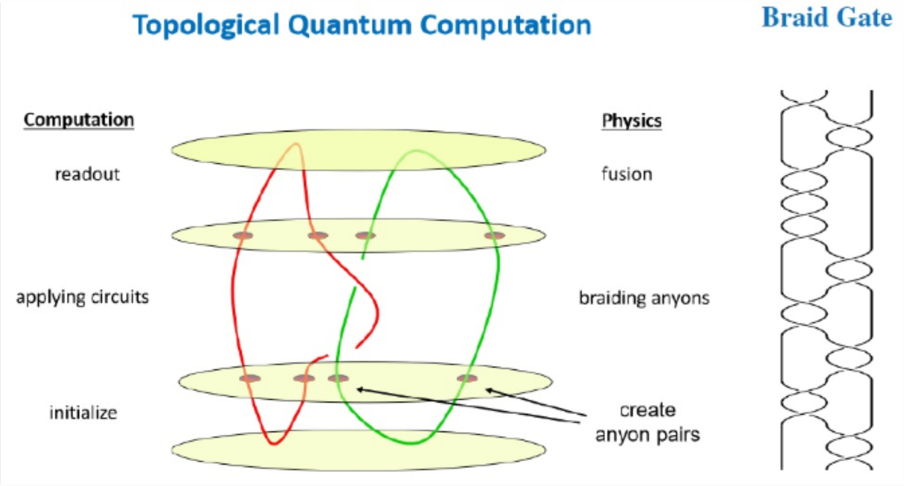
quantum computation

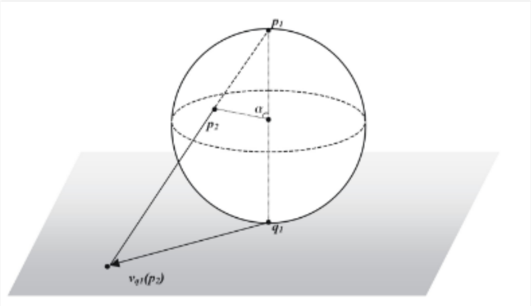
quantum invariants

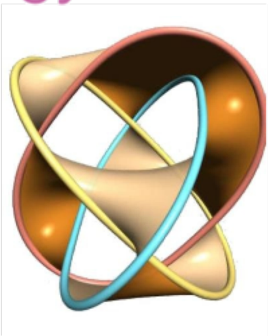
Low-Dim. Topology

Geometry

...







Categorification

L. Crane

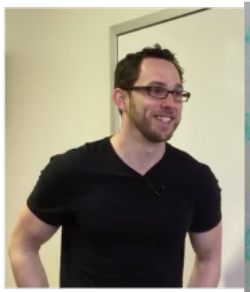
I. Frankel

M. Khovanov

L. Rozansky

B. Webster

A. Lauda



R. Rouquier

~94': Ideas from TQFT

$$n\text{-TQFT} \xrightarrow{\text{categorification}} (n+1)\text{-TQFT}$$

99' Khovanov: a categorification of the Jones polynomial

04' Khovanov-Rozansky: slN- and HOMFLYPT-link homologies

10' Webster: categorification of RTW link invariants (any (g,V))

08' Khovanov-Lauda-Rouquier

Categorification of quantum groups (& its representations)

What is a TQFT? ... for a mathematician

Main idea: Geometry $\overset{\text{TQFT}}{\rightsquigarrow}$ Algebra

topology

all compact

$$\mathrm{Bord}_n : \begin{cases} \mathrm{Ob}(\mathrm{Bord}_n) = (n-1) \text{ dim manifolds (closed)} \\ \mathrm{Hom}(M,N) = \{\text{n}-\text{dim manifolds } W \text{ s.t. } \partial W = (-N) \sqcup N\} \end{cases}$$

monoidal: $\sqcup$

n-TQFT: a symmetric, monoidal functor

$$\mathcal{Z} : \mathrm{Bord}_n \to \mathrm{Vect}_{\Bbbk}$$

TQFT Slogan: (1) Compute locally & glue; (2) It gives invariants of *n*-manifolds

Example: n=2

$$\mathcal{Z}(S^1) = H(\mathbb{C}P^1, \mathbb{Q}) = \mathbb{Q}[X]/(X^2) = V$$

$V \otimes V$

$\mathcal{Z}\Bigg($  $\Bigg)$ $= \begin{cases} X \mapsto X \otimes X \\ 1 \mapsto 1 \otimes X + X \otimes 1 \end{cases}$

V

$\mathcal{Z}(\Sigma)$

invariant

Example: Witten-Reshetikhin-Turaev, Chern-Simons

Categorifiying a TQFT: "categorify the target of $\mathcal{Z}$"

Interesting TQFTs only n<4

Challenge: Construct interesting TQFTs in dim 4

$(n-1)$-dim M	$\longrightarrow$	$\mathcal{C}_M$ (category)
n-dim $W \in \mathrm{Hom}(M, N)$	$\longrightarrow$	$\mathrm{F}_W : \mathcal{C}_M \to \mathcal{C}_N$ (functor)
$(n+1)$-dim $X \in 2\mathrm{Hom}(W, W')$	$\longrightarrow$	$\mu_X : \mathrm{F}_W \to \mathrm{F}_{W'}$ (natural transformation)

New information

⚠ Categorified TQFTs are not necessarily TQFTs !

Nevertheless: (n+1)-dimensional information

Khovanov homology

- Good integrality properties

invariants of 3 & 4-manifolds

- moduli spaces of solns of diff-geom equations
- evaded all attempts of combinatorial defn

Combinatorial

Geometric

WRT 3-manifold invariants

finite type invariants

Casson (homology 3-spheres)

Milnor torsion

...

Floer

Seiberg-Witten (3-folds)

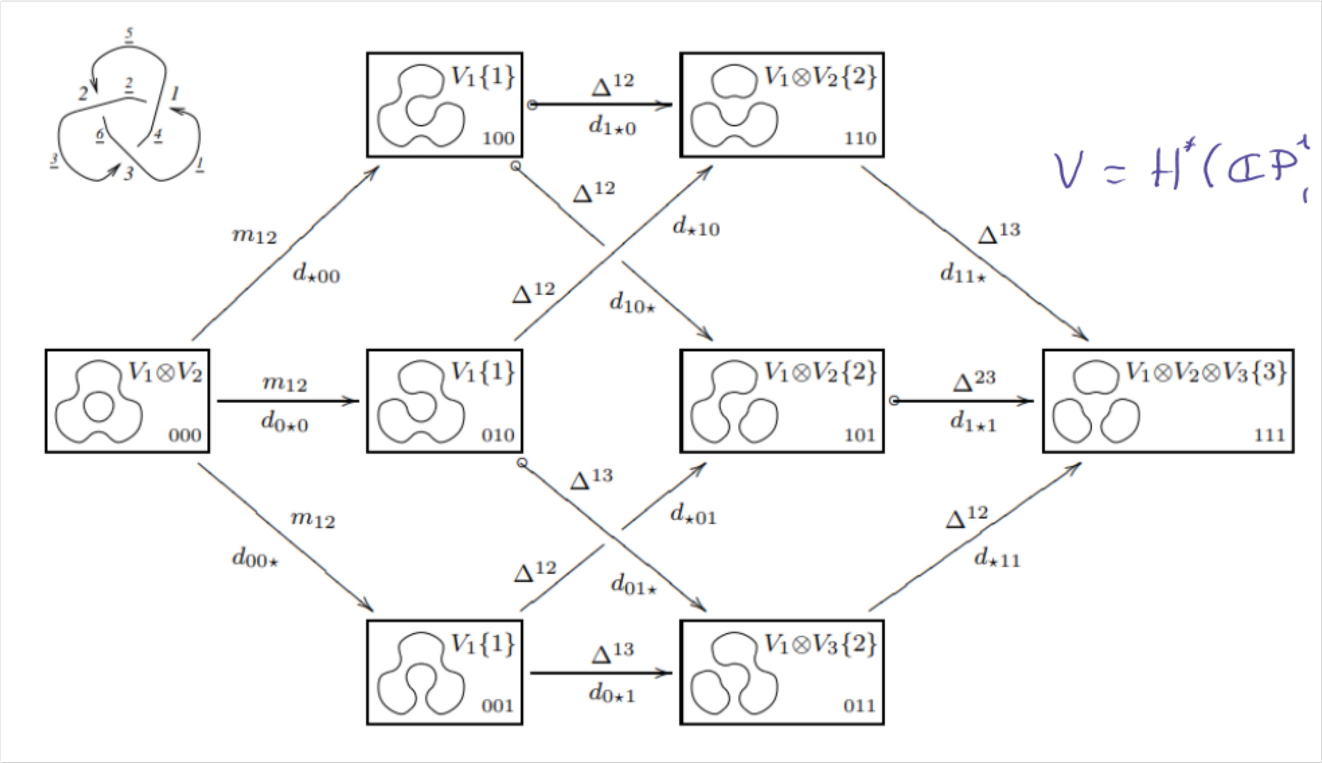
Donaldson-Seiberg-Witten (4-folds)

...

Speculative idea (Khovanov 99'):

quantum invariants of 3-manifolds $\xleftarrow{\chi}$ A homology theory of 3-manifolds ?

The Khovanov complex



$V = H^*(\mathbb{CP}^1; \mathbb{Q})$

$H^0(\;) \cong (\mathbb{Q}[x]/x^2)\{-1\}$

$H^{-1}(\;) = 0$

$H^{-2}(\;) \cong \mathbb{Q}\{-5\}$

$H^{-3}(\;) \cong \mathbb{Q}\{-9\}$

$Kh(\;) = q^{-1}(q + q^{-1}) + t^{-2}q^{-5} + t^{-3}q^{-9}$

Properties

- As invariant: stronger than Jones polynomial
- Detects the unknot
- It is a functor: higher dimensional information

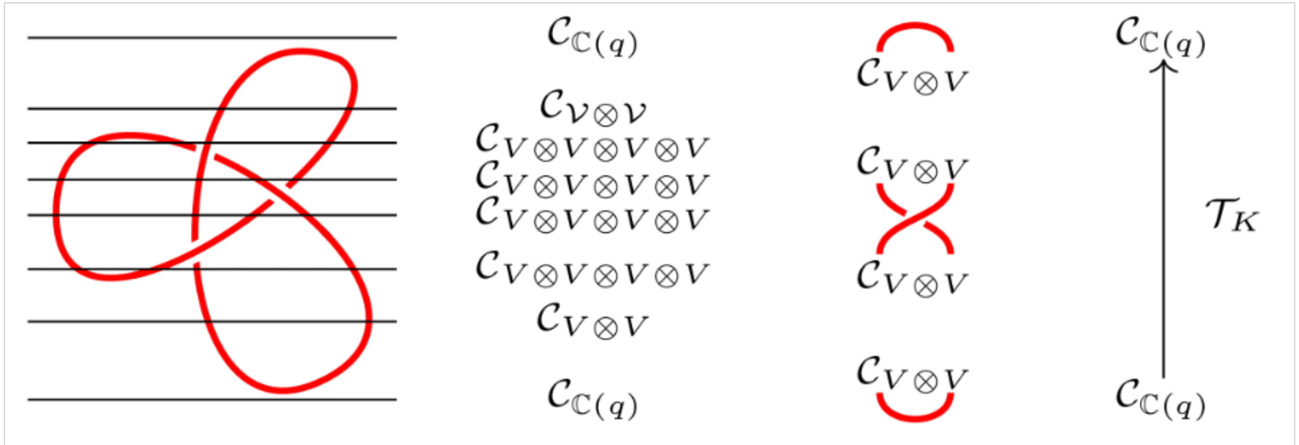
Generalisations: link homologies

Khovanov-Rozansky 04'+05' : slN & HOMFLYPT homologies (Matrix Factorizations)

Mackaay-Stošić-V. 07'+08' : slN (foams) & colored HOMFLYPT homologies (Soergel bimodules)

Webster 10' :

all $(\mathfrak{g}, V)$



Not really calculable!

Witten 11' : fivebranes and knots (Khovanov homology). A sort of categorification of CS theory - not a TQFT

Naisse-V. 17' : HOMFLYPT homology (categorified Verma modules)

Still missing: Kauffman 2-variable polynomial

Link homologies → The 4th dimension

2-representation theory

- J. Rasmussen (04'): s-invariant
  L. Piccirillo (18') Conway knot is slice
- M. Khovanov (05')' Q. You (12') : Hopfological algebra
- S. Morrison, K. Walker, P. Wedrich (19') :
  4-manifold invariants from HKhR (skein lasagna)
- Q. Ren, M. Willis (24') :
  Skein lasagna detects exotic structures
- Y. Liu, A. Mazel-Gee, D. Reutter,
  C. Stroppel, P. Wedrich (24') :
  Braided monoidal $(\infty,2)$- category of Soergel bimodules

- V. Mazorchuk, V. Miemietz (from 10'):
  2-representation theory
  (+ Mackaay, Tubbenhauer, Macpherson, ...)
- G. Naisse, P.V. (from 15') : 2-Verma modules
  (doesn't fit in MM framework)

- A. Manion, R. Rouquier (20'): Higher reps and cornered Heegaard-Floer homology

...

How strong are quantum invariants ?   Ask a computer !

D. Tubbenhauer, V. Zhang + D.T., A. Lacabanne, P.V. (W.I.P.) :

Jones & Khovanov: 100% detection on knots up to 10 crossings,
Then both start decreasing.
At 16 crossings, the Jones polynomial drops to 49.4%, and
Khovanov doesn't do much better: 54.7%

The situation on quantum invariants of 3-manifolds in far worse....

Nevertheless they allowed proving statements ....

Current trend : use big data/AI techniques on them.

Thank you !